Bremsstrahlung of a slow electron at a Coulomb center in an external electromagnetic field

V. P. Kraĭnov and S. P. Roshchupkin

Moscow Engineering Physics Institute
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A theory is developed of spontaneous bremsstrahlung of a slow electron scattered by a Coulomb center in the presence of a weak external monochromatic electromagnetic field. The external field is taken into account with the framework of first-order perturbation theory, and the unperturbed wave functions of the electron are used in the quasiclassical approximation. A simple analytic formula is obtained for the bremsstrahlung cross section. It is shown in which of the emitted-spontaneous-photon spectral intervals the theoretical results can be experimentally verified, without the strong slow-electron ordinary (Kramers) bremsstrahlung background that is produced in the absence of an external electromagnetic field. It is concluded that the obtained quasi-classical cross section, which is valid for electron energies that are low compared with the Rydberg energies, supplements the well known fast-electron bremsstrahlung cross section obtained in the Born approximation for interaction of an electron with a Coulomb center: the latter cross section is valid at electron energies that are high compared with the Rydberg energies. Numerical estimates are presented for the obtained bremsstrahlung cross sections. It is concluded that the quasiclassical cross section exceeds by many orders the Born cross section for bremsstrahlung in the presence of a weak electromagnetic field.

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§1. INTRODUCTION

Spontaneous bremsstrahlung of a nonrelativistic electron scattered by a Coulomb center is quite well known.[1] We consider here this radiation in the presence of a weak external classical electromagnetic field. We assume that this field is monochromatic. A solution of such a problem within the framework of the Born approximation in terms of the interaction of the electron with the Coulomb center is contained in Ref. 2. The Born approximation is valid only for electrons of sufficient velocity, whose energies are high compared with the characteristic atomic energy, i.e., with the Rydberg energy.

In this paper we deal with the opposite limiting case, when the energy of the electron scattered by the Coulomb center is low compared with the Rydberg energy. The wave functions of such a slow electron in the field of a Coulomb center can be described within the framework of the quasiclassical approximation. We consider spontaneous emission of such an electron in the case of scattering accompanied by absorption or emission of one photon of an external given electromagnetic field (weakness of the field is also understood in this sense). We note in this connection that in Ref. 3 was investigated stimulated bremsstrahlung of a quasiclassical electron scattered by a Coulomb center, i.e., emission of one photon of frequency equal to the frequency of the external electromagnetic field. We, however, consider emission of a spontaneous photon with a frequency different from that of the external field.

Since the initial and final states of the scattered electron are treated quasiclassically, the solution of our problem yields, apart from trivial notation changes, solutions of the following related problems: 1) spontaneous Raman scattering accompanied by transitions between highly excited atomic states; 2) photoionization by Rydberg states of the atoms, accompanied by bremsstrahlung of ionized electrons at the same atom; 3) spontaneous bremsstrahlung of slow electrons, accompanied by their recombination after absorbing a photon of the external electromagnetic field. Cross sections are obtained for all these processes.

We use the atomic system of units $\hbar = e = M_e = 1$ (the speed of light is $c = 137$).

§2. GENERAL THEORY

We assume that the external field $\mathscr{E}\cos\omega t$ ($\mathscr{E}$ is the amplitude of the electric field) is linearly polarized, and choose the polarization axis along the z direction. We consider first, for the sake of argument, absorption of one photon of this field by an electron, and it is more convenient to deal first with electron states in a discrete spectrum. We denote by $N = n,l,m$ the set of quantum numbers of the initial highly excited state of the atom (n—principal, l—orbital, m—magnetic), by N'' the set of intermediate-state quantum numbers, and by N' the set of the quantum numbers of the final state of the discrete spectrum, when a photon from the electromagnetic field has been absorbed and a spontaneous photon of frequency ν emitted. We assume initially, to be definite, that $n' > n$.

