Contribution to the theory of acoustic quantum generators based on nonequilibrium superconductors

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The possibility of realizing phonon instability in nonequilibrium superconductors is considered on the basis of the kinetic equations for electrons and phonons. This instability can lead to an acoustic quantum-generator (AQG) regime, i.e., to coherent and monochromatic phonon emission from a superconducting film. The general conditions for phonon instability are derived and the possibility of satisfying them is analyzed in the case of microwave pumping and tunnel injection at low temperatures. It is shown that when electrons are injected from a bulky superconductor into a thin superconducting film, conditions that realize the AQG regime are satisfied in the film. The analysis is carried out with effective allowance for the inelastic electron-electron collisions. The departure of the electron subsystem of the superconductor from equilibrium can lead to a number of other instabilities that compete with the phonon instability and whose development might impede the onset of the AQG regime. The most typical of these instabilities are analyzed and it is shown that a subthreshold AQG regime can in principle be attained in experiment.

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§1. INTRODUCTION

The degree of ordering of a physical system that is very far from thermodynamic equilibrium can become higher than in the equilibrium state (see, e.g., Ref. 1). Superconductors are no exception in this respect. Thus, factors that casue the electron subsystem of a superconductor to deviate from equilibrium can make the system nondissipative, inhomogeneous in space, and nonstationary in time.[2] Another manifestation of ordering of a nonequilibrium semiconductor is the "phonon instability" [1)] discussed in the present article. This instability can lead to a "quasi-laser" generation of phonons, wherein phonon emission from a nonequilibrium-superconductor film is coherent and monochromatic.

The possibility of transition into "quasi-lasing" is connected with a circumstance common to Bose fields, namely that the phonon source contains terms proportional to the phonon occupation numbers.[3] This possibility was discussed independently in Refs. 4 and 5 for the cases of tunnel injection and microwave pumping, respectively (we note also a paper by Bulyzhenkov,[6] which borders quite closely on our problem). As shown in these papers, phonon instability calls for population inversion ($n_\varepsilon > 1/2$ in some energy region above the gap, where n_ε is the distribution function of the electronic excitations), but this condition is still insufficient for instability development. Population inversion can result from both microwave pumping and tunnel injection, but in the former case no phonon instability is apparently produced,[5] whereas in the case of tunnel injection from a bulky superconductor into a thin film (in the so-called *S-I-S′* junction) this situation seems to be more favorable.[4]

We analyze in this paper the problem of reaching the threshold of "quasi-lasing" from the point of view of the kinetic equations for phonons and electrons. We note that back in 1974 Chang and Scalapino, assuming a quasi-Fermi distribution function of the electron distribution function (with a certain effective chemical potential), calculated the sound absorption coefficient and found it to be negative at a high degree of disequilibrium; this enabled them to deduce the feasibility of sound amplification by a nonequilibrium superconductor.[7] However, in their following paper, devoted to the stability of the resultant picture, they reached the conclusion that at such a disequilibrium the system ceases to be in a spatially homogeneous state, and therefore "phonon instability has no physical meaning."[8] Later papers, particularly Refs. 2 and 9–13, give grounds for assuming that the stability problem is by far not trivial and the conclusion of Ref. 8 seems in our opinion to be excessively categorical.

The complicated nonlinear relation between the electronic excitations, the Cooper condensate, and the phonons leads to a great variety of phenomena in nonequilibrium superconductors, including to onset of instabilities of various kinds. The kinetic equations that describe the nonequilibrium behavior of a superconductor admit of a multitude (depending on the external conditions) of sollutions, and this multitude is the internal source of the instabilities that evolve as a result of inevitable fluctuations. From the experimental viewpoint primary interest attaches to the possibility of choosing the external parameters such that a subthreshold acoustic quantum generator (AQG) regime is attained in the simplest manner. Obviously, the simplest case is one in which the phonon instability (more accurately, the subthreshold regime) is realized in a spatially homogeneous and stationary situation, and the role of fluctuations capable of causing some other instabilities can be neglected. It follows from the results presented here that the threshold AQG regime can apparently be attained in this case for a definite choice of the system parameters. The influence of the devel-

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opment of some instability on the investigated regime, as well as the dynamics of the AQG itself, is of course quite interesting, but calls for a separate examination.

§2. POPULATION INVERSION IN NONEQUILIBRIUM SUPERCONDUCTORS

Let us ascertain what general condition must be satisfied by the nonequilibrium electron distribution function if phonon instability is to occur in a superconductor.

We assume that external factors that cause the electron subsystem to deviate from equilibrium establish, in a superconductor connected to an external thermostat, a dissipative stationary state wherein the nonequilibrium electronic excitations are characterized by a distribution function n_ε, where ε is the energy of the quasiparticle excitation. It will be assumed below that the distribution function n_ε is even, as well as spatially homogeneous and isotropic (as is also the order parameter Δ).

Consider the influence exerted on the electron subsystem by an external phonon flux characterized by occupation numbers $N_{\omega q}$ (we assume that $N_{\omega q} \gg 1$). In nonequilibrium superconductors the phonons can be absorbed (and emitted) both in pair-breaking (respectively, recombination) processes and in relaxation processes. At small deviation of the electron subsystem from equilibrium, the relaxation processes predominate.[14,15] As shown by Aronov and Spivak,[12] there is no phonon instability in this case (at any rate for microwave pumping). At a strong deviation from equilibrium, the recombination processes in superconductors are much faster than the relaxation processes,[15] and in this case one must consider the stability to recombination emission of phonons.

