Nonlinear theory of surface polaritons

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(Submitted 28 December 1982)
Zh. Eksp. Teor. Fiz. **84**, 1907–1917 (May 1983)

A theoretical analysis of the propagation of surface polaritons along the interface between two dielectrics is carried out for the case in which the permittivity of one of the media depends quadratically on the wave field. A method involving the superposition of the phase portraits of the adjoining media is proposed for the solution of the problem. Several possible cases of the various relations between the signs of the nonlinear coefficient and the permittivities of the adjoining media are considered. The dependence of the energy flux on the effective refractive index for a surface wave is computed for each of them. The dispersion relations for surface polaritons are obtained for the case in which one of the adjoining media possesses a resonance permittivity. It is shown that, when the nonlinearity is taken into account, the surface polaritons occupy some region in the frequency-wave vector plane, one of the boundaries of which is the dispersion curve for the linear surface polaritons.

PACS numbers: 71.36. + c, 73.40. − c, 77.20. + y

1. INTRODUCTION

The possibility of the existence at the interface between two media of nonlinear surface waves is theoretically investigated in a number of papers published in recent years.[1–11] The analyses in these papers are for both the case of *TM* modes[1–3] and the case of *TE* modes[3–6] and, in particular, for thin films located between semi-infinite media.[3,6] The propagation of nonlinear *TM* waves at a plasma boundary is investigated in Refs. 7–10 for various mechanisms of the nonlinearity. But the dispersion relations for nonlinear surface polaritons (NSP) propagating at the boundary between two dielectrics were derived in the case of special anisotropic nonlinear media,[1–3] for which the Maxwell equations, as written for nonlinear media, are amenable to exact analytic solution. In the case of an isotropic nonlinear medium a complete analytic solution does not exist, and in Ref. 11 the first terms of the expansion of the dispersion relation in powers of the field amplitude are found.

In the present paper we consider the problem of the propagation of *NSP* at the interface between two dielectrics in the case when one of the adjoining media possesses a linear permittivity ε_l and the second medium has a permittivity that depends quadratically on the applied optical field:

$$\varepsilon=\varepsilon_0+\alpha|E|^2, \tag{1}$$

where the coefficients ε_0 and α are assumed to be isotropic.

We propose for the solution of the problem a method involving the superposition of the phase portraits of the adjoining media, which allows us to determine graphically on the phase plane which types of surface waves can exist at the interface between the two media in question, and in which parameter ranges they can exist. Here each of the permittivities ε_0 and ε_l may be frequency dependent, and we show that, in a situation where one of the them is a resonance permittivity and becomes negative in some frequency region, the NSP in the frequency-effective refractive index plane can exist in a region one of whose boundaries is the dispersion curve for the linear surface polaritons. The possibilities of dispersion of the nonlinear susceptibility α is also taken into account. In the case when the nonlinear medium is the resonant medium, the dispersion of α can bring about a significant reduction in the radiation energy fluxes necessary for the observation of the NSP.

2. THE WAVE EQUATIONS FOR THE NONLINEAR MEDIUM

Let us choose the system of coordinates such that the set of Maxwell equations for a *TM* wave reduces to a set of equations for the components E_x and E_z of the electric field and the component H_y of the magnetic field. We shall assume that the wave propagates along the x axis. Then the solution for the fields has a form $\sim\exp(ik_x x - i\omega t)$, and we obtain the following system of equations:

$$\begin{gathered} dE_x/dz=ik_xE_z+ik_0H_y,\\ dH_y/dz=ik_0E_x[\varepsilon_0+\alpha(|E_x|^2+|E_z|^2)],\\ k_xH_y+k_0E_z[\varepsilon_0+\alpha(|E_x|^2+|E_z|^2)]=0, \end{gathered} \tag{2}$$

where $k_0=\omega/c$. Let us introduce in these equations the new variables:

$$\tau=k_0z,\quad X(\tau)=-i(|\alpha|)^{1/2}E_x,$$
$$Y(\tau)=(|\alpha|)^{1/2}H_y,\quad Z(\tau)=(|\alpha|)^{1/2}E_z.$$