The two-photon matrix element describing the considered transition has the well known form

$$V_{NN'}^{(2)} = \sum_{N''}\left(\frac{V_{NN''}V'_{N''N'}}{\omega_{n''n}-\omega} + \frac{V'_{NN''}V_{N''N'}}{\omega_{n''n}+\nu}\right). \tag{1}$$

Here $V = z\mathscr{E}/2$ is the amplitude of the perturbation due to the external electromagnetic field, and $V' = (2\pi\nu/v)^{1/2}\,(\mathbf{r}\cdot\mathbf{e})$

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is the amplitude of the perturbation due to the electromagnetic field of the spontaneous emission (v is the normalization volume). Both perturbations are written in the dipole approximation, and $\mathbf{e}$ is the unit vector of the spontaneous-photon polarization. We assume hereafter $v = 1$.

The transition probability per unit time, according to the Fermi "golden rule," is

$$dw_{NN'}=2\pi|V_{NN'}^{(2)}|^2\delta(\omega_{n'n}-\omega+\nu)\,d\rho_{\mathbf{k}}, \tag{2}$$

where $d\rho_{\mathbf{k}} = d\,\mathbf{k}/(2\pi)^3$ is the number of final states of the spontaneous photon, and $k = \nu/c$ is the wave number of this photon.

Summing (2) over the final states of the electron N', which are located in a narrow interval near the values corresponding to the energy conservation law $\omega_{n'n} = \omega - \nu$, we use the known rule for highly excited Rydberg atomic states:

$$\sum_{n'}(\ldots)=n'^3\int d\omega_{n'n}(\ldots). \tag{3}$$

We then obtain from (2), taking (1) and (3) into account,

$$\frac{dw_{NN'}}{d\nu}=\frac{n'^3\nu^3\mathscr{E}^2}{8\pi c^3}\int d\Omega_{\mathbf{k}}|z_{NN'}^{(2)}|^2, \tag{4}$$

where $d\Omega_{\mathbf{k}}$ is the solid angle of the emitted spontaneous photon and

$$z_{NN'}^{(2)}=\sum_{N''}\left\{\frac{z_{NN''}(\mathbf{r}_{N''N'}\mathbf{e})}{\omega_{n''n}-\omega}+\frac{z_{N''N'}(\mathbf{r}_{NN''}\mathbf{e})}{\omega_{n''n}+\nu}\right). \tag{5}$$

Equation (4) determines the probability of emission of the spontaneous photon per unit interval of its frequencies.

We proceed next to calculate the two-photon matrix dipole element (5) for transitions with different quantum numbers.

We conclude this section by noting that the employed perturbation theory in terms of $\mathscr{E}$ in lowest order is valid 4 if the condition $\mathscr{E}n'\ll 1$ is satisfied.

§3. CALCULATION OF TWO-PHOTON MATRIX ELEMENTS

We consider first the two-photon matrix element (5) for the transition $nlm\to n'l'm'$. Then

$$z_{NN'}^{(2)}=e_z v_{NN'},$$

$$v_{NN'}=\sum_{n''l''}\left(\frac{1}{\omega_{n''n}-\omega}+\frac{1}{\omega_{n''n}+\nu}\right)z_{nlm}^{n''l''m}\,z_{n''l''m}^{n'l'm}. \tag{6}$$

Summing (4) over two independent polarizations of the spontaneously emitted photon and integrating over the solid angle $d\Omega_{\mathbf{k}}$ of its emission, we obtain

$$\frac{dw_{nlm}^{n'l'm}}{d\nu}=\frac{n'^3\nu^3\mathscr{E}^2}{3c^3}|v_{NN'}|^2. \tag{7}$$

We assume that the quasiclassical orbital quantum numbers are large enough, so that $l\gg 1$. The dipole matrix element is then of the form

$$z_{nlm}^{n''l''m}=\frac{1}{2}\left[1-\left(\frac{m}{l}\right)^2\right]^{1/2}R_{nl}^{n''l''}, \tag{8}$$

where $R_{nl}^{n''l''}$ is a radial dipole matrix element. By the same token, the dependence of the two-photon matrix element (6) on the magnetic quantum number m is explicitly separated. Averaging (7) over this magnetic quantum number of the initial state of the electron, we obtain the transition probability averaged over m:

$$\left\langle\frac{dw_{nlm}^{n'l'm}}{d\nu}\right\rangle=\frac{n'^3\nu^3\mathscr{E}^2}{90c^3}|u_{nl,n'l'}|^2, \tag{9}$$

where

$$u_{nl,n'l'}=\sum_{n''l''}\left(\frac{1}{\omega_{n''n}-\omega}+\frac{1}{\omega_{n''n}+\nu}\right)R_{nl}^{n''l''}R_{n''l''}^{n'l'}. \tag{10}$$