The number of phonons absorbed per unit time at a frequency ω_q in an interval $d\omega_q$ is given by ($\hbar = 1$)

$$d\dot{N}_{\omega_q}=\rho(\omega_q)I(\omega_q)d\omega_q, \tag{1}$$

where $\rho(\omega_q) = V\omega_q^2/2\pi^2u^3$, V is the volume of the superconductor, u is the speed of sound, and $I(\omega_q)$ is the phonon-electron collision operator.[3] In the case of interest to us, when the electronic excitations in the presence of an external perturbation can be described in terms of the distribution function n_ε, this operator takes the form

$$I(\omega_q)= \iint_\Delta^\infty d\varepsilon_1\, d\varepsilon_2 L(\varepsilon_1, \varepsilon_2) \Big\{ [(1-n_{\varepsilon_1})(1-n_{\varepsilon_2})-n_{\varepsilon_1}n_{\varepsilon_2}] \times\left(1+\frac{\Delta^2}{\varepsilon_1\varepsilon_2}\right)\delta(\varepsilon_1+\varepsilon_2-\omega_q)-2[n_{\varepsilon_1}(1-n_{\varepsilon_2})-(1-n_{\varepsilon_1})n_{\varepsilon_2}] \times\left(1-\frac{\Delta^2}{\varepsilon_1\varepsilon_2}\right)\delta(\varepsilon_2-\varepsilon_1+\omega_q)\Big\}, \tag{2}$$

$$L(\varepsilon_1, \varepsilon_2)=N_{\omega_q}\frac{\pi\lambda}{2}\frac{\omega_D}{\varepsilon_F}\frac{\varepsilon_1}{(\varepsilon_1^2-\Delta^2)^{1/2}}\frac{\varepsilon_2}{(\varepsilon_2^2-\Delta^2)^{1/2}} \theta(\varepsilon_1^2-\Delta^2)\theta(\varepsilon_2^2-\Delta^2)$$

(we recall that $N_{\omega q} \gg 1$; $\lambda \sim 1$ is the dimensionless electron-phonon interaction constant). The recombination part of the collision integral (2) can be represented in the form[2]

$$I(\omega_q)^{rec}= \int_\Delta^{\omega_q-\Delta} L(\varepsilon, \omega_q-\varepsilon)\left(1+\frac{\Delta^2}{\varepsilon(\omega_q-\varepsilon)}\right)(1-2n_\varepsilon)d\varepsilon, \tag{3}$$

and the relaxation part in the form

$$I(\omega_q)^{rel}=2\int_\Delta^\infty L(\varepsilon, \varepsilon+\omega_q)\left(1-\frac{\Delta^2}{\varepsilon(\varepsilon+\omega_q)}\right)(n_\varepsilon-n_{\varepsilon+\omega_q})d\varepsilon. \tag{4}$$

It can be seen from (3) and (4) that phonon instability ($I(\omega_q)<0$) due to recombination processes (at frequencies $\omega_q \gtrsim 2\Delta$) can occur only if the condition

$$n_\varepsilon>1/2 \tag{5}$$

is satisfied in some region of values of ε above the gap.[3] By virtue of Elesin's known theorem,[2,15] such a situation cannot occur when pumping a "broad"source, when the external action generates quasiparticles in a large energy region ($\tilde{\varepsilon}_{max} - \Delta \gg \Delta$). The picture is different when a "narrow" quasiparticle source acts, wherein the quasiparticles are generated in a narrow energy region ($\tilde{\varepsilon}_{max} - \Delta \ll \Delta$). In this case, as shown by Aronov and Spivak[12] as well as by Genkin and Protogenov[11] (see also Refs. 4 and 5), n_ε can exceed 1/2 and therefore the necessary condition for "phonon instability" can in principle be satisfied. Assuming "narrowness" of the quasiparticle distribution, we simplify expressions (3) and (4) and represent the "sufficient" condition for phonon instability in the form

$$I(\omega_q) = \frac{\pi}{2}\frac{\omega_D}{\varepsilon_F}\Delta N_{\omega_q}\left\{\int_\Delta^{\omega_q-\Delta}\frac{(1-2n_\varepsilon)\,d\varepsilon}{(\varepsilon-\Delta)^{1/2}(\omega_q-\varepsilon-\Delta)^{1/2}} + \frac{1}{\Delta^{1/2}}\int_\Delta^{\tilde{\varepsilon}_{max}}\frac{n_\varepsilon\,d\varepsilon}{(\varepsilon-\Delta)^{1/2}}\right\}<0 \tag{6}$$

for some frequency $\omega_q \geqslant 2\Delta$. An analysis of (6) shows that this condition can be satisfied if the deviation from equilibrium is strong enough: $n_{\tilde{\varepsilon}} \sim 1$, $\tilde{\varepsilon} \leqslant \tilde{\varepsilon}_{max}$. Indeed, in this case, for the frequencies $\omega_q \approx \tilde{\varepsilon}_{max} + \Delta$ the first integral in the curly brackets is equal to ($-\pi$), whereas the second does not exceed $2(\tilde{\varepsilon}_{max}/\Delta - 1)^{1/2}$ and is small by virtue of the "narrowness" of the electron distribution. In this case we have therefore not absorption but amplification of the sound wave.

