Atoms in the field of strong multimode synchronized radiation

E. E. Fradkin

Leningrad State University

(Submitted 19 October 1982)
Zh. Eksp. Teor. Fiz. **84**, 1654–1663 (May 1983)

The behavior is considered of a two-level atom system in a high-frequency field of periodically varying intensity. An example of such a field is synchronized multimode laser radiation. Expressions for the quasi-energy levels of the nuclei in the periodic-amplitude field are obtained in an approximation adiabatic in the phase, and are suitable for an arbitrary number of modes of the synchronized radiation. The spectrum of the resonance frequencies Ω_k ($k = 1,2,3,...$) corresponding to the crossing of the quasilevels of the system is determined. The frequencies Ω_k determine the positions of the parametric resonances. With a biharmonic field as an example, it is shown that if the intermode frequency interval Ω becomes equal to Ω_k the absorption (gain) of individual modes and of their combination tones becomes extremal.

PACS numbers: 32.80.Bx, 42.50. + q

1. INTRODUCTION

The simplest case of a field whose intensity is modulated at a frequency Ω is a biharmonic field with mode-frequency difference Ω. The behavior of a two-level system in a quasiresonant strong biharmonic field has been well investigated experimentally and theoretically.

A case investigated experimentally is when the frequency of one field component (E_0) coincided with the absorption line center ω_0 and the frequency $\omega_1 = \omega_0 + \Omega$ of the other component (E_1) was scanned in the frequency interval from $\omega_0 - \omega_R$ to $\omega_0 + \omega_R$, where ω_R is the frequency of the optical nutations (the Rabi frequency) of the biharmonic field. The absorption coefficient K_1 of the scanned field was measured as a function of the difference frequency Ω (Refs. 2 and 3). When both field components were strong, two new phenomena were observed: a subradiative structure of the absorption line,[2] wherein absorption maxima appear at frequencies[1)] Ω_R/k ($k = 2,3,4$), and an anomalously strong dependence of the absorption[3] near $\Omega = 0$ on the amplitude ratio E_1/E_0.

The onset of absorption maxima was attributed in Ref. 2 to multiphoton transitions, and an approximate theory of the resonances was given there for the case $E_1 \ll E_0$. Braun and Miroshnichenko[4] investigated theoretically the case of equal mode amplitudes, $E_1 = E_0$. Absorption in the region $\Omega \approx 0$ was not calculated in Ref. 2 and 4. In Refs. 5 and 6 was constructed a quantitative theory for arbitrary mode amplitudes E_0 and E_1, describing both phenomena in good agreement with experiment.[2,3]

The purpose of the present paper is to interpret the resonant structure of the absorption of individual field components as the result of the crossing of the atom quasi-energy levels.[7] Such an interpretation makes the physical results more lucid and makes possible a generalization of the theory to include the case of any number of synchronized modes. Just as the crossing of the magnetic sublevels of the fine or hyperfine structure of an excited state of an atom leads to a redistribution of the intensities among spontaneous-emission components having different polarizations,[8] quasi-level crossing leads to a redistribution of the stimulated gain or absorption of individual modes of the synchronized radiation (see Fig. 1). As seen from Fig. 1, the absorption (gain) of the field E_1 (curve 1) coincides with the minima of the ab-

