Thermodynamic potentials of the superfluid phases of helium-3 in the region of strong critical fluctuations

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The critical thermodynamics of superfluid helium-3 in zero magnetic field is theoretically studied. The free energies of the superfluid phases A and B are computed in the region of strong fluctuations by summing the ring diagrams. The equations of the binodals and the high-temperature spinodals for the two phases are derived, and the magnitudes of the condensate amplitude jumps on these curves are determined.

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The tendency towards fluctuational destabilization of second-order phase transitions in systems with many-component order parameters was discovered by Wilson and Fisher about ten years ago.[1] Since then several scores of models have been developed in which the interaction of the critical fluctuations leads to the conversion of the continuous phase transitions into first-order transitions. In solving the problem of the order of a transition, researchers in this field limit themselves in the overwhelming majority of cases to the investigation of the stability of the quartic form in the Landau expansion for the free energy of the system under consideration, and the inference of fluctuational "first-orderness" of a phase transition is made on the basis of the fact that this form loses its positive definiteness when $T \to T_c$.

It is difficult to regard such a solution to the problem as comprehensive. The complete analysis should clearly include the computation of the free energy of the ordered phase and the determination of the equation of state, the shape of the first-order phase transition line, the magnitude of the order-parameter jump, etc. The problem of carrying out such an analysis is especially critical in those cases when the interaction of the order-parameter fluctuations determines not only the order of the transition, but also the structure of the low-temperature phase. As is well known, for systems that can be in one of several ordered phases at $T < T_c$, allowance for the critical fluctuations can modify the relations, predicted by the Landau theory, between the free energies of these phases, and make thermodynamically stable a phase different from the one predicted by the phenomenological theory.[2–5] In such situations the computation of the free energy of the system below T_c is an operation absolutely necessary for the establishment of the form of the system's phase diagram. Moreover, the solution of problems of the present type is now of direct practical interest, since the first experiments on the study of first-order phase transitions having a fluctuational character have already been published.

In the present paper we consider the critical behavior of superfluid helium-3, a system which is characterized by fluctuational instability of its second-order phase transitions and fluctuational competition between the low-temperature phases. Below we shall compute the free energies of the phases A and B as functions of the condensate amplitude, derive the equations of the binodals and high-temperature spinodals, and determine the magnitudes of the order-parameter jumps on these curves. Here for the free energies F_A and F_B we shall use an approximate representation in the form of Gaussian path integrals, which will enable us to carry all the computations through in the case of the nontrivial five-charge model with an eighteen-component order parameter.

Thus, the fluctuation Hamiltonian of a superfluid Fermi liquid with p pairing without allowance for the very weak dipole interaction and the fluctuation-spectrum anisotropy,[4] which is unimportant here, has the form[6]

$$H = \frac{1}{2}\int d\mathrm{x}\Big[\varkappa_0^2 \varphi_{ij}\varphi_{ij}^* + (\nabla_i \varphi_{jk})(\nabla_i \varphi_{jk})^* + \frac{1}{2}(\beta_1 \varphi_{ij}\varphi_{ij}\varphi_{kl}^*\varphi_{kl}^* + \beta_2 \varphi_{ij}\varphi_{ij}^*\varphi_{kl}\varphi_{kl}^* + \beta_3 \varphi_{ij}\varphi_{kj}\varphi_{kl}^*\varphi_{il}^* + \beta_4 \varphi_{ij}\varphi_{kl}\varphi_{kj}^*\varphi_{il}^* + \beta_5 \varphi_{ij}\varphi_{il}\varphi_{kj}^*\varphi_{kl}^*)\Big], \quad (1)$$

where the first index of φ_{ij} is a spin index, while the second is an orbital index.

Let us briefly recall the principal relations characterizing the thermodynamics of liquid helium-3 in the region of applicability of the Landau theory. The order parameter in the Anderson-Morel and Balian-Werthamer phases has the form[7]

$$\langle \varphi_{ij} \rangle_A = \frac{\Delta}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \langle \varphi_{ij} \rangle_B = \frac{\Delta}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

The free energies of these phases are equal to

$$F_A = \frac{\varkappa_0^2}{2}\Delta^2 + \frac{\beta_2+\beta_4+\beta_5}{4}\Delta^4,$$

$$F_B = \frac{\varkappa_0^2}{2}\Delta^2 + \left(\frac{\beta_1+\beta_2}{4} + \frac{\beta_3+\beta_4+\beta_5}{12}\right)\Delta^4. \quad (3)$$