We shall analyze in the following sections the extent to which the condition (6) can be satisfied in the case of microwave pumping or tunnel injection.

§3. SELF-CONSISTENT KINETIC EQUATIONS

We are interested in the possible stationary energy distributions of the electrons in a superconducting film connected to a thermostat, following the action of various external perturbations which take the electron subsystem out of equilibrium. Let the film thickness be small enough to prevent back action of the nonequilibrium phonons on the electrons. If the film is ideally acoustically matched to the thermostat (for details see Refs. 16 and 3), this situation obtains already for films of thickness $d \sim \xi_0$(ξ_0 is the correlation radius of the superconductor). In this case the kinetics of the electron and phonon subsystems of the superconductor are no longer coupled and the usual model with a phonon thermostat can be used to investigate the electron kinetics. We assume in addition that the film contains enough electron-elastic-scattering centers to make the electron distribution function and the order parameter isotropic. The Éliashberg kinetic equations, which determine the electron distribution, then take the form

$$0=Q(\varepsilon,\ \Delta)+J(\varepsilon), \tag{7}$$

where $Q(\varepsilon,\Delta)$ is the external source of the nonequilibrium electronic excitations (it will be specified for the different cases below), and the collision integral contains an electron-phonon and an electron-electron part, $J(\varepsilon)=J_\varepsilon^{(\mathrm{ph})}+J_\varepsilon^{(e)}$. The electron-phonon part takes at $T=0$ the form

$$J_\varepsilon^{(ph)}=\frac{\pi\lambda}{2(up_F)^2}\int_\Delta^\infty d\varepsilon'\int_0^\infty d\omega\,\frac{\omega^2\varepsilon\varepsilon'}{(\varepsilon^2-\Delta^2)^{1/2}(\varepsilon'^2-\Delta^2)^{1/2}}$$
$$\times\left[n'(1-n)\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\delta(\varepsilon'-\varepsilon-\omega)-n(1-n')\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\right.$$
$$\left.\times\,\delta(\varepsilon-\varepsilon'-\omega)-nn'\left(1+\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\delta(\varepsilon+\varepsilon'-\omega)\right]. \tag{8}$$

The electron-electron part is

$$J_\varepsilon^{(e)}=\frac{1}{2\varepsilon_F}\iiint_\Delta^\infty\frac{d\varepsilon_1\,d\varepsilon_2\,d\varepsilon_3}{(\varepsilon^2-\Delta^2)^{1/2}(\varepsilon_1^2-\Delta^2)^{1/2}(\varepsilon_2^2-\Delta^2)^{1/2}(\varepsilon_3^2-\Delta^2)^{1/2}}$$
$$\times\{M_1[(1-n)n_1n_2n_3-n(1-n_1)(1-n_2)(1-n_3)]\delta(\varepsilon-\varepsilon_1-\varepsilon_2-\varepsilon_3)$$
$$+3M_2[n_1n_2(1-n)(1-n_3)-nn_3(1-n_1)(1-n_2)]\delta(\varepsilon+\varepsilon_3-\varepsilon_1-\varepsilon_2)$$
$$+3M_3[n_1(1-n)(1-n_2)(1-n_3)-(1-n_1)nn_2n_3]\delta(\varepsilon+\varepsilon_2+\varepsilon_3-\varepsilon_1)\},$$
$$M_1=a(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon-\Delta^4)-{}^1/_3 b(-\varepsilon^2+\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_3)\Delta^2,$$
$$M_2=-M_1(-\varepsilon_3),\quad M_3=M_1(-\varepsilon_2,\ -\varepsilon_3), \tag{9}$$

where a and b are numbers of the order of unity, and are connected with the electron-electron interaction potential.[16] The quantity Δ in (7)–(9) should be determined from the self-consistency equation

$$1=\lambda\int_\Delta^\infty\frac{(1-2n_\varepsilon)\,d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}. \tag{10}$$

The self-consistent system of equations is strongly nonlinear even if no account is taken of the inelastic electron-electron collisions described by the term (9). However, the (assumed!) "narrowness" of the energy distribution of the electronic excitations makes it possible to find the solution effectively even when (9) is taken into account.

We introduce the variables $\delta=\tilde{\varepsilon}_{\max}-\Delta$ and $Z=(\varepsilon-\Delta)/\delta$, $\delta>0$. We put (our definition differs somewhat from that used in the literature, cf. Ref. 2)

$$\bar{n}=\int_\Delta^\infty\frac{n_\varepsilon d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}=\left(\frac{\delta}{2\Delta}\right)^{1/2}\int_0^1\frac{n_z dz}{z^{1/2}} \tag{11}$$

and represent the self-consistency equation (10) in the form [the reduction of (10) to (12) calls only for satisfaction of the conditions $(\Delta,\Delta_0)\ll\omega_D$]

$$\ln(\Delta/\Delta_0)=-2\bar{n}, \tag{12}$$

where Δ_0 is the gap at $T=0$ in the absence of an external perturbation. The narrowness of the quasiparticle distribution means, as can be seen from (11), smallness of $\bar{n}$, therefore the gap

$$\Delta\approx\Delta_0(1-2\bar{n}) \tag{13}$$

varies little.