In this case the system (2) assumes the form

$$\dot X=nZ+Y, \tag{3a}$$

$$\dot Y=-X(\varepsilon_0+\beta X^2+\beta Z^2), \tag{3b}$$

$$nY+Z(\varepsilon_0+\beta X^2+\beta Z^2)=0, \tag{3c}$$

where $n=k_x/k_0$, $\beta=\mathrm{sgn}(\alpha)$ all the quantities are real, and the dot denotes differentiation with respect to τ. The system (3) possesses the first integral

$$Y^2=\varepsilon(\varepsilon^2-C^2)/2\beta(2n^2-\varepsilon), \tag{4}$$

where $\varepsilon=\varepsilon_0+\beta(X^2+Z^2)$, and C is a constant of integration, that, for the trajectories passing through the origin, is equal to ε_0 With the aid of (4), we can immediately write out

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the dispersion relation[8] for the surface waves:

$$n^2=[2\varepsilon_l-(\varepsilon_1+\varepsilon_0)]\varepsilon_1^2\varepsilon_l/[(3\varepsilon_1-\varepsilon_0)\varepsilon_l^2-\varepsilon_1^2(\varepsilon_1+\varepsilon_0)], \quad (5)$$

where ε_1, the permittivity of the nonlinear medium at the interface, depends on the fields at the boundary. But this formula does not solve the problem as a whole, since, to determine the fields at the boundary in each particular case, we must carry out numerical computations. Furthermore, the quantity that can be prescribed in experiment is not the field at the boundary, but the integrated energy flux S in the surface wave, and the dispersion relation into which S enters is the more convenient one. Therefore, we carry out the analysis of the NSP, proceeding directly from the equations (3).

The equations (3) constitute a system of two differential equations, given the additional relation (3c) and they can be analyzed on the phase plane of any pair of the variables X, Y, and Z.[12–15] For our purposes, however, it is convenient to choose the variables X and Y, since they are proportional to the tangential components of the fields, and therefore we can, when considering layered media, match the solution trajectories for the various media on one and the same phase plane so as to obtain a complete continuous trajectory. Such a superposition of phase portraits turns out to be especially convenient in the investigation of surface waves in layered media with several interfaces.

Solving the last of the equations (3) for Z, and substituting the resulting function $Z=Z(X,Y)$ into Eqs. (3a) and (3b), we obtain a system of two independent equations for the variables X and Y. Here it is necessary to consider the following cases separately:

1. The parameters ε_0 and α have the same sign, i.e., $\varepsilon_0\alpha>0$. The discriminant

$$Q=[{}^1/_3((\varepsilon_0/\beta)+X^2)]^3+(nY/2)^2$$

of Eq. (3a) in this case is positive, and (3c) has one real solution:

$$Z(X,Y)=(nY/2+\sqrt{Q})^{1/3}+(nY/2-\sqrt{Q})^{1/3}. \quad (6)$$

The function (6) is single-valued, and the system (3) has a unique phase portrait in the plane of the variable X and Y.

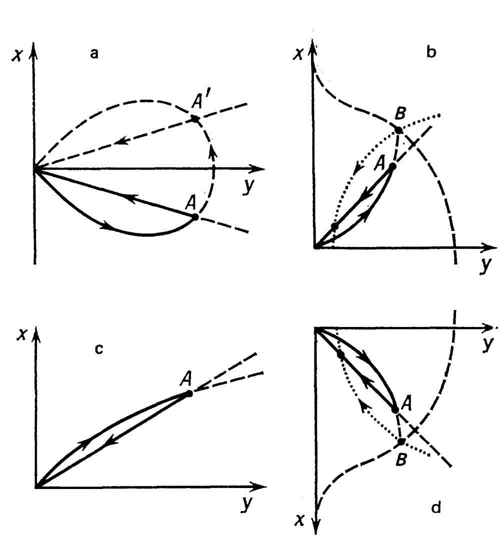

FIG. 1. Phase trajectories of nonlinear surface waves.

2. The parameters ε_0 and α have different signs: $\varepsilon_0\alpha<0$. In this case the discriminant Q is negative in some region of the (X,Y) plane within the limits of the dotted lines in Figs. 1b and 1d (in each figure we show only the part of the line located in one of the quadrants), and Eq. (3c) in this region has three real roots, which it is convenient to write in the trigonometric form:

$$Z_{1,2}=-2\left[-\frac{1}{3}\left(\frac{\varepsilon_0}{\beta}+X^2\right)\right]^{1/2}\cos\left(\frac{\varphi}{3}\pm\frac{\pi}{3}\right), \quad (7a)$$

$$Z_3=2\left[-\frac{1}{3}\left(\frac{\varepsilon_0}{\beta}+X^2\right)\right]^{1/2}\cos\frac{\varphi}{3}, \quad (7b)$$

$$\cos\varphi=-\frac{nY}{2\beta}\left[-\frac{1}{3}\left(\frac{\varepsilon_0}{\beta}+X^2\right)\right]^{-3/2}.$$