We calculate now the radial two-photon matrix element (10). We consider first in sufficient detail the case $l' = l + 2$. The main contribution to the sum (10) is made by terms with n'' in the region where the energy denominators vanish: $\omega_{n''n} - \omega\approx 0$ for the first term and $\omega_{n''n} + \nu\approx 0$ for the second. This agrees with the general concept of transition to the classical limit, according to which transitions via intermediate state are transformed from virtual (without energy-conservation law) to real (with energy conservation for the intermediate states). Of course, the number of states n'' is quite large in the region where the energy conservation law is satisfied for the intermediate states.

The radial quantum numbers n, n', and n'' of the states differ very greatly. The corresponding quasiclassical radial dipole matrix elements were obtained in Ref. 5 and take at $n' > n$ the form

$$R_{nl}^{n'l\pm1}=\frac{l^2}{\pi\sqrt{3}\,(nn')^{3/2}}\,\omega_{n'n}^{-1}\left[K_{2/3}\left(\omega_{n'n}\frac{l^3}{3}\right)\pm K_{1/3}\left(\omega_{n'n}\frac{l^3}{3}\right)\right]. \tag{11}$$

Here $K_s(x)$ is a Macdonald function. Equation (11) is valid[5] if $l\ll n,n'$ (otherwise the radial dipole matrix elements are exponentially small). We note that, as already mentioned, the orbital momenta are assumed to be quasiclassically large, i.e., $l\gg 1$).

Substituting (11) in (10) we obtain from the first term of (10) the expression

$$u_{nl,n'l+2}^{(1)}=\frac{l^4(nn')^{-3/2}}{3\pi^2\omega\nu}\left[K_{2/3}\left(\omega\frac{l^3}{3}\right)+K_{1/3}\left(\omega\frac{l^3}{3}\right)\right]$$

$$\times\left[K_{2/3}\left(\nu\frac{l^3}{3}\right)-K_{1/3}\left(\nu\frac{l^3}{3}\right)\right]\sum_{n''=-\infty}^{\infty}\frac{1}{n''-N_1}, \tag{12}$$

where

$$N_1=(n^{-2}-2\omega)^{-1/2}. \tag{13}$$

In the derivation of (12) we took it into account the $n'' > n' > n$. The quantity N_1 characterizes the region of quantum numbers n'' of the intermediate state in the first term of (19), a region influencing the sum significantly.

The sum in (12) is easily separated and is equal to $-\pi\cot\pi N_1$. The second term in the sum (10) is similarly calculated, and the values of n'' for it are close to

$$N_2=(n^{-2}+2\nu)^{-1/2}. \tag{14}$$

In the upshot expression (10) with $l' = l + 2$ takes the form

$$u_{nl,n'l+2}=-[\mathrm{ctg}\,\pi N_1+\mathrm{ctg}\,\pi N_2]\,l^4(3\pi\omega\nu)^{-1}(nn')^{-3/2}\left[K_{2/3}\left(\omega\frac{l^3}{3}\right)+K_{1/3}\left(\omega\frac{l^3}{3}\right)\right]\left[K_{2/3}\left(\nu\frac{l^3}{3}\right)-K_{1/3}\left(\nu\frac{l^3}{3}\right)\right]. \quad (15)$$

In the resonant case we have

$$\mathrm{ctg}\,\pi N_1\to\pi^{-1}(N_1-[N_1])^{-1}, \quad (16)$$

where [...] denotes the integer part of the number. This is tantamount to stating that only one resonant term with $n''=[N_1]$ remains in the sum (10).