We turn now to the collision integrals. If $n_\varepsilon\sim 1$ and is concentrated in the immediate region above the gap, the relaxation terms in (8) are negligibly small compared with the recombination terms, so that the operator $J_\varepsilon^{(\mathrm{ph})}$ can be simplified to

$$J^{(ph)}\approx-\frac{4\pi\lambda\Delta^4}{(up_F)^2}\bar{n}\frac{n_\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}. \tag{14}$$

Electron-electron collisions can lead to a substantial change of the distribution function if they are intense enough. In the case considered by us, when the quasiparticles are concentrated in a narrow layer near the Fermi surface and $n_\varepsilon\sim 1$ in this layer, it is necessary primarily to take into account the "impact pairing" processes (the second member in the term proportional to the factor M_3 in the collision integral $J_\varepsilon^{(e)}$), when three electronic excitations with energies $\varepsilon\geqslant\Delta$ collide and form a bound state (a Cooper pair) and a free quasiparticle with energy $\geqslant 3\Delta$. The inverse processes of "impact multiplication" are not very effective in this case, so that the collision integral $J_\varepsilon^{(e)}$ can be reduced to the form

$$J_\varepsilon^{(e)}\approx-\frac{3}{2^{7/2}}(a+b)\frac{\Delta^3}{\varepsilon}\bar{n}^2\frac{n_\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}. \tag{15}$$

Comparing (14) and (15) we can verify that the electron-electron collisions are inessential if the parameter

$$c=c_0\frac{\bar{n}\Delta_0}{\Delta},\quad c_0=\frac{3(a+b)}{2^{7/2}\pi\lambda}\frac{\omega_D^2}{\varepsilon_F\Delta_0} \tag{16}$$

is small (we have put $\omega_D\approx up_F$). For metals with relatively high Debye frequencies (such as aluminum) c_0 may be not very small and allowance for c becomes essential.[17,18] Unfortunately, the factors a and b in (16) are not well known from experiment, and we therefore confine ourselves below the values $c_0=0$, 1, and 10.

§4. RESONANT PUMPING BY AN ELECTROMAGNETIC FIELD

Let the external action that takes the electron subsystem out of equilibrium be a high-frequency electromagnetic field of frequency $\omega_0\geqslant 2\Delta$. The quasiparticle source in (7) takes then the form

$$Q(\varepsilon)=2\alpha[U_-(n_{\varepsilon-\omega_0}-n_\varepsilon)-U_+(n_\varepsilon-n_{\varepsilon+\omega_0})+V(1-n_\varepsilon-n_{\omega_0-\varepsilon})],$$
$$\varepsilon\geqslant\Delta, \tag{17}$$

$$U_\pm=\frac{[\varepsilon(\varepsilon\pm\omega_0)+\Delta^2]\theta(\varepsilon\pm\omega_0-\Delta)}{(\varepsilon^2-\Delta^2)^{1/2}[(\varepsilon\pm\omega_0)^2-\Delta^2]^{1/2}}$$
$$V=\frac{[\varepsilon(\omega_0-\varepsilon)-\Delta^2]\theta(\omega_0-\varepsilon-\Delta)}{(\varepsilon^2-\Delta^2)^{1/2}[(\omega_0-\varepsilon)^2-\Delta^2]^{1/2}}. \tag{18}$$

Here $\alpha=(e/c)^2A_{\omega_0}A_{-\omega_0}D$, $D=\frac{1}{3}v_F^2\tau_{\mathrm{imp}}$ is the diffusion coefficient, and A_{ω_0} is the vector potential of the electromagnetic field, the later assumed to be monochromatic. Equations (7), (17), and (18) were solved in the "recombination approximation" (14), without allowance for electron-electron collisions, by Aronov and Spivak.[12] Also neglected in the solution was also the Éliashberg mechanism [the terms $U_\pm$ in (17)], which leads to a field redistribution of the generated particles. This neglect valid in the case of weak fields (with small parameter α/γ, where γ is the electron energy damping). In the approximation indicated, it is easy to obtain from (7), (17), and (14) the distribution function (we are interested only in the part of n_ε above the gap, while the far and small tail is approximated by the equilibrium equation)

$$n_z = \frac{Az^{1/2}\theta(z)\theta(1-z)}{A(z^{1/2}+(1-z)^{1/2})+z^{1/2}(1-z)^{1/2}}, \qquad (19)$$

where $A = (\Gamma/\pi)^{1/2}$, $\Gamma = \alpha\omega_D^2/2\pi\lambda\Delta^3$, with $\tilde{\varepsilon}_{\max} = \omega_0 - \Delta$. The quantity Γ is proportional to the factor α/γ. It follows from (19) that $n_z = 1$ at $z = 1$. In the remaining region $z < 1$ the quantity n_z is small if the value of A is small. In the case of high-intensity fields one can expect (19) to yield an overestimate of n_z (by virtue of the aforementioned approximations). Nonetheless, approximating (19) in the region $A \gg 1$, we have a saturation regime

$$n_z = \frac{z^{1/2}}{z^{1/2}+(1-z)^{1/2}}\theta(z)\theta(1-z). \qquad (20)$$

Substitution of (29) in (6) leads to vanishing of the first of the integrals at the frequency $\omega_9 = \omega_0$, and while the second integral is small, it is positive so that the instability condition (6) is not satisfied (at other frequencies ω_1 the first integral is also positive, therefore the frequency $\omega_q = \omega_0$ is the least stable).