The qualitative analysis of the system (3) should be carried out after the substitution of each of the solutions (7) into the system. Thus, there exist three phase portraits of the system (3) in the case $\varepsilon_0\alpha<0$, a transition from one phase portrait to another being possible during the motion along the trajectories on the $Q=0$ line.[10,14,15] In the present paper, however, we shall limit ourselves to the considerations of the solutions that do not go outside the limits of one phase portrait: To the solutions in the form of surface waves correspond phase trajectories that begin and end at the coordinate origin, and below we shall consider only such trajectories. This condition is fulfilled by the root Z_1 with the plus sign in (7a). The parameter φ is assumed to be positive.

3. PHASE TRAJECTORIES OF THE SURFACE WAVES

Let us consider the possible types of phase portraits corresponding to the solutions in the form of surface waves as functions of the parameters of the layered system. Here we shall consider straight away the trajectories that arise as a result of the superposition of the phase portraits of the linear and nonlinear media. The equations for the linear medium are obtained from (3) by setting $\beta=0$ in them and replacing ε_0 by the permittivity ε_l of the linear medium. For $n^2>\varepsilon_l$ these equations have a solution in the form of an exponential function $Y=A\exp(\pm\gamma_l\tau)$, where $\gamma_l=(n^2-\varepsilon_l)^{1/2}$ and A is a constant that can be determined from the boundary conditions. The phase trajectory of interest to us is given by the formula

$$X=\pm(\gamma_l/\varepsilon_l)Y, \quad (8)$$

and is reprsented in all the cases in Fig. 1 in the form of a straight line passing through the coordinate origin. The direction of the trajectory in a given quadrant depends on the sign of ε_l.

For the nonlinear medium, the shape of the phase trajectory depends on two parameters: ε_0 and β. It is assumed that $n^2>\varepsilon_0$. Let us consider various particular cases:

1. Let $\varepsilon_0>0$, $\alpha>0$, and $\varepsilon_l<0$. The phase trajectory for this case is shown in Fig. 1a. The complete trajectory of the nonlinear solution, that goes out of the coordinate origin, is an eightshaped horizontal figure that is symmetrical about the X and Y axes. The figure shows half of this trajectory. The continuous line depicts the surface polariton's trajec-

tory obtained as a result of the superposition of the linear and nonlinear trajectories. At the point A ($\tau = 0$), which corresponds to the interface, there occurs a transition from the nonlinear to the linear medium, and, as can be seen from the figure, the magnetic field for this solution attains its maximum at the boundary between the media.

Let us compute the parameter values at which a surface wave of the type in question exists. The asymptotic behavior of the nonlinear solution near the coordinate origin has the form $Y \approx \exp(\pm\gamma\tau)$, where $\gamma = (n^2 - \varepsilon_0)^{1/2}$ the variables X and Y are linearly related ($X \approx -\gamma Y/\varepsilon_0$) and the slope of the trajectory is equal to $-\gamma/\varepsilon_0$. A surface wave can exist only in the case when the slope of the linear trajectory is smaller than that of the nonlinear trajectory at small X, Y values, i.e., when $|\gamma_1/\varepsilon_1| < |\gamma/\varepsilon_0|$. It follows from this inequality that the permittivities should satisfy the condition $\varepsilon_1 < -\varepsilon_0$ and that the effective refractive index for the surface wave should be greater than the value $n_1 = [\varepsilon_0\varepsilon_1/(\varepsilon_0 + \varepsilon_1)]^{1/2}$ corresponding to a linear surface polariton: $n > n_1 > \varepsilon_0^{1/2}$. When the slopes are nearly equal, the small-amplitude solutions degenerate into linear solutions. Thus, surface waves of the type in question can be obtained from ordinary linear surface polaritons by increasing the wave amplitude.