For the constants β_α a microscopic theory of the type of the BCS theory, but without allowance for the paramagnon exchange, yields[8]

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$$\beta_1=-(1+0.1\delta)C,\quad \beta_2=(2+0.2\delta)C,\quad \beta_3=(2-0.05\delta)C,$$

$$\beta_4=(2-0.55\delta)C,\quad \beta_5=-(2+0.7\delta)C,$$

$$C={}^{21}/_{20}\zeta(3)N(0)/(\pi T_c)^2, \tag{4}$$

where $N(0)$ is the density of states at the Fermi surface and δ is the paramagnon coupling constant. This constant has a threshold value $\delta_c\approx0.4$, such that for $\varkappa_0^2<0$ and $\delta<\delta_c$ the B phase is the thermodynamically stable phase; for $\delta>\delta_c$, the A phase.

Let us now discuss the role of the critical fluctuations. As is well known, the interaction of the fluctuations leads to the renormalization of the "mass" $\varkappa_0$ and the coupling constants β_α. The nature of this renormalization is such that, as $\varkappa^2$ decreases, the coefficients of Δ^4 in the expressions (3) decrease until they vanish at some finite values of $\varkappa^2$ (Refs. 4 and 6); this is a harbinger of a first-order phase transition into one of the superfluid phases. To describe these phases and the phase transition itself, we need to go beyond the lowest approximation in terms of dressed charges. The contribution of the next order of perturbation theory to the free energy is given by a sum of ring diagrams.[9,10] In the ring approximation the terms of the type $\varphi^3\langle\varphi\rangle$ are completely ignored, and the Hamiltonian H_Δ describing the interaction of the fluctuations with the condensate is similar in structure to the Hamiltonian of a free field. For the phase A it has the form

$$H_\Delta{}^A=\frac{\Delta^2}{2}\int d\mathbf{x}[(2\beta_1+\beta_2+2\beta_3+\beta_4+\beta_5)\psi_{11}\psi_{11}^*$$
$$+(\beta_1+\beta_4+\beta_5)(3\psi_{12}'^2$$
$$+\psi_{12}''^2+\psi_{13}\psi_{13}^*+\psi_{22}'^2+\psi_{33}'^2)+(\beta_2+\beta_3)(\psi_{21}\psi_{21}^*+\psi_{31}\psi_{31}^*)$$
$$+\beta_2(\psi_{23}\psi_{23}^*+\psi_{33}\psi_{33}^*)+(\beta_2+\beta_4-\beta_5)(\psi_{22}''^2+\psi_{32}''^2)], \tag{5}$$

where

$$\psi_{k1,\,k2}=2^{-1/2}(\varphi_{k1}\pm i\varphi_{k2}),\quad \psi_{k3}=\varphi_{k3},\quad \psi_{ij}'=\mathrm{Re}\,\psi_{ij},\quad \psi_{ij}''=\mathrm{Im}\,\psi_{ij}. \tag{6}$$

The Hamiltonian H_Δ for the B phase is equal to

$$H_\Delta{}^B=\frac{\Delta^2}{2}\int d\mathbf{x}\Big\{\Big[\frac{\beta_1}{2}+\frac{\beta_3}{6}+\frac{\beta_5}{6}\Big](\varphi_{ij}\varphi_{ij}+\varphi_{ij}^*\varphi_{ij}^*)$$
$$+\Big(\beta_2+\frac{\beta_3+\beta_5}{3}+\frac{2}{3}\beta_4\Big)\varphi_{ij}\varphi_{ij}^*+\frac{\beta_2}{6}(\varphi_{ii}\varphi_{jj}+\varphi_{ii}^*\varphi_{jj}^*)$$
$$+\frac{2\beta_1+\beta_2}{3}\varphi_{ii}\varphi_{jj}^*+\frac{\beta_4}{6}(\varphi_{ij}\varphi_{ji}+\varphi_{ij}^*\varphi_{ji}^*)+\frac{\beta_3+\beta_5}{3}\varphi_{ij}\varphi_{ji}^*\Big\}. \tag{7}$$

For the free energy in the single-loop representation we have the following representation[11]:

$$F=\frac{\varkappa^2\Delta^2}{2}+\frac{\gamma\Delta^4}{4}-\ln_R\int\exp(-\tilde H_0-\tilde H_\Delta)D\{\tilde\varphi(\underline{x})\}, \tag{8}$$

where $\tilde H_0$ is the Hamiltonian of a free field with mass $\varkappa$, $\tilde H_\Delta$ is a Hamiltonian of the type (5), (7) with the bare coupling constants replaced by dressed charges γ_α, and γ is the linear combination of the γ_α, that figures in the corresponding Landau expansion. For the A and B phases $\gamma_A=\gamma_2+\gamma_4+\gamma_5$, and $\gamma_B=\gamma_1+\gamma_2+(\gamma_3+\gamma_4+\gamma_5)/3$. The subscript R of the logarithm indicates that we have to make three subtractions at the point $\Delta^2=0$ when computing this term.