Thus, in a microwave field there is no phonon instability even if the field is strong.[4)]

§5. PHONON STABILITY IN SYMMETRIC TUNNEL INJECTION

In this case (*S-I-S* junction) the electron kinetics is also described by the Éliashberg equations, except that the factors $U\pm$ and V in the quasiparticle source (17) are now equal to[20–22]

$$U_\pm = \frac{\varepsilon(\varepsilon\pm V_0)\theta(\varepsilon\pm V_0-\Delta)}{(\varepsilon^2-\Delta^2)^{1/2}[(\varepsilon\pm V_0)^2-\Delta^2]^{1/2}},$$
$$V = \frac{\varepsilon(V_0-\varepsilon)\theta(V_0-\varepsilon-\Delta)}{(\varepsilon^2-\Delta^2)^{1/2}[(V_0-\varepsilon)^2-\Delta^2]^{1/2}}, \qquad (21)$$

where V_0 is the electric potential applied to the junction (the charge e is included in this quantity), and the factor 2α in the field term (17) is replaced by $I_0 = (8e^2RN(0)Sd)^{-1}$, where R is the normal resistance of the dielectric film of the junction, S is its area, while d is the thickness of the injected film, and $N(0) = mp_F/2\pi^2$. is the density of the electron levels in it on the Fermi level. We note that single-particle tunneling leads, generally speaking, to unbalancing of the branches in the superconductors, 23 an unbalancing not accounted for by Eqs. (7)–(9), (17), and (21). But under the action of a "narrow" injection source at low temperatures, when the electronic excitations are injected directly into the region above the gap, the role of the unbalancing process can be neglected. The equations are analyzed in the same manner as in the case of a microwave field. The distribution function of the nonequilibrium electrons is again given by (19), but A is now equal to[5)]

$$A = \begin{cases} \Gamma\Delta/2\delta, & \Gamma \gg \delta/\Delta \\ (\Gamma\Delta/\pi\delta)^{1/2}, & \Gamma \ll \delta/\Delta \end{cases}; \quad \delta = (V_0 - 2\Delta) \ll \Delta. \qquad (22)$$

Even though the value of Γ (now $\Gamma \sim I_0/\gamma$) is assumed as before to be small, narrowness of the injection source can lead the distribution function to the saturation regime (20) if $\delta/\Delta \ll \Gamma \ll 1$; this substantial difference is the result of the difference between the factors V in (18) and (21).[2] As already noted, (§4), even in the saturation regime there is no phonon instability. It can be easily seen that allowance for the electron-electron inelastic collisions [in the effective approximation (15)] likewise does not change this result.

§6. ASYMMETRIC *S-I-S′* JUNCTION

The picture changes radically in the case of an *S-I-S′* junction, when the quasiparticles are injected from a bulky superconductor into a thin film. This case was considered by Genkin and Protogenov,[11] who have shown that the function n_ε in the region of ε above the gap can be close to unity. Since their analysis is incomplete,[6)] we present here a solution of Eqs. (7)–(10) in a more accurate approximation needed for our purposes.

The initial equations for the case of an *S-I-S′* junction are similar to those of a symmetrical *S-I-S* junction, except that the distribution function contained in Eq. (17) for the source $Q(\varepsilon)$ with shifted arguments pertains to the bulky superconductor, while the factors $U\pm$ and V are[20]

$$U_\pm = \frac{\varepsilon(\varepsilon\pm V_0)}{(\varepsilon^2-\Delta^2)^{1/2}[(\varepsilon\pm V_0)^2-\Delta'^2]^{1/2}},$$
$$V = \frac{\varepsilon(V_0-\varepsilon)}{(\varepsilon^2-\Delta^2)^{1/2}[(V_0-\varepsilon)^2-\Delta'^2]^{1/2}} \qquad (23)$$

(Δ' is the gap of the bulky injector).

If the injector thickness is much larger than that of the thin film, which in turn does not exceed the quasiparticle diffusion length, the electron subsystem of the bulky superconductor can be regarded as unperturbed even in the case of a strong deviation of the thin film from equilibrium. Assuming "narrowness" of the resultant strong-nonequilibrium ($n_\varepsilon \sim 1$) distribution of the quasiparticles at $T = 0$, the solution of Eqs. (7)–(10) with allowance for (12), (14), (15), (17), and (23) can be represented in the form

$$n_z = \frac{B\theta(1-z)}{B+(1-z)^{1/2}}, \quad z \geqslant 0, \qquad (24)$$

where

$$B = A/\bar{n}(1+c), \quad A = (\Delta'/2\delta)^{1/2}\Gamma, \quad \Gamma = \Gamma_0/\Delta^3,$$
$$\Gamma_0 = I_0\omega_D^2/4\pi\lambda, \quad \Delta = \Delta_0\exp(-2\bar{n}), \qquad (25)$$

and the parameter $\bar{n}$ must be determined from the "self-consistency" equation

$$\bar{n} = \left(\frac{\delta}{2\Delta}\right)^{1/2} A \int_0^1 \frac{dz}{z^{1/2}[A+(1-z)^{1/2}(1+c)\bar{n}]}. \qquad (26)$$