The energy flux in a surface wave can be computed from the formula

$$S = -\int_{-\infty}^{\infty} ZY\,d\tau, \tag{9}$$

which gives a dimensionless value for the flux. In order to obtain the dimensional energy flux, we must multiply S by the coefficient $c/8\pi|\alpha|k_0$. To find the magnitude of the energy for each n, we compute numerically, using (3) and (7), the dependence of Z and Y on τ, and perform the integration in (9). Figure 2a shows the effective refractive index (n) dependence of S thus obtained for $\varepsilon_0 = 2.25$ and several values of ε_1. It can be seen that, as n increases, the energy flux increases up to some limiting value and then decreases to zero. This is due to the fact that, in a medium with negative permittivity, the energy flux and the wave vector have opposite directions, and this gives rise to the inverse dependence of the energy flux on the effective refractive index n in some region of n values. The energy density in the region around the boundary increases monotonically with increasing n.

A negative permittivity can be realized at frequencies close to resonance frequencies. Let the frequency dependence of the permittivity of the linear medium have the form

$$\varepsilon(\xi) = \varepsilon_\infty + p/(1-\xi^2), \tag{10}$$

where $\xi = \omega/\omega_0$ is the frequency reduced by the resonance frequency, p is the "oscillator strength," and ε_∞ is the contribution from the other resonances to the permittivity. The longitudinal frequency in this representation is equal to $\xi_L = (1 + p/\xi_\infty)^{1/2}$, and the surface frequency $\xi_S = [1 + p/(\varepsilon_\infty + \varepsilon_0)]^{1/2}$ under the condition that the permittivity of the neighboring medium is equal to ε_0. Since, as we have seen, the NSP in the case of negative ε_1 exist in some interval of n values, it is clear that these polaritons will occupy in the (ξ, n) plane a band lying to the right of the dispersion curve for the linear surface polaritons (the hatched region in Fig. 3a). To each point in this band corresponds a definite energy flux.

2. Now let $\varepsilon_0 < 0$, $\alpha > 0$, and $\varepsilon_l > 0$. The phase trajectory is shown in Fig. 1b. The trajectory of the nonlinear solution in the present case turns out to be open. It terminates on the $Q = 0$ line, which is represented in the figure by the broken lines. Solving the equation $Q = 0$ together with (4), we find the coordinates of the point B and then the greatest possible slope of the linear trajectory:

$$\gamma_1/\varepsilon_1 < [(2\varepsilon_0 - 3\varepsilon_c)/(3\varepsilon_c^2 - \varepsilon_0^2)]^{1/2}, \tag{11}$$

where

$$\varepsilon_c = n^2\left[1 - 2\cos\left(\frac{\varphi_c}{3} + \frac{\pi}{3}\right)\right] \qquad \sin\varphi_c = \frac{\varepsilon_0}{2n^2}.$$

In the present case the surface waves exist in a bounded range of variation of n. The upper limit for n, which we denote by n_c can be found numerically from the relation (11). The lower bound is given by the inequality $\gamma_1/\varepsilon_1 > |\gamma/\varepsilon_0|$, which can be satisfied only when $\varepsilon_1 < -\varepsilon_0$ and whose solution is the inequality $n > n_1 > \varepsilon_1^{1/2}$. Figure 2b shows the n dependence of the energy flux, as computed from the formula (9), for $\varepsilon_1 = 2.25$ and several values of ε_0. The curves have a maximum in the interval $n_1 < n < n_c$. Notice that in the present case the nonlinear equations (3) allow the continuation of the trajectory beyond the point B on another sheet of the nonlinear phase portrait, which is indicated by dots in Fig. 1b.[14,15] A waveguide channel is then formed near the surface, which requires an increase in the wave power. But, as calculations show, the effective refractive index decreases from the value n_c as the energy flux increases, so that two values of S will correspond to one n value, and the wave with the lower energy flux value should be realized.