The subsequent computations are, in principle, simple. The path integral in (8) is Gaussian, and its evaluation amounts to the calculation of some 18×18 determinant. In the case of the phase A the matrix corresponding to this determinant is diagonal, while in the case of the phase B the matrix can be diagonalized by elementary methods. As a result, we obtain for the two superfluid phases the free energies

$$\tilde F_A=2d^2-L^Ad^4+{}^8/_{13}|g_1|[2f(d^2,\,2+v+x+2y)+4f(d^2,\,v+y)$$
$$+2f(d^2,\,v+x-2y-2z)+4f(d^2,\,v)], \tag{9}$$

$$\tilde F_B=2d^2-L^Bd^4+{}^8/_{13}|g_1|[5f(d^2,\,1+v+x+y)$$
$$+5f(d^2,\,v+{}^1/_3(x+y)-1)+3f(d^2,\,v+x-{}^5/_3y-{}^4/_3z-1)], \tag{10}$$

where

$$\tilde F=\frac{32\pi|g_1|}{13\varkappa^3}F,\quad d=\frac{\Delta}{\varkappa}|\gamma_1|^{1/2},\quad L^A=\frac{\gamma_A}{\gamma_1},\quad L^B=\frac{\gamma_B}{\gamma_1},$$
$$v=\frac{\gamma_2}{\gamma_1},\quad x=\frac{\gamma_4+\gamma_5}{\gamma_1},\quad y=\frac{\gamma_3}{\gamma_1},\quad z=\frac{\gamma_5-\gamma_3}{\gamma_1},\quad g_\alpha=\frac{13\gamma_\alpha}{8\pi\varkappa}, \tag{11}$$
$$f(d^2,\xi)=\frac{1}{3}+\frac{|\xi|d^2}{2}+\frac{\xi^2d^4}{8}-\frac{1}{3}(1+|\xi|d^2)^{3/2}.$$

In writing down (9) and (10), we dropped in the square brackets the terms corresponding to the "soft" and Goldstone branches in the excitation spectrum of the system. It can be shown that the contribution of these terms to $\tilde F$ is actually proportional to g_1^2, and allowance for them would lead in the present case to our exceeding the computational accuracy.

The resulting formulas (9) and (10) are quite unwieldy, and their analytical investigation is possible only when $|g_\alpha|\ll1$. In this case the dimensionless order parameter $d\gg1$ on the binodal and the high-temperature spinodal, and to find d and the form of these curves themselves we can use the following simplified expressions for the $\tilde F$:

$$\tilde F_{A,B}\approx2d^2-\frac{8|g_1|}{39}M_3^{A,B}d^3+\Big[\frac{|g_1|}{13}M_4^{A,B}-L^{A,B}\Big]d^4, \tag{12}$$

where

$$M_{2\mu}{}^A=2|2+v+x+2y|^\mu+4|v+y|^\mu+2|v+x-2y-2z|^\mu+4|v|^\mu,$$

$$M_{2\mu}{}^B$$
$$=5|1+v+x+y|^\mu+5|v+{}^1/_3(x+y)-1|^\mu+3|v+x-{}^5/_3y-{}^4/_3z-1|^\mu. \tag{13}$$

Hence we can now easily determine the equations for the binodals,

$$L^{A,B}=\frac{M_4^{A,B}|g_1|}{13}\Big[1-\frac{8|g_1|(M_3^{A,B})^2}{117M_4^{A,B}}\Big], \tag{14}$$

and spinodals,

$$L^{A,B}=\frac{M_4^{A,B}|g_1|}{13}\Big[1-\frac{|g_1|(M_3^{A,B})^2}{13}\Big] \tag{15}$$

and also find the magnitudes of the order-parameter jumps

TABLE I. The values of the parameters $M_3^{A,B}$ and $M_4^{A,B}$ at two points located in the $L^A = L^B$ hyperplane, and corresponding to first-order phase transitions into the superfluid state in the weak-coupling regime.