We recall that the quantities Δ, δ, c, and A in (25) depend on $\bar{n}$ and that Eq. (18) determines the values of B for the given injection parameters. We shall find it more convenient to transform (26) into

$$\bar{n}\left(\frac{\Delta}{2\delta}\right)^{1/2}=B\frac{\pi}{2}-\frac{B^2}{|1-B^2|^{1/2}}\cdot\begin{cases}\ln\left|\dfrac{1+B+(1+B^2)^{1/2}}{-1-B+(1-B^2)^{1/2}}\right|, & B<1,\\ 2\,\mathrm{arctg}\left(\dfrac{B-1}{B+1}\right)^{1/2}, & B>1.\end{cases}\tag{27}$$

Equation (27) can be analytically investigated in limiting case (for simplicity we put for the time being $c=0$).

a) Let the external parameters V_0 and Γ_0 be such that a self-consistent value of $\bar{n}$ leads to $B\ll 1$. Then (27), with account taken of (25), reduces to the form

$$\bar{n}^2\exp(-7\bar{n})=\frac{\pi}{2}\left(\frac{\Delta'}{\Delta_0}\right)^{1/2}\frac{\Gamma_0}{\Delta_0^3}.\tag{28}$$

We note that in this case the roots of (28) do not depend on the parameter δ, but what depends on δ is the region of applicability of Eq. (28), inasmuch as by virtue of the condition $B\ll 1$ it is necessary that the steady-state solution satisfy the inequality

$$\Gamma_0\ll(2\delta/\Delta')^{1/2}\Delta^3\bar{n}.\tag{29}$$

The left-hand side of (28) has a maximum at $n=2/7$. This value of n is a root if

$$\Gamma_0=\Gamma_0^*\equiv\left(\frac{2}{7e}\right)^2\frac{\Delta_0^3}{\pi}\left(\frac{\Delta_0}{\Delta'}\right)^{1/2}\tag{30}$$

(here e is the base of the natural logarithms). At large values of Γ_0 there are obviously no solutions with small B, at smaller values there are two roots, and with decreasing Γ_0 the smaller of the roots varies like

$$\bar{n}\approx\left[\frac{\pi}{2}\left(\frac{\Delta'}{\Delta_0}\right)^{1/2}\frac{\Gamma_0}{\Delta_0^3}\right]^{1/2},\tag{31}$$

while the other increases quite rapidly and goes ultimately beyond the applicability of our analysis (we recall that we assume $\bar{n}$ to be small). Near the characteristic value $\Gamma_0\approx\Gamma_0^*$, using (29) and (30), we have the following condition for δ:

$$\delta/\Delta_0\gg 10^{-2}.\tag{32}$$

Thus, if $\Gamma_0\lesssim\Gamma_0^*$, the self-consistent kinetic equations always have two solutions with $B\ll 1$, since it is easily seen that the condition (32) is satisfied only if δ_0 is not too small (δ_0 is the initial excess above threshold, $\delta_0=V_0-\Delta'-\Delta$; for small $\bar{n}$ we have $\delta-\delta_0\approx 2\bar{n}$).

b) In the opposite limiting case $B\gg 1$, the right-hand side of (27) becomes equal to unity, so that (27) has a small root only at small δ. Using (13) we obtain from (27) the following expression for the small root:

$$\bar{n}\approx 2[1-(1+\delta_0/2\Delta_0)^{1/2}],\tag{33}$$

which is meaningless at $\delta_0>0$. Thus, at a negative initial displacement ($\delta_0<0$) the self-consistent system of equations has a solution with

$$\bar{n}\approx-{}^1/_2\delta_0;\tag{34}$$

in this case $B\gg 1$ if $|\delta_0|$ is small enough.

c) Of greatest interest is the case $B\approx 1$, for it is precisely at these values of the parameter that phonon instability is attained. In this case the right-hand side of (27) is equal to $(\pi/2-1)$ and we obtain for the small root the same value (34). Thus, the roots of interest to us coincide in the cases b) and c), and differ only in the value of B, which is determined by the external parameters δ_0 and Γ_0.

A numerical analysis of the transcendental equation (27) confirms the arguments advanced here. Thus, for the parameters $\Gamma_0=10^{-3}$, $\delta_0=-0.02$, $c_0=0$, $\Delta'=2$ (in units of Δ_0) we have three roots of Eq. (27): $\bar{n}_1\approx 0.01$ ($B\approx 14$), $\bar{n}_2\approx 0.05$ ($B\approx 0.1$); $\bar{n}_3\approx 0.95$ ($B\approx 0.3$). At $\Gamma_0>\Gamma_0^*$, however, e.g., $\Gamma_0=0.05$, we have (at $\delta_0=-0.3$, $\Delta'=0.5$) only one solution $\bar{n}_1\approx 0.2$ (in this case $B\approx 2.9$; $\delta\approx 0.02$; $\Delta\approx 0.67$) for the case $c_0=0$. With increasing c_0 the value of B decreases ($B\approx 1.9$ at $c_0=1$) and at sufficiently large c_0 (e.g., $c_0=10$) there are no roots at all. Thus, the behavior of an *S-I-S'* junction is found to be extremely sensitive to the initial parameters of the injection and of the junction itself.