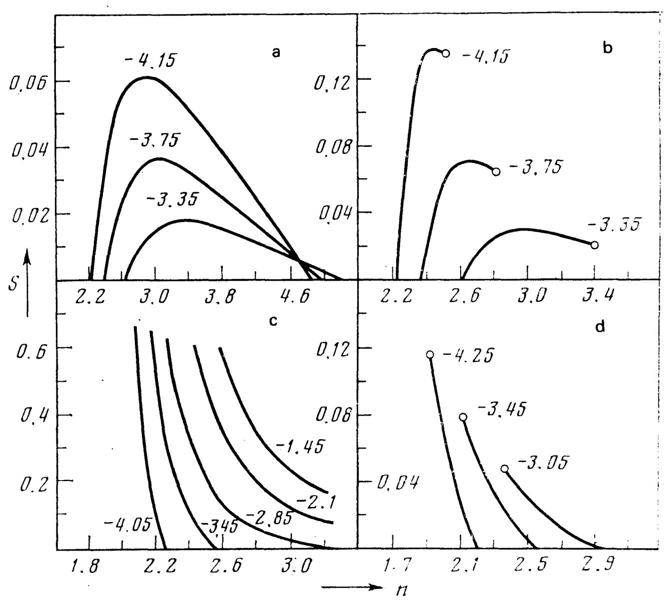

FIG. 2. Dependence of the energy flux of a surface wave on the effective refractive index in four different cases. For the medium with the positive permittivity (ε_0 or ε_1) we chose the value $\varepsilon = 2.25$. The values of the permittivity of the adjoining medium are indicated above the curves.

Let the permittivity of the nonlinear medium in the present case be frequency dependent: $\varepsilon_0 = \varepsilon(\xi)$. The nonlinear susceptibility α does not enter directly into the equations (3), and we can for the present neglect its dispersion. It is easy to see that, in the plane (ξ,n), the NSP of the type in question also occupy a band lying to the right of the dispersion branch for the linear surface polaritons, but that this band is narrower than in the preceding case. Thus, for positive nonlinearity (the cases 1 and 2), the effective refractive index for the NSP is always greater than the value n_l corresponding to the linear surface polaritons, irrespective of which of the media is nonlinear.

3. Let us now turn to the case $\varepsilon_0 < 0$, $\alpha < 0$, and $\varepsilon_1 > 0$. The phase trajectory is shown in Fig. 1c. The complete trajectory of the nonlinear solution in the present case is also open, and goes out to infinity; therefore, the surface-wave amplitude determined by the coordinates of the point A can also increase without restriction as $n \to \varepsilon_1^{1/2}$. For this type of surface wave to exist, the inequality $\gamma_1/\varepsilon_1 > |\gamma/\varepsilon_0|$, which has a solution for all $\varepsilon_l > 0$:

$$\begin{aligned} \varepsilon_1^{1/2} < n < n_l \quad &\text{for} \quad 0<\varepsilon_1<-\varepsilon_0, \\ \varepsilon_1^{1/2} < n < \infty \quad &\text{for} \quad \varepsilon_1>-\varepsilon_0, \end{aligned} \tag{12}$$

must be satisfied. Figure 2c shows for the present case the n dependence of the energy flux computed from the formula (9). As was to be expected, the energy flux is equal to zero when $n = n_1$, and increases without restriction as $n \to \varepsilon_1^{1/2}$. Of special interest is the situation in which $\varepsilon_1 > -\varepsilon_0$. For such a relation between the permittivities, surface waves do not exist in the linear case, and NSP can exist only when the energy flux exceeds some finite value (the curves for $\varepsilon_0 = -2.1$ and $\varepsilon_0 = -1.45$). The minimum energy flux is realized at $n \to \infty$.

We shall assume that the permittivity ε_0 of the nonlinear medium has a resonance character, and that its frequency dependence is given by the formula (10). Then in the plane (ξ,n) the NSP occupy a triangular region lying to the left of the dispersion curve for the linear surface polaritons and also the band lying between the longitudinal ξ_L and surface ξ_S frequencies, in which surface waves do not, in the linear case, exist at all. Inside this band, however, surface polaritons can exist only at finite energy flux values, and not beginning with the zero value, as obtains in all the remaining cases.

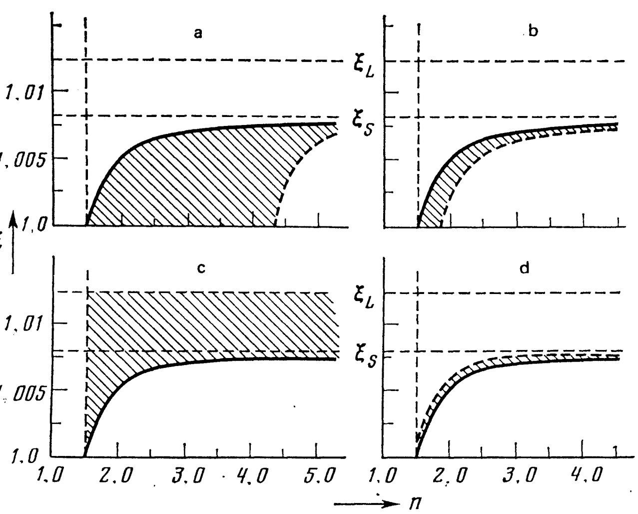

FIG. 3. Admissible regions (hatched) for the dispersion curves of the nonlinear surface polaritons. The medium with the negative permittivity was chosen as the resonant medium. The continuous line represents the dispersion curve of the linear surface polaritons. The computations were performed with the parameters $\varepsilon_\infty = 4$, $p = 0.1$, and $\xi = \omega/\omega_0$.