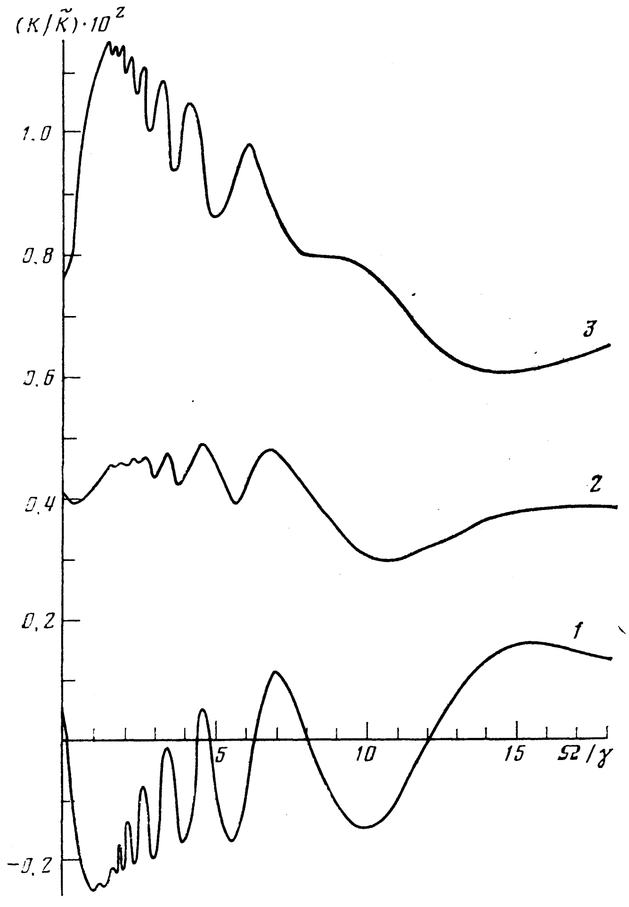

FIG. 1. Dependences of the absorption coefficients of individual components of a bichromatic field $K_1/\tilde{K}$ (curve 1), $K_0/\tilde{K}$ (curve 3) and of the average absorption coefficient $K/\tilde{K} = (K_1 + K_0)/2\tilde{K}$ (curve 2) on the intermode frequency interval Ω/γ ($\tilde{K}$ is the linear absorption coefficient at the line centers, the field frequencies are $\omega_0 = \omega_{12}$, $\omega_1 = \omega_{12} + \Omega$, $dE_0/\sim\hbar = 11.1$, $E_1/E_0 = 0.8$).

 0038-5646/83/050965-06$04.00

sorption (gain) of the field E_0 (curve 3). The average absorption (gain) of the fields (curve 2) changes little.

The onset of resonances can be explained in the following manner: the elements of the atomic density matrix oscillate at a frequency determined by the difference between the quasi-energy values. At the quasi-level crossing points the oscillations take place at zero frequency and enter into resonance with the constant "pumping force." These resonance appear in the absorption spectrum $K_1(\Omega)$ when the difference frequency Ω coinides with the frequency Ω_k corresponding to the quasi-level crossing. The transition scheme is shown in Fig. 2; the transitions d occur at the crossing of quasi-level of the lower or upper states of the atom, and the c transitions do not oocur at the quasi-level crossing.

In Sec. 4 is determined the number of natural oscillations that appear in the spectrum K_1. It depends on the ratio $\rho = E_1/E_0$ of the field amplitudes. This fact makes it possible to explain the anomalous dependence of $K_1(0)$ on ρ near $\rho = 1$. As ρ approaches unity in the range $\rho_1 < \rho < \rho_2$ the number of resonances in the spectrum K_1 increases rapidly, but they are not resolved separately and form a common maximum with center at $\Omega = 0$. (The values of ρ_1 and ρ_2 are defined in Eq. (37).) At $\rho < \rho_1$ and $\rho > \rho_2$ there appear in K_1 a limited number of oscillations $k < k_m$, where k_m is defined in (36). In this case there are no resonances in the absorption coefficient $K_1(\Omega)$ at $|\Omega| < \Omega_R/k_m$ and the values of K_1 in the region $\Omega \approx 0$ is much lower than in the interval $\rho_1 < \rho < \rho_2$ (see Fig. 3).

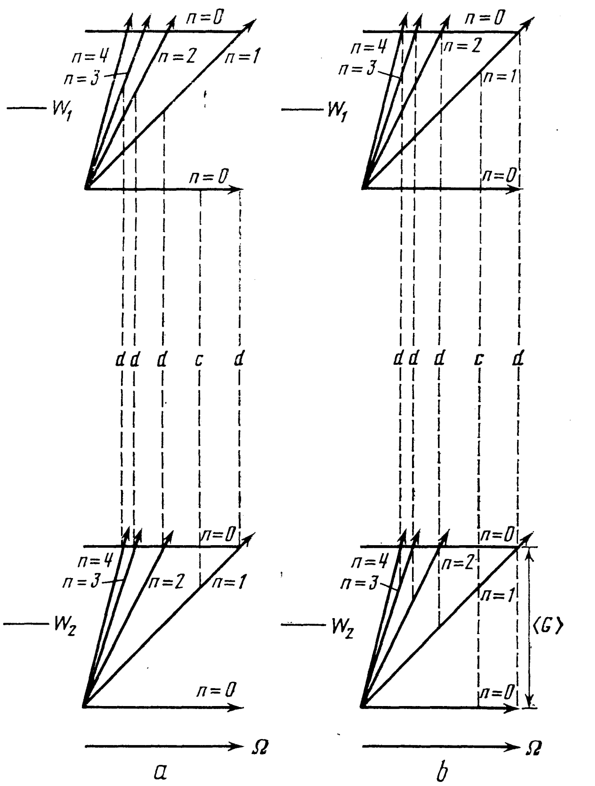

FIG. 2. Examples of transitions between quasi-levels of upper and lower atomic state. Corresponding to each state are two systems of quasi-levels (see Ref. 14a). Several quasi-levels are shown as functions of Ω; a—low-frequency wing of the line, b—high-frequency wing, c—nonresonant transitions, d—resonant transitions.