The situation we encounter now has been under thorough study in recent years in the theory of nonequilibrium superconductivity. In particular, for symmetric tunnel injection there is also a multiplicity of solutions of the self-consistent kinetic equations (see, e.g., Refs. 2 and 13). We note that besides the solutions $\bar{n}_{1-3}$ obtained by us there exists at $\delta_0<0$ also the solution $n_0=0$ corresponding the situation when there is no excess of quasiparticles, as well as a solution corresponding to the normal state ($\Delta=0$). Disregarding solutions with large $\bar{n}$,[7)] we assert that in the general case there exist only three solutions, and the case $B\gtrsim 1$ corresponds to the intermediate value $\bar{n}_1$. We have in mind the analogy with the *S-I-S* junction, we can expect this state to be unstable, the *S-I-S'* junction current-voltage characteristic to be *S*-shaped, and a spatially inhomogeneous state to be realized in the superconducting film. The possibility of realizing an AQG regime in this case calls for additional analysis. We, however, shall not deal with this question since, as noted above, at $\Gamma_0\gg\Gamma_0^*$ and $\delta_0<0$ (see, in particular, the case mentioned above with $\Gamma_0=0.05$) there is only one solution corresponding to a superconducting state with an excess of electronic excitations. This simplifies greatly the analysis of the stability needed to attain the AQG threshold regime.

§7. THE STABILITY PROBLEM

Analysis of the stability problem is exceedingly important for the attainment of the AQG regime. Many factors influence the stationary state in nonequilibrium superconductors (e.g., fluctuations of the superfluid velocity,[10] high-frequency fluctuations of the order parameter[9] or of the electromagnetic field,[12] and others).

We begin with the stability to high-frequency fluctuations of Δ. The corresponding instability criterion was obtained by Aronov and Gurevich[9] and is connected with the reversal of the sign of the damping coefficient of the modes of the natural collective oscillations of the semiconductor's electron subsystem. The limiting stable value of Δ is determined by the reversal of the sign of the expression

$$P_i(\Delta)=\int_\Delta^\infty\frac{(1-2n_\varepsilon)\,d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}(\varepsilon^2-\omega_i^2/4)},\tag{35}$$

where the frequency of the ith mode is determined by the equation

$$n(\omega_i/2)={}^1/_2.\tag{36}$$

In addition, the mode $\omega_0=0$ can be unstable. In the case $B \gg 1$, which we have in mind, expression (35) takes (for a narrow source) the form[8)]

$$P_0(\Delta)=(1-2\bar{n})/\Delta^2, \qquad (37)$$

and consequently the rapid "disintegration" of the Cooper pairs can take place only at $\bar{n} \geqslant 1/2$. In other words, no instability sets in at small n.

We consider now the instability to fluctuations of the superfluid velocity $\mathbf{V}_s$ (Ref. 10), which is due to the fact that the response of a nonequilibrium superconductor to an external magnetic field becomes paramagnetic,[24] in which case the sign of the "superfluid-component density" N_s reverses sign:

$$N_s=1+2\int_\Delta^\infty \frac{\partial n}{\partial \varepsilon}\frac{\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}d\varepsilon=1+\left(\frac{2\Delta}{\delta}\right)^{1/2}\int_0^\infty \frac{\partial n}{\partial z}\frac{dz}{z^{1/2}}. \qquad (38)$$

It follows from (38) that at the small values of δ used by us and at $B \sim 1$ we have negative N_s. Actually, however, the reversal of the sign of N_s still does not mean development of instability. A rigorous analysis based on the solution of the Maxwell-London equations and carried out by Genkin and Protogenov,[11] shows that the contact with the bulky superconductor stabilizes the situation in a thin film if

$$\lambda' d<|\lambda^2| \qquad (39)$$

(here λ' and λ are the penetration depths of the magnetic field and are connected with the London relation $\lambda^{-2}=\lambda_L^{-2}N_s$). This instability should therefore likewise not occur if the parameters are suitably chosen.

Slow spatial and temporal fluctuations of the distribution function of the electronic excitations and of the order parameter, with allowance for the multiplicity of the self-consistent solutions of the Éliashberg kinetic equations, can lead, as noted in the preceding section, to the onset of a spatially inhomogeneous state. In the case when a single small value $\bar{n} \neq 0$ exists for a distribution of the form (24), the problem was considered (in the relaxation-time approximation) by Genkin and Protogenov.[11] The results of Ref. 11 offer evidence of stability of a homogeneous distribution (24) at arbitrary B.

As for stability to high-frequency fluctuations of the electromagnetic field, it was indicated in Ref. 11 (see also Ref. 25) that instability to generation of electromagnetic waves is indeed possible in London superconductors. It must be kept in mind, however, that such an "instability" still does not mean destruction of the homogeneous superconducting state. The reason is[12] that the conductivity has a large imaginary part that suppresses the possible instability. Nonetheless, such a system can operate as an amplifier of electromagnetic waves[25] in reflection.

We note finally that the longitudinal-electric-field fluctuations due to the unbalance of the branches of the electronic-excitation spectrum are also possible in superconductors. In the case $T=0$ investigated by us (of "narrow" distributions), however, the unbalance is negligibly small[14,21] and it can apparently be assumed that no instability to longitudinal-field fluctuations develop. (In our approximation, there is no longitudinal field at all.)

Of course, the foregoing analysis cannot be regarded as exhaustive, since it is difficult to investigate all the instability-development channels possible in the system. We regard, however, the AQG problem as sufficiently interesting and important, and it would therefore be of interest, in addition to further analysis, to perform the pertinent experiments.