Thus far, we have assumed that $\alpha = \text{const}$ in the case when ε_0 is frequency dependent. In point of fact this is not so, and the nonlinear susceptibility α also exhibits dispersion. What changes will occur in our results if we take this fact into account? Since only the sign of the coefficient α, and not its magnitude, enters into the basic equations (3), for α that does not change sign in the frequency region where $\varepsilon_0 < 0$, the analysis for our dimensionless quantities remains unchanged. But if because of dispersion the coefficient α changes sign at some frequency in the frequency region where $\varepsilon_0 < 0$, then a transition from the case 2 to the case 3 will occur at this frequency. Notice, however, that the magnitude $cS/8\pi|\alpha|k_0$, which depends on the magnitude of α, of the actual energy flux will change at each point of the hatched regions in Figs. 3b and 3c. Because of the resonance character of the dependence $\alpha(\xi)$, this circumstance can significantly lower the energy flux values necessary for the observation of NSP. Notice also that, since the quantity α enters into the expression for the energy flux simply as a factor in the denominator, we can, by comparing the theoretical and experimental frequency dependence of the energy flux, obtain the α dispersion at frequencies close to the resonance frequency.

4. Now let $\varepsilon_0 > 0$, $\alpha < 0$, and $\varepsilon_1 < 0$. The phase trajectory is shown in Fig. 1d, and is a symmetrical replication of the trajectory shown in Fig. 1b. The slope of the linear trajectory here is also bounded from both sides. The lower bound is determined by the inequality $|\gamma_l/\varepsilon_1| > \gamma/\varepsilon_0$, which has solutions for any $\varepsilon_1 < 0$:

$$\begin{aligned} \varepsilon_0^{1/2} < n < n_l \quad &\text{for} \quad \varepsilon_1<-\varepsilon_0, \\ \varepsilon_0^{1/2} < n < \infty \quad &\text{for} \quad \varepsilon_1>-\varepsilon_0. \end{aligned} \tag{13}$$

We find an additional limitation on the range of variation of n from the following inequality, the analogue of (11):

$$-\gamma_1/\varepsilon_1 > [(2\varepsilon_0 - 3\varepsilon_c)/(3\varepsilon_c^2 - \varepsilon_0^2)]^{1/2}, \tag{14}$$

where ε_c is given by the formula (11a). The case $\varepsilon_1 > -\varepsilon_0$ is excluded by the inequality (14), and therefore the NSP can exist only when $\varepsilon_1 < -\varepsilon_0$. The effective refractive index can then vary within the limits $n_c < n < n_1$, where n_c can be found numerically from (14). Figure 2d shows the n dependence of the energy flux for this case when $\varepsilon_1 = -3.05$, -3.45, and -4.25. The energy flux increases as n decreases from the value $n = n_1$, and the curves terminate abruptly at $n = n_c$, when the point A merges with the point B in Fig. 1d. In this situation the nonlinear trajectory can be continued on the other sheet of the nonlinear phase portrait determined by the root Z_2 in (7), which is illustrated in Fig. 1d by the dotted curve. But as the energy flux increases, the quantity n will decrease, starting from $n = n_c$, so that the two energy flux values will correspond to one effective refractive index. Since those branches of the curves which start from linear surface waves are stable in the case of small amplitudes, the NSP having, for a given n, the higher S value should break away and form a lower branch.

If we assume that the frequency dependence of ε_1 has the resonance character (10), then, in the plane (ξ, n), the given type of NSP occupies a region lying to the left of the dispersion curve for the linear surface polaritons, which is illustrated by the hatched region in Fig. 3d. Thus, for negative nonlinearity (the cases 3 and 4), irrespective of which of the media is nonlinear, as the energy flux increases, the effective refractive index for the NSP decreases, starting from the value n_1 corresponding to the linear surface polaritons.