Substituting (15) in (9) we obtain the probability of the transition $l\to l+2$, averaged over the magnetic quantum numbers m of the initial state:

$$\left\langle\frac{dw_{nlm}^{n'l+2m}}{d\nu}\right\rangle=\frac{\nu l^8\mathscr{E}^2(\mathrm{ctg}\,\pi N_1+\mathrm{ctg}\,\pi N_2)^2}{810\pi^2c^3\omega^2n^3}\times\left[K_{2/3}\left(\omega\frac{l^3}{3}\right)+K_{1/3}\left(\omega\frac{l^3}{3}\right)\right]^2\left[K_{2/3}\left(\nu\frac{l^3}{3}\right)-K_{1/3}\left(\nu\frac{l^3}{3}\right)\right]^2. \quad (17)$$

We note that a structure of the type $\cot\pi N_1$ appears also when multiphoton ionization of quasiclassical levels is considered.

On going to the case when the final state n' is in the continuous spectrum, we must substitute in (17) $\cot\pi N_1\to i$, since the sum in (12) is replaced by an integral with respect to n'' and is equal to πi when account is taken of the usual circuiting around the pole $n''-N_1\to n''-N_1-i\delta$.

The probability of the transition $l\to l-2$ can be calculated similarly. The result differs from (17) only in that the function $K_{1/2}$ reverses sign in both factors of (17).

We turn now to the transition $l\to l$. It is effected via two intermediate orbital angular momenta $l''=l\pm1$. Calculations similar to those carried out above in the derivation of (17) yield

$$\left\langle\frac{dw_{nlm}^{n'lm}}{d\nu}\right\rangle=\frac{\nu l^8\mathscr{E}^2(\mathrm{ctg}\,\pi N_1+\mathrm{ctg}\,\pi N_2)^2}{810\pi^2c^3\omega^2n^3}\times4\left[K_{2/3}\left(\omega\frac{l^3}{3}\right)K_{2/3}\left(\nu\frac{l^3}{3}\right)+K_{1/3}\left(\omega\frac{l^3}{3}\right)K_{1/3}\left(\nu\frac{l^3}{3}\right)\right]^2. \quad (18)$$

From (17) and (18) we easily obtain the transition probability summed over the orbital momenta $l'=l, l\pm2$ of the final states. We shall not present the corresponding formula.

We have considered so far the probability of a transition with conservation of the magnetic quantum number, $m\to m$. Actually spontaneous emission of a photon can change the magnetic quantum number by unity, so that $m'=m\pm1$. The treatment of such transitions is perfectly analogous to that of the transitions analyzed above. Leaving out the rather laborious manipulations, we present only the result for the sum over $m'=m, m\pm1$ and $l'=l, l\pm2$:

$$\sum\left\langle\frac{dw_{nlm}^{n'l'm'}}{d\nu}\right\rangle=\frac{\nu l^8\mathscr{E}^2(\mathrm{ctg}\,\pi N_1+\mathrm{ctg}\,\pi N_2)^2}{81\pi^2c^3\omega^2n^3}\times\left[K_{2/3}^2\left(\omega\frac{l^3}{3}\right)+K_{1/3}^2\left(\omega\frac{l^3}{3}\right)\right]\left[K_{2/3}^2\left(\nu\frac{l^3}{3}\right)+K_{1/3}^2\left(\nu\frac{l^3}{3}\right)\right]. \quad (19)$$

This result takes into acount also the transitions considered above with $m'=m$.

We modify Eq. (19) for the case when the initial state of the electron is in the continuous spectrum (with momentum p), and the final state of the electron also belongs to the continuous spectrum. We are interested here in emission of spontaneous photons of frequency $\nu>p^2/2$, for at lower frequencies ν there exists the usual spontaneous bremsstrahlung in which the external electromagnetic field $\mathscr{E}$ does not take part. This radiation has a much higher probability than (19), since we assume the external magnetic field $\mathscr{E}$ to be weak and consider it in first-order perturbation theory. The process of absorption of a photon ω and subsequent emission of a photon ν (the first term in (10)) proceeds via an intermediate state in the continuous spectrum, thus calling for the substitution $\cot\pi N_1\to i$ in (19). At the same time, the emission of the spontaneous photon ν and subsequent absorption of the photon ω proceeds via a discrete-spectrum state, so that $\cot\pi N_2$ should be retained in (19). Furthermore, replacing $|n|$ in (19) by p^{-1}, where p is the momentum of the incident electron, we obtain