§8. CONCLUSION

Analysis shows thus that, notwithstanding the presence of "competing" instabilities when the electron subsystem of the superconductor deviates from equilibrium, that these instabilities do not arise in a number of experimentally realizable situations and a threshold AQG regime can be attained in a spatially homogeneous and stationary state. We present numerical values of some possible parameters. Most frequently and easiest, as shown by experiment, nonequilibrium states are reached in aluminum films, owing to the relatively long lifetimes of the electronic excitations. Typical values of the damping γ for aluminum are estimated at 10^7–$10^8\ \mathrm{sec}^{-1}$ (Ref. 26). Returning $\hbar$ to the equations, we have for the junction resistance

$$R=\hbar/8e^2N(0)SdI_0, \qquad (40)$$

and if $I_0 \sim 0.01\gamma$, we obtain from (40) for aluminum at $d \sim 10^{-4}$ cm and $N(0) \sim 10^{34}\ \mathrm{erg}^{-1}\,\mathrm{cm}^{-3}$ the estimate $RS \sim 10^{-5}\ \Omega\ \mathrm{cm}^2$. Junctions with such low resistances can be fully realized (see, e.g., Refs. 27 and 28), but in the experimental studies known to us the junctions were symmetrical and this is probably one of the reasons for not observing phonon instability. Another factor to be considered is the need for satisfying the inequality (39) that imposes additional conditions on the pair making up the S-I-S' junction. Even though N_s may not be small (e.g., $|N_s| \sim 0.1$), there are no grounds for expecting the condition (39) to hinder greatly the attainment of the AQG threshold. To avoid misunderstandings, we note once more than we are dealing with the possibility of just the AQG threshold. It is natural to expect that when this threshold is reached and phonon instability develops, a radical restructuring of the system will take place and will be accompanied, in particular, by a change of the distribution function of the electrons and the onset of a dynamic regime. The investigation of the AQG dynamics, however, is not the purpose of the present study.

We note that the speed of sound in a superconductor is much lower than the speed of light, and the dimensionless electron-phonon interaction constant exceeds the corresponding electromagnetic interaction constant. As a result, the AQG gain is quite high. Indeed, using the collision operator (2), we easily obtain for this gain the expression

$$K(\omega_q)=\frac{\pi\lambda}{2}\frac{\omega_D}{\varepsilon_F}\frac{\Delta}{u}I^0(\omega_q), \qquad (41)$$

where $-I^0(\omega_q)$ is the quantity in the curly brackets of (6), and for typical metals (Pb, Sn, Al) we have from (41) $K(\omega_q) \sim 10^3\ \mathrm{cm}^{-1}$ at pumps corresponding to $I^0(\omega_q) \sim 1$. This is higher than the working gain of gas, solid-state, and semiconductor lasers. Therefore in contrast to the ordinary laser, the AQG regime can in principle be attained without the use of a cavity, and the coherent phonon flux will propagate along a thin film. In addition, if necessary one can use the circumstance that at the interface of a metal and liquid helium the phonons undergo strong internal reflection, and a ring AQG regime can be realized (likewise in the plane of the film).

Finally, we wish to note one circumstance that can make AQG most attractive from the practical point of view. The wavelength of the phonons emitted by an AQG is of the order of 10^{-5} cm, whereas the energy of light of the same wavelength is larger by five orders of magnitude. This energy feature of AQG based on superconductors is most interesting.

1)Phonon instability can be understood as the reversal of the sign of the sound absorption coefficient of the nonequilibrium electron system of the superconductor.

2)The factor $(1 - n_\varepsilon - n_{\omega-\varepsilon})$ under the integral sign in (3) can be reduced by changing the variables in the last term to the form $(1 - 2n_\varepsilon)$.

3)We note that the distribution n_ε produced in the superconductor need not be a monotonically decreasing function, so that the mode $\omega_q = 2\Delta$ is not the least stable one and, in addition, the condition $n_\Delta > 1/2$ indicated in Ref. 4 is not, generally speaking, necessary. What is necessary is the weaker condition (5).

4)In principle, the situation can change at finite temperatures, if account is taken of the contribution of the inelastic electron-electron collisions simultaneously with the Éliashberg mechanism (cf. Ref. 19). Unfortunately, this problem has so far not been sufficiently studied for the case $\omega_0 \gg 2\Delta$.

5)We assume that δ and Δ in (22) are self-consistent values if the external parameters are given, but in view of the absence of the sought effect (i.e., of the phonon instability) we shall not go into the details of the behavior of an *S-I-S* junction (for details see Ref. 2).

6)Unfortunately, no detailed analysis was made in Ref. 11 of the kinetic equations with allowance for the self-consistency equation and it was not discovered that the latter causes the solutions of the kinetic equations to be multiply valued. Therefore without going outside the framework of the approach of Ref. 11 it is impossible to make unequivocal predictions concerning the state of a nonequilibrium superconductor.

7)Cases corresponding to large $\bar{n}$ must be treated outside the framework of the approximations employed. It must be borne in mind, however, that states with large $\bar{n}$ are separated from states with small $\bar{n}$ by a large energy barrier, and transitions between them are usually exceedingly improbable.[13]

8)No instability sets in at $i \neq 0$ in the case $B > 1$ (cf. Ref. 9).

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