5. Let us now consider the case in which all the parameters ε_0, α, and ε_1 are positive. The phase trajectory of the surface wave terminates in the fourth and first quadrants (the broken lines in Fig. 1a). There occurs a transition to the linear trajectory at the point A′. The magnetic field attains its maximum in the nonlinear medium, and not at the interface between the two media. In this sense, this wave is similar to the surface TE wave considered in Ref. 2. Such a self-focused solution is possible only if $\varepsilon_1 > \varepsilon_0$, and the resonance conditions for the permittivities are not required here. This wave can exist only when the energy flux exceeds some threshold value (Fig. 4), and therefore it does not have an analogue in the linear optics of surface waves.

Let us note that we do not cover all the possible cases in the present paper. In particular, NSP that go over from one sheet of the phase portrait of the nonlinear equations (3) to another[10] are possible in the case when ε_0 and ε_1 have the same sign. The special type of solution that the nonlinear equations possess in this situation requires a separate analysis.

4. SMALL-AMPLITUDE EXPANSIONS

The first four types of NSP considered above develop from the linear surface polaritons as the wave amplitude increases, and, for them, it is convenient, in computing the energy flux, to use the small-amplitude expansions. Since, as we have seen, the NSP exist in a finite range of variation of n, the curves in Fig. 2 can be described with sufficient accuracy by a finite expansion. To reproduce the curves in Fig. 2 with a high accuracy it is sufficient to limit ourselves in the expansions of the nonlinear equations to terms of third order in the amplitude when determining the energy flux (only terms of odd order enter into the expansions), and to find the field amplitude at the boundary with the same degree of accuracy we must expand the phase trajectories up to terms of fifth order in the amplitude.

The solution of the cubic equation for small X and Y up to terms of fifth order has the form

$$Z=-\frac{n}{\varepsilon_0}Y+\beta\frac{n^3}{\varepsilon_0^4}Y^3+\beta\frac{n}{\varepsilon_0^2}X^2Y-\frac{3n^5}{\varepsilon_0^7}Y^5-\frac{4n^3}{\varepsilon_0^5}X^2Y^3-\frac{n}{\varepsilon_0^3}X^4Y. \tag{15}$$

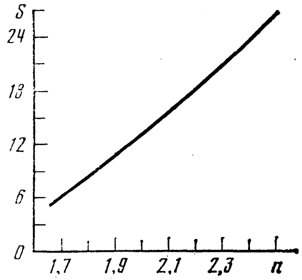

FIG. 4. Dependence of the energy flux of a surface wave on the effective refractive index in the case when $\varepsilon_0 = 2.25$, $\varepsilon_1 = 2.5$, and $\alpha > 0$.

Substituting this solution into the system of equations (3a) and (3b), we obtain the following system with the same degree of accuracy:

$$\dot{X}=-\frac{\gamma^2}{\varepsilon_0}Y+\beta\frac{n^4}{\varepsilon_0^4}Y^3+\beta\frac{n^2}{\varepsilon_0^2}X^2Y-\frac{3n^6}{\varepsilon_0^7}Y^5-\frac{4n^4}{\varepsilon_0^5}X^2Y^3-\frac{n^2}{\varepsilon_0^3}X^4Y, \tag{16a}$$

$$\dot{Y}=-\varepsilon_0X-\beta X^3-\beta\frac{n^2}{\varepsilon_0^2}XY^2+2\frac{n^4}{\varepsilon_0^5}XY^4+2\frac{n^2}{\varepsilon_0^3}X^3Y^2. \tag{16b}$$

The phase trajectories of those solutions to this system which pass through the origin can be represented in the form of two expansions: as an X dependence of Y:

$$Y=-\frac{\varepsilon_0}{\gamma}X-\frac{\beta f^2}{4\gamma^5}X^3+\frac{f^2v}{32\varepsilon_0\gamma^9}X^5, \tag{17}$$

where

$$f=2n^2-\varepsilon_0,\quad v=4n^4-12n^2\varepsilon_0+\varepsilon_0^2,$$

or in the form of a Y dependence of X:

$$X=-\frac{\gamma}{\varepsilon_0}Y+\frac{\beta f^2}{4\varepsilon_0^4\gamma}Y^3-\frac{f^2u}{32\varepsilon_0^7\gamma^3}Y^5, \tag{18}$$

where $u = 28n^4 - 36n^2\varepsilon_0 + 7\varepsilon_0^2$. The signs of the expansion are correct for the same quadrants in which we carried out the analysis of the phase trajectories in Fig. 1. Substituting (17) and (18) into Eqs. (16a) and (16b) respectively, and limiting ourselves now to the cubic terms, we obtain two ordinary differential equations with one variable:

$$\dot{X}=\gamma X-\beta f(2n^2+\varepsilon_0)X^3/4\varepsilon_0\gamma^3, \tag{19a}$$

$$\dot{Y}=\gamma Y-\beta f(3\varepsilon_0-2n^2)Y^3/4\varepsilon_0^3\gamma. \tag{19b}$$