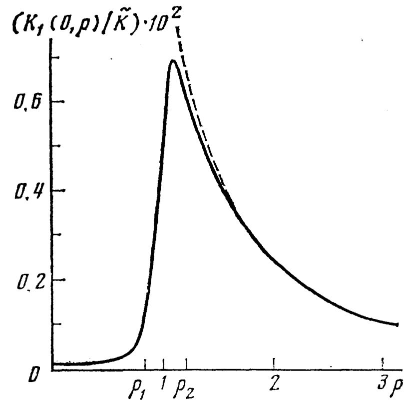

FIG. 3. Absorption coefficient $K_1(0,\rho)/\tilde{K}$ of the component E_1 of a bichromatic field at $\Omega = 0$ vs the amplitude ratio ρ at $dE_0/\hbar\gamma = 10$ (solid curve). The values of ρ_1 and ρ_2 were calculated from Eq. (37). Dashed curve—the asymptotic expression $K_1(0,\rho)/K = (1 + G_1^2/\gamma^2)^{-1}$ at $E_0^2 \ll E_1^2$.

The connection between the natural oscillations of the atomic system and the subradiative structure of the absorption of an individual field component can be generalized to include the case of synchronized multimode radiation. In this case, too, the resonant-frequency spectrum is of the form $\Omega_k = \Omega_R/k$ ($k = 1,2,3...$), and the following simple expression is obtained for the Rabi frequency:

$$\Omega_R = d\langle E(t)\rangle/\hbar, \tag{1}$$

where d an off-diagonal dipole-moment matrix element, $\langle E(t)\rangle$ is the field amplitude averaged over the period $2\pi/\Omega$,

$$\mathscr{E}(t) = \mathrm{Re}\,[E(t)\exp\{-i(\omega_{12}t+\varphi(t))\}],$$

ω_{12} is the frequency of the atomic transition, and $\varphi(t)$ is the slow phase of the field.

Expression (1) was obtained in an approximation adiabatic in the phase shift. In the case when the radiation is modulated only in amplitude and the carrier frequency coincides with the transition frequency, the phase shift $\varphi(t)$ undergoes a jump at the instants of time $\Omega t_n = (2n+1)$ and the adiabatic approximation no longer holds. In this case, however, an exact solution exists.

Thus, for harmonic amplitude modulation, $\mathscr{E}(t)$ can be represented in the form

$$\mathscr{E}(t) = \mathrm{Re}\,\{[E_0 + 2E_1\cos\Omega t]\exp(-i\omega_{12}t)\}, \tag{2}$$

and Ω_R is determined only by the amplitude of the carrier, $\Omega_R = dE_0/\hbar$.

Parametric resonances in fluorescence were theoretically and experimentally investigated for the field (2) in Refs. 9 and 10. In the experiment, measurement of the fluorescence of an atomic sodium beam at the $3\,^2S_{1/2} - 3\,^2\Delta_{3/2}$ resonance line yielded two resonances ($k = 1,2$).

3. QUASI-ENERGY SPECTRUM OF TWO-LEVEL SYSTEM IN AN INTENSE PERIODIC FIELD

A two-level atomic system is described by the wave function

$$\Psi = C_1(t)\exp(-iW_1t/\hbar)\psi_1 + C_2(t)\exp(-iW_2t/\hbar)\psi_2, \tag{3}$$

where ψ_1, ψ_2 and W_1, W_2 are the wave functions and energies of the levels 1 (upper) and 2.

The equations for the probability amplitudes C_1 and C_2 are correspondingly[11]

$$i\hbar dC_1/dt = V_{12}C_2 \exp(i\omega_{12}t), \quad (4)$$

$$i\hbar dC_2/dt = V_{21}C_1 \exp(-i\omega_{12}t). \quad (5)$$

In the dipole approximation, the matrix element of the Hamiltonian of the interaction with the field can be written in the form

$$V_{12} = V_{21} = -{}^1/{}_2\hbar G(t) \exp\{-i(\omega_{12}t + \varphi(t))\} + \text{c.c.}, \quad (6)$$

where $\omega_{12} = (W_1 - W_2)/\hbar$ and $G = dE(t)/\hbar$. From (5) and (6) it follows in the rotating-field approximation that

$$-idC_1/dt = (G(t)C_2/2) \exp(-i\varphi(t)),$$
$$-idC_2/dt = (G(t)C_1/2) \exp(i\varphi(t)). \quad (7)$$

We introduce a new variable

$$\tau = \frac{1}{2}\int_0^t G(t')\,dt' \quad (8)$$

and change from the system (7) to the second-order equation

$$\frac{d^2C_{1,2}}{d\tau^2} \pm i\frac{d\varphi}{d\tau}\frac{dC_{1,2}}{d\tau} + C_{1,2} = 0. \quad (9)$$