v	x	y	z	$L^A = L^B$	M_3^A	M_3^B	M_4^A	M_4^B
−0.857	0.857	−1.286	2.60	0	24.9	25.5	35.8	36.6
−0.78	0.90	−1.20	2.44	0.12	21.5	22.4	29.4	31.4

on these curves:

$$d_b^{A,B} = \frac{39}{2|g_1|M_3^{A,B}}, \quad d_s^{A,B} = \frac{13}{|g_1|M_3^{A,B}}. \tag{16}$$

Knowing the expressions for the free energies of the superfluid phases, we can determine to what extent the form of the phase diagram of liquid helium-3 as predicted in Ref. 4 on the basis of an analysis of a quartic form of the type (3) will be modified when allowance is made for the ring diagrams. To do this, we need not, in principle, study the thermodynamics of the system in the entire region of physically accessible values of g_1, v, x, y, and z. For $|g_1| \ll 1$, it is sufficient to consider only a small neighborhood of the $L^A = L^B$ hyperplane, in which $F_A = F_B$ when the rings are neglected, and in the vicinity of which we can, consequently, observe the fluctuational competition of the superfluid phases. Moreover, to answer the question posed, it is not at all necessary to compute the magnitudes of the free energies of the A and B phases for thermodynamic-equilibrium values of d. The cause of this good luck lies in the virtually exact coincidence of the numerical values of the parameters M_3 and M_4 for both phases when $L^A \approx L^B \ll 1$. Let us, to demonstrate this fact, give the values of M_3 and M_4 at two points lying in the $L^A = L^B$ hyperplane in the (v, x, y, z) phase space. The first of these points, for which $L^A = L^B = 0$, corresponds to the stability boundary of biquadratic forms of the type (3), while the second point, where $L^A = L^B = 0.12$, corresponds to a weak instability of these forms, which can be compensated by the sum of the ring diagrams even at small charge values (i.e., with $|g_1| \approx 0.05$). The numerical data are given in Table I. The coordinates given there for the two indicated points were obtained through a computer solution of renormalization-group equations for v, x, y, and z (Ref. 4) with initial conditions dictated by the formulas (4) and paramagnon-condensate values close to the "critical" value $\delta = 0.27$, i.e., close to that value at which the phase trajectories cross or touch the $L^A = L^B$ hyperplane at the first-order phase transition point.

Thus, the "radiation" component of the free energy, while guaranteeing the thermodynamic stability of the superfluid phases when $L^A, L^B > 0$, makes at the same time virtually no contribution to the difference $F_A - F_B$ when $L^A \approx L^B$. This means that the structure of the low-temperature phase in liquid helium-3 is determined only by the action of the fourth-order invariants in the expressions for F. Consequently, all the results obtained earlier, including the conclusion that the phase diagram contains a narrow "beak" formed by first-order phase-transition lines, remain valid within the framework of the above-developed more complete theory.

The validity of the last assertion is, of course, closely tied-in with the fulfillment of the condition $|g_\alpha| \ll 1$. For $|g_\alpha| \sim 1$ the actual picture of the critical behavior of superfluid helium-3 can differ appreciably from the one that we have painted. Since it is not possible for us to find with any accuracy the magnitudes of the dimensionless charges on the first-order phase transition lines (even the bare vertices are known only approximately), we are not in a position to choose judiciously between the weak- and strong-coupling regimes. And although we can arguably assert that the qualitative results of the theory do not depend on which regime is realized,[4] it is nevertheless useful to consider a system for which problems connected with the possible transition into the strong-coupling regime do not arise. We have in mind the four-dimensional model with the Hamiltonian (1). In this case the ring approximation gives

$$F_{A,B} \approx \frac{\varkappa^4}{4|\gamma_1|}\left[2d^2 - L^{A,B}d^4 + \frac{|\gamma_1|M_4^{A,B}}{16\pi^2} d^4 \ln d^2\right], \tag{17}$$

where the approximate-equality sign reminds us of the fact that these formulas are valid with logarithmic accuracy. Since in the case under consideration the first-order phase transitions occur at exponentially small values of $L^{A,B}$ and the pattern of phase trajectories of the renormalization-group equations for the ratios of v, x, y, and z are the same as in three dimensions, the coordinates of the triple point, and the values of the parameters $M_4^{A,B}$ there, coincide practically exactly with those given in the first row in Table I. Consequently, everything said earlier about the character of the critical behavior in the weak-coupling limit remains valid. And the resulting formulas of the type (14)–(16) can easily be obtained from (17); they are, for brevity, not written out here.

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