$$\sum\left\langle\frac{dw_{plm}^{p'l'm'}}{d\nu}\right\rangle=\frac{\nu l^8\mathscr{E}^2(1+\mathrm{ctg}^2\,\pi N_2)\,p^3}{81\pi^2c^3\omega^2}\times\left[K_{2/3}^2\left(\omega\frac{l^3}{3}\right)+K_{1/3}^2\left(\omega\frac{l^3}{3}\right)\right]\left[K_{2/3}^2\left(\nu\frac{l^3}{3}\right)+K_{1/3}^2\left(\nu\frac{l^3}{3}\right)\right]; \quad (20)$$

The meaning of the sum here is the same as in (10).

We now average the probability (20) over the orbital momentum l of the initial level, assuming all the states of this level to be equally populated. We use to this end the relation

$$\langle(\ldots.)_l\rangle=\frac{2}{n^2}\int_0^\infty(\ldots.)\,l\,dl. \quad (21)$$

The upper integration limit extends here to infinity in view of the rapid convergence of the integral. We then obtain from (20)

$$\frac{dw_{pp'}}{d\nu}=\frac{2\cdot3^{1/3}}{9\pi^2}\frac{\nu p^5\mathscr{E}^2}{c^2\omega^{10/3}}f\left(\frac{\nu}{\omega}\right)(1+\mathrm{ctg}^2\,\pi N_2), \quad (22)$$

where

$$f(x)=\int_0^\infty t^{7/3}[K_{2/3}^2(t)+K_{1/3}^2(t)][K_{2/3}^2(xt)+K_{1/3}^2(xt)]dt. \quad (23)$$

The function $f(x)$ has a simple analytic form at $x\gg1$ and $x\ll1$. At $x\gg1$

$$f(x)=\frac{\pi\Gamma^2(2/3)}{4^{1/3}\sqrt{3}}x^{-2}. \quad (24)$$

In addition, this function satisfies a condition that follows directly from its definition (23):

$$f(x^{-1})=x^{10/3}f(x). \quad (25)$$

Its value at $x\ll1$ can therefore be easily obtained from (24) with the aid of (25). We shall not cite it here. Figure 1 shows the numerically calculated function $f(x)$ at $x\geqslant1$. The value of

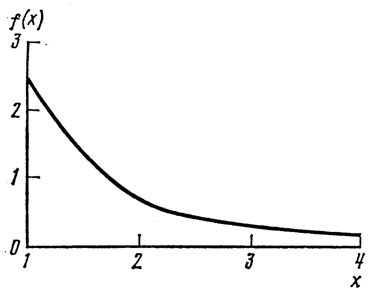

FIG. 1. The universal function $f(x)$ [see (23)] that enters in the expression (26) for the bremsstrahlung cross section.

this function in the interval $0<x<1$ can be easily obtained from the condition (25).

The bremsstrahlung cross section is given by (22) divided by p. We ultimately obtain for the cross section (summed over the polarizations of the emitted spontaneous photon, integrated over its emission angles, summed over the magnetic and orbital quantum numbers of the electron final state, and averaged over the magnetic and orbital quantum number of the initial electron):

$$\frac{d\sigma_{pp'}(\mathscr{E})}{d\nu}=\frac{2\cdot 3^{1/6}}{9\pi^2}\frac{\nu p^4\mathscr{E}^2}{c^3\omega^{10/3}}(1+\operatorname{ctg}^2\pi N_2)f\left(\frac{\nu}{\omega}\right) \tag{26}$$

(in contrast to the probability, the cross section does not contain the normalization volume, always taken equal to unity above).