In the approximation adiabatic in the phase shift we discard the small term $i(d\varphi/d\tau)(dC_{1/2}/d\tau)$. This can be done in the case of a strong quasi-resonant field:

$$|d\varphi/d\tau| = |2d\varphi/G(t)\,dt| \ll 1. \quad (10)$$

In the approximation (10), the solution of Eq. (9) takes the form (after returning to the variable t)

$$C_1 = C_{10}^{(\pm)} \exp[\pm(i/2)\int_0^t G(t')\,dt'],$$
$$C_2 = C_{20}^{(\pm)} \exp[\pm(i/2)\int_0^t G(t')\,dt'], \quad (11)$$

$C_{10}^{(\pm)}$ and $C_{20}^{(\pm)}$ are constants, and it follows from (7) that $C_1^{(+)} = C_2^{(+)}$ and $C_1^{(-)} = -C_2^{(-)}$.

If the field amplitude $G(t)$ is a periodic function of the time with frequency Ω,

$$G(t + 2\pi/\Omega) = G(t), \quad (12)$$

one can determine from the solutions (11) the quasi-energy levels of the atom.[7] To this end we obtain the mean value $\langle G\rangle$ (averaged over the period) of the field amplitude $G(t)$:

$$\langle G\rangle = (\Omega/2\pi)\int_0^{2\pi/\Omega} G(t)\,dt. \quad (13)$$

We recall the amplitude is always positive, and consequently $\langle G\rangle > 0$.

The shifts of the quasi-levels for the upper and lower states are

$$\hbar\varepsilon_{1n} = [\langle G\rangle/2 \pm n\Omega]\hbar, \quad \hbar\varepsilon_{2m} = -[\langle G\rangle/2 \pm m\Omega]\hbar,$$
$$n, m = 0, 1, 2, \ldots \quad (14)$$

The quasi-energy levels themselves are

$$W_1 + \hbar\varepsilon_{1n} = W_1 + [\langle G\rangle/2 \pm n\Omega]\hbar,$$
$$W_1 + \hbar\varepsilon_{2m} = W_1 - [\langle G\rangle/2 \pm m\Omega]\hbar,$$
$$W_2 + \hbar\varepsilon_{1n} = W_2 + [\langle G\rangle/2 \pm n\Omega]\hbar, \quad (14a)$$
$$W_2 + \hbar\varepsilon_{2m} = W_2 - [\langle G\rangle/2 \pm m\Omega]\hbar.$$

The equations for the quasi-energies were obtained for a rather general case when the field amplitude $G(t)$ is an arbitrary periodic function of the time and the condition (10) is satisfied. Precisely such is strong synchronous laser radiation with an arbitrary number of modes. This radiation has a discrete spectrum with frequency interval Ω. A special role is played in the interaction of radiation with matter by the frequencies Ω_k determined from the quasi-level crossing condition $\varepsilon_{1,n} = \varepsilon_{2,m}$

$$\Omega_k = \langle G\rangle/k, \quad k = m - n = 1, 2, 3. \quad (15)$$

In fact, the general solution of the system (7) contains the two linearly independent solutions (11). Therefore the density-matrix elements $\sigma_{ik} = C_i C_k^*$ $(i,k = 1,2)$ contain oscillating terms with frequencies $\nu_{nm} = \varepsilon_{1,n} - \varepsilon_{2,m}$. Only at $\nu_{nm} = 0$, i.e., in the case when the intermode frequency $\Omega = \Omega_k$ (15), do the oscillation occur at zero frequency and enter into resonance with the constant "pumping force."

As will be shown in Sec. 4, in this case the absorption of the radiation in the medium becomes extremal and the frequencies Ω_k (15) are singled out.

We consider two concrete examples of the determination of $\langle G\rangle$ for different fields.

1. Biharmonic field:

$$V_{12}(t)/\hbar = -\text{Re}\{\exp(-i(\omega_{12} - \mu)t)[G_0 + G_1\exp(-i\Omega t)]\}, \quad (16)$$

$$\langle G(t)\rangle = (\Omega/\pi)\int_{-\pi/2\Omega}^{\pi/2\Omega} (G_0^2 + G_1^2 + 2G_0G_1\cos\Omega t)^{1/2} dt = 2(G_0 + G_1)E(k)/\pi, \quad (17)$$

where $E(k)$ is a complete elliptic integral of the second kind, and the modulus of the integral is $k = 2(G_0G_1)^{1/2}/(G_0 + G_1)$. The value of $\langle G(t)\rangle$ in the interval $\Omega/2 \geqslant \mu \geqslant 0$ is independent of μ. The most dangerous instants of time for the satisfaction of the inequality (10) are $\Omega t = (2n + 1)\pi$ (n is an integer). At these instants the condition (10) at $G_1 \approx G_0$ takes the form