The solutions to these equations can also be found in the form of expansions. But to find the energy flux in a wave we can proceed somewhat differently. Let us go over in the formula (9) from integration over τ to integration over the variable Y. Then

$$S=\frac{nY_0^2}{2\varepsilon_l\gamma_l^2}-\int_0^{Y_0}\frac{ZY}{\dot{Y}}dY, \tag{20}$$

where Y_0 is the value of Y at the boundary and the first term in (20) is the already integrated energy flux in the linear medium. After substituting into (20) $\dot{Y}$ from (19b) and Z from the formula (15) with the variable X expressed in terms of Y with the aid of (18), we can easily perform the integration in (20), and for the energy flux we obtain the formula

$$S=\frac{n}{2}\left(\frac{1}{\varepsilon_1\gamma_1}+\frac{1}{\varepsilon_0\gamma}\right)Y_0^2+\frac{\beta nf(6n^2-7\varepsilon_0)}{16\gamma^3\varepsilon_0^4}Y_0^4. \qquad (21)$$

The field strength Y_0 at the boundary is the coordinate of the point A in Fig. 1, and it is found as the intersection of the straight line $X=-(\gamma_1/\varepsilon_1)Y$ with the phase trajectory (18):

$$Y_0^2=\frac{4\beta\gamma^2\varepsilon_0^3}{u}\left\{1-\left[1-\frac{2u\varepsilon_0^2}{\gamma f^2}\left(\frac{\gamma}{\varepsilon_0}+\frac{\gamma_1}{\varepsilon_1}\right)\right]^{1/2}\right\}. \qquad (22)$$

The formula (22) gives a positive value for Y_0^2 in the regions of existence of the NSP. Let $G=(\gamma/\varepsilon_0+\gamma_1/\varepsilon_1)$, so that $G=0$ is the dispersion relation for the linear surface waves. Then, substituting (22) into (21), we obtain the expansion of the energy flux in powers of the quantity G:

$$S=\frac{2\beta n\gamma\varepsilon_0^4}{f^2}\left(\frac{1}{\varepsilon_1\gamma_1}+\frac{1}{\varepsilon_0\gamma}\right)G+\frac{\beta n\varepsilon_0^4}{f^4}\left(\frac{\varepsilon_0 u}{\varepsilon_1\gamma_1}+16n^2\gamma\right)G^2. \qquad (23)$$

The formula (23) is valid in all the four cases of surface waves considered above, and it reproduces all the characteristics of the curves shown in Fig. 2. Therefore, it can be used [with allowance for the limitations (11)] to carry out the calculations when making a comparison with the experimental data.

5. CONCLUSIONS

We have considered here the influence of the wave intensity in the nonlinear medium on the dispersion characteristics of the surface polaritons. As we have seen, the energy flux, the frequency, and the effective refractive index turn out to be connected by different relations in the cases of the various relations between the signs of the permittivities and the nonlinear susceptibility, and a separate analysis is required in each specific case. Of greatest interest in connection with the experimental investigation of NSP are the cases 2) and 3), since not only the permittivity, but also the nonlinear susceptibility α, exhibits dispersion in the vicinity of the resonance lines, and the extent of the dispersion allows us to observe the NSP-induced effects at low radiation powers. In particular, the measurement of the frequency dependence of the energy flux and the comparison of it with the theoretical curves would allow us to measure thereby the α dispersion in the frequency region of the longitudinal-transverse splitting, which it is, as is well known, quite difficult to do by other methods. Chen and Carter[16] have measured with the aid of NSP the nonlinear susceptibility in the opposite situation, i.e., in the case when the linear medium has the negative permittivity. Let us also note that we can artificially increase the nonlinear cubic susceptibility α by many orders of magnitude in layered media.[17] This opens up the possibility of significantly reducing the energy fluxes necessary for the manifestation of nonlinearity in the dispersion characteristics of surface waves and, consequently, of their wide use both for measurement purposes and in integral-optics devices.

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Translated by A. K. Agyei