The result (26) obtained in the quaisclassical approximation supplements the result of the Born approximation[2]

$$d\sigma_{pp'}(\mathscr{E})/d\nu=2p'\mathscr{E}^2/9\nu\omega^4pc^3, \tag{27}$$

which is valid at $p,p'\gg 1$, whereas (26) is valid when the reverse conditions $p,p'\ll 1$ hold. Comparing the two cross sections we see that generally speaking the quasiclassical cross section exceeds the Born cross section appreciably.

It is advantageous to express the result (26) in units of the usual cross section for bremsstrahlung in the absence of the external electromagnetic field $\mathscr{E}$ and calculated by the Kramers formula

$$d\sigma_{pp'}/d\nu=16\pi/3\sqrt{3}p^2\nu c^3. \tag{28}$$

The ratio of the cross sections is

$$\frac{d\sigma_{pp'}(\mathscr{E})}{d\sigma_{pp'}}=\frac{3^{-1/6}}{8\pi^3}\frac{\nu^2p^6\mathscr{E}^2}{\omega^{16/3}}(1+\operatorname{ctg}^2\pi N_2)f\left(\frac{\nu}{\omega}\right). \tag{29}$$

The result (26) could be verified experimentally by investigating the spectral distribution of the emitted spontaneous photons (in this case, the energy conservation law

$$p'^2/2-p^2/2=\omega-\nu \tag{30}$$

sorts out automatically electrons with definite final energy). For separation from the ordinary bremsstrahlung of electrons in the absence of the external electromagnetic field $\mathscr{E}$ it must be noted that the continuous spectrum of the ordinary bremsstrahlung (28) has a short-wave end point (at $\nu=E_p=p^2/2$), for when ν is increased further the electron lands in the discrete spectrum (recombination) and the spontaneous emission has a line rather continuous spectrum. In the presence of an external electromagnetic field of frequency ω, however, the short-wave end point of the continuous spectrum of the emitted spontaneous photons is located much farther, namely at $\nu=E_p+\omega$, so that in the interval of the spontaneous-emission frequencies

$$E_p<\nu<E_p+\omega$$

the background due to the ordinary bremsstrahlung (28) is absent (apart from the line structure connected with the electron recombination). As for the order of magnitude, according to (29) this cross section is of the order of $E_p^{-1/3}\mathscr{E}^2$ of the Kramers cross section (the field $\mathscr{E}$ is measured in atomic units $\mathscr{E}_0=5\times 10^9$ V/cm). To obtain this estimate we have assumed that the energy of the spontaneous photon and the energy ω of the external-field photon are of the order of the electron energy E_p.

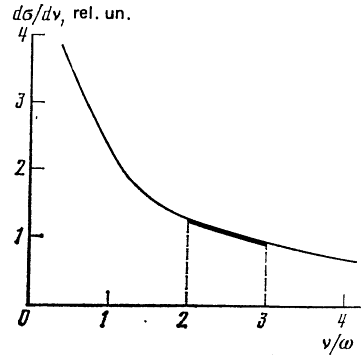

FIG. 2. Functional dependence of the spectral distribution of the cross section for bremsstrahlung in the presence of an external electromagnetic field $\mathscr{E}\cos\omega t$ on ν/ω, where ν is the spontaneous-emission frequency. The thick line corresponds to the example $E_p=2\omega$, where E_p is the energy of the incident electron. The long-wave boundary of the spectrum corresponds to the region where ordinary (Kramers) bremsstrahlung sets in; the short-wave boundary of the region is determined by the law of energy conservation in the presence of a weak external electromagnetic field.

The dependence of the cross section (26) on the ratio ν/ω is shown in Fig. 2. According to the foregoing, this dependence should be considered in the interval

$$E_p/\omega<\nu/\omega<1+E_p/\omega. \tag{31}$$

Both cases $E_p>\omega$ and $E_p<\omega$ are possible. By way of example, the thick line of Fig. 2 shows the case $E_p=2\omega$. Inside such a line, the recombination structure of the usual electron-recombination process corresponds in (26) to frequencies ν at which $\cot\pi N^2\rightarrow\infty$, i.e., where it is generally speaking incorrect.

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