$$\Omega \ll (G_1 - G_0)^2/(G_1 + G_0). \quad (18)$$

For other instants of time we can replace (10) by the weaker condition

$$\mu + \Omega/2 \ll \langle G\rangle. \quad (19)$$

The inequality (18) means that if $G_1 = G_0$ the approximation (10) turns out to be invalid, but in the case $\mu = 0$, when the frequency of the field G_1 coincides with the line center ω_{12}, solutions at $G_1 = G_0$ were obtained in Refs. 4 and 6 outside the framework of the approximation (10). In Ref. 6 it was also shown that the solutions are continuous in the parameter $\rho = G_1/G_0$. We therefore extend in this paper the solutions obtained at $\mu = 0$ to include also the case of equal fields $G_1 = G_0$. It follows from (17) that in this case $\langle G\rangle = 4G_1/\pi$.

We note that the spectrum of the resonant frequencies (15) and (17) does not depend on the frequency shift μ, but this statement does not hold in the vicinity of equal fields $G_1 = G_0$ if the condition (18) is violated. Thus, no parametric resonances appear in the symmetric case $\mu = \Omega W/2$ and $G_1 = G_0$.

2. Three-mode field:

$$V_{12}(t)/\hbar = -\mathrm{Re}\,\{\exp(-i\omega_{12}t)\,[G_{-1}\exp(i\Omega t) + G_0 + G_1\exp(-i\Omega t)]\},$$

$$\langle G(t)\rangle = (\Omega/2\pi)\int_0^{2\pi/\Omega} dt\,[G_1^2 + G_0^2 + G_{-1}^2 + 2G_1G_{-1}\cos 2\Omega t + 2G_0(G_1+G_{-1})\cos\Omega t]^{1/2}. \tag{20}$$

At arbitrary amplitudes of the modes G_1, G_0, and G_{-1} the integral (20) must be calculated numerically. Under the condition $4G_1G_{-1} = G_0^2$, however, the integral can be evaluated analytically: $G(t) = G_1 + G_{-1}$.

3. CONNECTION OF THE RESONANT FREQUENCIES Ω_k WITH THE POSITIONS OF THE PARAMETRIC RESONANCES

The interaction of an electromagnetic field with a medium, with account taken of the pumping and the relaxation, are described by the density-matrix equations. We shall use these equations in Sec. 4 to establish the connection between the resonant frequencies Ω_k with the extrema of the absorption of individual components of the synchronized radiation. In the present section we describe a method of determining the parametric-resonance frequencies which, as will be shown, conicide with the frequencies Ω_k (15).

We consider the system of equations of a two-level matrix with time-constant pumping, describing the interaction of a medium (in the immobile-atom model) with quasi-resonant radiation:

$$\begin{aligned} &d\rho_{12}/dt + (\gamma + i\omega_{12})\rho_{12} = iV_{12}(t)N/\hbar, \quad \rho_{12} = \rho_{21}^*, \\ &dN/dt + \gamma N = i2V_{12}(\rho_{12} - \rho_{21})/\hbar + \gamma N_0, \quad N = \rho_{11} - \rho_{22}, \end{aligned} \tag{21}$$

where we consider for simplicity the case of equal relaxation constants, γN_0 is the pumping per unit time, and V_{12} and ω_{12} are defined by expression (5).

We transform to the slow variables

$$\begin{aligned} &\rho_{12} = \exp(-i\omega_{12}t)\,\sigma, \\ &\sigma = \exp\{-\gamma t - i\varphi(t)\}(y + iz), \quad N = 2xe^{-\gamma t}. \end{aligned} \tag{22}$$

The system (21) goes over then into the system of real equations

$$\begin{aligned} &\frac{dx}{dt} = G(t)z + \left(\frac{\gamma}{2}\right)e^{\gamma t}N_0, \\ &\frac{dy}{dt} = -\frac{d\varphi(t)}{dt}z, \quad \frac{dz}{dt} = -G(t)x + \frac{d\varphi}{dt}y. \end{aligned} \tag{23}$$

It follows from (6) that the instantaneous field frequency is $\omega_{12} + d\varphi/dt$, i.e., $d\varphi/dt$ plays the role of the detuning of the instantaneous field frequency from the center ω_{12} of the absorption line.

From the system (23) we obtain in the approximation (10) a second-order differential equation for the function $z(t)$:

$$\frac{d^2z}{dt^2} - \frac{1}{G}\frac{dG}{dt}\frac{dz}{dt} + G^2z = -\gamma G(t)e^{\gamma t}\frac{N_0}{2}. \tag{24}$$

Equation (24) takes into account only the amplitude modulation and admits, as is known, of an exact analytic solution. This, however, does not mean that we neglect the phase modulation completely. The variable $y(t)$ determined from the second equation of (23) is proportional to $d\varphi/dt$. Accordingly, the off-diagonal element σ (22), which determines the response of the two-level system, takes into account the phase modulation accurate to terms of second order in the small parameter $(d\varphi/dt)G^{-1}$ (10).

Equation (24) for the case of a biharmonic field, in terms of the dimensionless temporal coordinate $\tau = \Omega t$, is written in the form

$$\{d^2/d\tau^2 + \lambda + a\cos\tau\}z(\tau) = f(\tau), \tag{25}$$

$$\lambda = (G_0^2 + G_1^2)/\Omega^2, \quad a = 2G_0G_1/\Omega^2, \quad \rho = G_1/G_0, \tag{26}$$

$$f(\tau) = -\frac{\gamma(2G_0G_1)^{1/2}}{2\Omega^2}[{}^1/_2(\rho + \rho^{-1}) + \cos\tau]^{1/2}\exp\left(\frac{\gamma\tau}{\Omega}\right)N_0. \tag{27}$$

We consider Eq. (25) at $f(\tau) = 0$. This homogeneous equation is called the Mathieu equation. We seek its solution in the form

$$z(\tau) = \exp(is\tau)\sum_{n=-\infty}^{\infty} C_n e^{in\tau}, \tag{28}$$

with $C_n \to 0$ as $|n| \to \infty$. Substitution of (28) in (25) yields the known recurrence relation for the Mathieu equation:

$$(\lambda - (k+s)^2)C_k + (a/2)(C_{k+1} + C_{k-1}) = 0. \tag{29}$$

We seek a purely periodic solution of the Mathieu equation. According to (28), such solutions exist at $s = 0$. It is possible to find a discrete set of such solutions, corresponding to the eigenvalues λ_k. It can be seen from the expression (26) for λ that by varying the parameters of the problem (the intermode frequency Ω or the intensities of the modes G_0^2 and G_1^2) it is possible to make the values of λ equal to any of the eigenvalues λ_k, in which case parameteric resonance sets in. The frequencies Ω at which resonances occur are determined by the condition

$$\Omega_k = [(G_0^2 + G_1^2)/\lambda_k]^{1/2}. \tag{30}$$

The eigenvalues λ_k for the Mathieu equation were determined by Braun[12] under the conditions $a \gg 1$, i.e., at $\Omega^2 \ll 2G_0G_1$. This condition is close to the inequality (19). The frequencies obtained by substituting in (30) the eigenvalues λ_k from Ref. 12 are

$$\Omega_k = \frac{2(G_0 + G_1)}{\pi k}E\left[\frac{2(G_1G_0)^{1/2}}{G_1 + G_0}\right] = \frac{\langle G\rangle}{k}, \tag{31}$$

where $\langle G\rangle$ is defined by (17) and E is an elliptic integral.

The spectrum of the frequencies (31) coincides with the spectrum (15). This is not accidental. After all, $s\hbar\Omega$ is the difference between the quasi-energy levels. The frequencies (31) were obtained under the condition $s = 0$, i.e., under the condition that the quasi-levels cross. The same condition was also used to determine the frequencies (15).

Another possibility of determining λ_k (meaning also Ω_k) follows from the condition that the homogeneous system of algebraic equations have a solution if the determinant of the system (29) is set equal to zero as $s = 0$.

The determination of an eigenvalue of an infinite determinant is in the general case quite a complicated matter, but for large $k \gg 1$ the diagonal aproximation $\lambda_k = k^2$ is found to be sufficient. According to (30), we obtain in this case

$$\Omega_k = (G_0^2 + G_1^2)^{1/2}/k. \tag{32}$$

The same estimate of the resonant frequencies was given also in Ref. 6. Expressions (31) and (32) depend on k in the same manner and are symmetric about G_0 and G_1, i.e., the Ω_k are not changed by the interchange $G_0 \leftrightarrow G_1$. It suffices therefore to compare (31) with (32) for $\Omega_R/G_0 = k\Omega_k/G_0$ as functions of $\rho = G_1/G_0$ in the interval $0 < \rho \leqslant 1$. Such a comparison in Fig. 3 shows that the discrepancy does not exceed 10%.

In analogy with the procedure used here for a biharmonic field, a theory can be constructed for an arbitrary field whose amplitude is aperiodic function, and a relation similar to (30) can be found between the values of the modulation frequency Ω and the eigenvalues λ_k of the corresponding equation. Just as in (31), the spectrum of the frequencies Ω_k, which determine the positions of the parametric resonance, coincides with the spectrum (15).

4. PARAMETRIC RESONANCES AND ABSORPTION (AMPLIFICATION) OF LIGHT

In the preceding section we obtained the eigenvalues λ_k of the Mathieu equation; corresponding to them are the strictly periodic albeit nonharmonic eigenfunctions z_k (28) at $s = 0$. We ask now which of the amplitudes $C_n^{(k)}$ in the series (28) will be the large ones for the oscillation z_k. The answer to this question can be obtained by turning again to Ref. 12, in which a modified WKB method was developed and, by way of example, the eigenvalues and eigenfunctions of the Mathieu equation were obtained at $s = 0$. It follows from Ref. 12 that the numbers of the harmonics $C_n^{(k)}$ with large amplitudes lie between the following limits (turning points):

$$[\lambda_k - a(\Omega_k)]^{1/2} \leqslant |n| \leqslant [\lambda_k + a(\Omega_k)]^{1/2}. \tag{33}$$

All the remaining harmonics are small. Substituting $\lambda_k = k^2$, a (26) and Ω_k (30) in (33), we obtain in first-order approximation

$$k|1-\rho|/(1+\rho^2)^{1/2} < |n| < k(1+\rho)/(1+\rho^2)^{1/2}. \tag{34}$$

We note that for arbitrary ρ and k the amplitude of the zeroth harmonic $C_0^{(k)}$ for $\Omega = \Omega_k$ is small.

Turning now to the solution of the inhomogeneous equation (25) we note that the force $f(\tau)$ (27) is proportional to the amplitude $G(\tau)$ and is consequently a periodic function containing all the harmonics that are present in (28). The solution (25) will therefore increase as $\Omega - \Omega_k$, i.e., oscillations of the system will build up at the resonant frequency.

Returning to the initial system of equations (21) for the density-matrix elements, we see that it contains only a time-constant "pump force," and resonance between the oscillations of the density matrix elements and the "pump force" take place, as already noted in Sec. 2, at zero frequency.

Soluton of the system (21) involves integration of Eq (23) for the variable y and determination of σ in accord with Eq. (22). Thus, in the Fourier expansion of σ

$$\sigma = \sum_{n=-\infty}^{\infty} d_n e^{-in\tau} \tag{35}$$

the amplitude d_n of the harmonics whose numbers n satisfy (34) are maximal at $\Omega - \Omega_k$, and all the other amplitudes are minimal at $\Omega = \Omega_k$. As shown in Ref. 6, the coeffficients d_n of the Fourier series determine the coefficient of absorption (gain) K_n at the frequency $\omega_0 + n\Omega$. Consequently, K_n are extremal at $\Omega = \Omega_k$, with K_n maximal for the values of n that satisfy (34) and minimal for all other n. This conclusion is fully confirmed by calculation, in Ref. 6, of the absorption coefficient K_1 at the frequency $\omega_0 + \Omega$, and by the experimental determination of K_1 in Ref. 3 (see Fig. 1). It can be seen in Fig. 1 that the minima of K_0 coincide with the maxima of K_1, a confirmation of the deduced smallness of the zeroth harmonic $C_0^{(k)}$ at $\Omega = \Omega_k$.

The first inequality of (34) can be used to determine the number k_m of the maxima that appear in the absorption spectrum $k_n(\Omega)$ at a given ratio ρ of the field amplitudes:

$$k_m \leqslant |n|(1+\rho^2)^{1/2}|1-\rho|^{-1}C(\rho), \tag{36}$$

where

$$C(\rho) = 2(1+\rho)E(2\sqrt{\rho}/(1+\rho))/(1+\rho^2)^{1/2},$$

ρ varies in the range $0 < \rho < \infty$, and $C(\rho) \approx 1$ in the entire range of ρ (see Fig. 4).

It follows from the inequality (36) that the number of maxima is the same for the absorption coefficients K_n at the frequencies $\omega_0 + n\Omega$ and K_{-n} at the frequencies $\omega_0 - n\Omega$, and increases with increasing $|n|$. The number of k_m of maxima increases in any absorption coefficient K_n as $\rho \to 1$. The last fact can explain the anomalous growth of $K_1(\Omega,\rho)$ when ρ changes from 0.9 to 1.1 near $\Omega = 0$ (Ref. 3). The point is that $\Omega_k \to 0$ with increasing k, and the frequency interval $\Omega_k - \Omega_{k-1}$ between the neighboring maxima decreases. At large k the neighboring maxima are not resolved. The number n^* of the resolved maxima is determined by Eq. (27) of Ref. 6. At ρ close to unity the number of maxima $k_m > n^*$ and the merging extrema form a common maximum of the absorption coefficient at $\Omega = 0$. From the equality $k_m = n^*$ we obtain the limits of the interval $\rho_1 < \rho < \rho_2$ of the anomalous growth of $K_1(0,\rho)$ with increasing ρ:

$$\rho_{1,2} = 1 \mp \frac{2\gamma}{G_0}\left[1 - \frac{\gamma}{\sqrt{2}G_0}\frac{1}{(1\mp 2\gamma/G_0)^{1/2}}\right]. \tag{37}$$

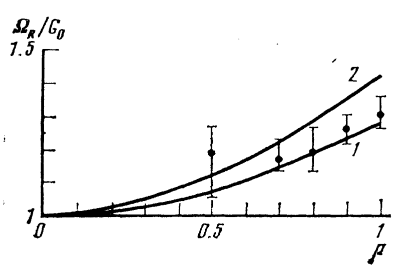

FIG. 4. Dependence of the Rabi frequency Ω_R/G_0 on ρ. The values of Ω_R for curve 1 were calculated from Eq. (31), and for curve 2 from the approximate formula (32). The points were obtained by averaging $\langle n\Omega_n \rangle/G_0$ over eight resonances, from $n = 2$ to $n = 9$. The positions of the maxima were obtained with a computer for different values of G_0 from the exact formulas of Ref. 6. The scatter is determined by the maximum and minimum values of $n\Omega_n/G_0$ at a given value of ρ.

The dependence of $K_1(0,\rho)$ on ρ at $G_0/\gamma = 10$ is shown in Fig. 3.

On the other hand, it follows from (34) that no maxima with numbers $k_m \leqslant k$ will appear in the absorption coefficient $K_{|n|}$ at the frequency $\omega_0 \pm n\Omega$, where $|n| < k(1+\rho)(1+\rho^2)^{-1/2}$.

Thus, at $k = 2$ and $\rho = 0.5$ or $\rho = 1.5$ there will be no second maximum in the absorption coefficient $K_{\pm 3}$ at the frequency $\omega_0 + 3\Omega$, but the subsequent maxima appear starting with $k = 3$. This also agrees with calculations by the continued-fraction method.[6]

We indicate in conclusion that the connection between the resonant fequencies $\Omega_k = \Omega_R/k$ of an atomic system in a field and the positions of the extrema of the absorption coefficients of individual harmonics can be generalized to include synchronized radiation with an arbitrary number of modes. Thus, for a field whose amplitude is modulated in the manner indicated in Eq. (2) at the frequencies Ω_R/k, where $\Omega_R = dE_0/h$, extrema of absorption (gain) of individual modes will appear. A simple analtytic equation for $\Omega_R = \langle G(t)\rangle$ was obtained also for the particular case of three-mode radiation in Eq. (20).

The author is grateful for V. I. Perel' for valuable advice and fruitful discussions and G. I. Toptygina for numerically calculating the plots in Figs. 1, 3, and 4.

[1)] A maximum at the Rabi frequency Ω_R (single-photon resonance) exists also at low amplitude of the scanned field and was experimentally investigated earlier.[1]

[1]A. M. Bonch-Bruevich, V. A. Khodovoĭ, and N. A. Chigir', Zh. Eksp. Teor. Fiz. **67**, 2069 (1974) [Sov. Phys. JETP **40**, 1027 (1975)].
[2]A. M Bonch-Bruevich, T. A. Vartanyan, and N. A. Chigir', Zh. Eksp. Teor. Fiz. **77**, 1899 (1979) [Sov. Phys. JETP **50**, 901 (1979)].
[3]A. M. Bonch-Bruevich, T. A. Vartanyan, and N. A. Chigir', Abstracts, 10-th All-Union Conf. on Coherent and Nonlinear Optics, Vol. 1, M. 1980, p. 325.
[4]P. A. Braun and G. P. Miroshnichenko, Zh. Eksp. Teor. Fiz. **81**, 63 (1981) [Sov. Phys. JETP **54**, 27 (1981)].
[5]G. I. Toptygina and É. E. Fradkin, All-Union Conf. on the Theory of Atoms and Atomic Spectra, Abstracts, p. 99, Tbilisi, 1981.
[6]G. I. Toptygina and É. E. Fradkin, Zh. Eksp. Teor. Fiz. **82**, 429 (1982) [Sov. Phys. JETP **55**, (1982)].
[7]Ya. B. Zel'dovich, Usp. Fiz. Nauk **110**, 139 (1973) [Sov. Phys. Usp. **16**, 427 (1973)].
[8]M. P. Chaĭka, Interferentsiya vyrozhdennykh atomnykh sostoyaniĭ (Interference of Degenerate Atomic States), Leningrad State Univ., 1975.
[9]P. J. Thomann, J. Phys. B **9**, 2411 (1976).
[10]P. J. Thomann, J. Phys. B **13**, 1111 (1980).
[11]L. D. Landau and E. M. Lifshitz, Kvantovaya Mekhanika, Vol. 1, Gostekhizdat, 1948, p. 170 [Quantum Mechanics, Nonrelativistic Theory, Pergamon, 1958].
[12]P. A. Braun, Teor. Mat. Fiz. **37**, 355 (1978).

Translated by J. G. Adashko