Quantum measurements and the reliability of information transfer

V. B. Braginskiĭ and F. Ya. Khalili

Moscow State University
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The presently known quantum nondemolition energy measurements (including quanta counting without absorption) allow one to solve the problem of reliability of information transfer. It is shown that there exist optimal conditions for the recording of electromagnetic pulses of given energy and finite duration, conditions for which the reliability of information transfer is the highest. It is also shown that the use of quantum states of definite energy leads to a considerable gain in information transfer reliability for identical energy expenditure, or a significant energy gain, compared to those obtained using coherent quantum states.

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1. The interest in the problems of quantum measurements was revived in recent years, on the one hand, by the development of gravitational antennas, where the recorded response of a mechanical oscillator must be smaller than the indeterminacy of the coordinate in a coherent quantum state (see the reviews,[1,2] as well as Ref. 3), and on the other hand by the advent of electromagnetic radiation sources based on a new principle, sources which produce incoherent states (e.g., pure energy states or so-called squeezed quantum states[4–6]). The proposed "Gedanken"-procedures,[7,8] as well as the quite realizable procedures[9,10] for nondemolition counting of optical and microwave electromagnetic quanta allow one to create qualitatively new radiation receivers, where the number of quanta in an electromagnetic pulse can be repeatedly measured without modifying the pulse in each measurement. It becomes possible in principle to realize the fundamental recommendation of quantum theory of communications,[11] namely the use of orthogonal, in particular pure energy eigenstates for the transfer of information.

Our purpose is the following: to analyze a concrete model of a quantum communications channel and to demonstrate that the transition from now used, coherent emitters and receivers to emitters of energy states and quantum nondemolition (QND) counters will allow one to obtain either a substantial gain in the reliability of information transfer at the same energy expenditure, or to obtain a significant gain in energy.

2. All applicable methods of information reception and transmission make use of coded states which in the limit of zero temperature, i.e., in the presence of only quantum fluctuations, go over into coherent states. A signal of duration τ and energy $\bar{n}\hbar\omega$ has the indeterminacy $(\bar{n})^{1/2}\hbar\omega$ which is proper both to the preparation method (i.e., is characteristic for the transmitter) and the measuring device (i.e., the receiver). In this case the only method for improving the reliability of communication reliability is an increase in the energy of the signal. If, on the other hand, a signal is encoded in pure energy states, the indeterminacy in the energy both in the preparation and in the detection by means of a quantum nondemolition receiver will be of the order of $\hbar/\tau$, independently of $\bar{n}$ (here $\tilde{\tau}$ is the duration of the measurement). An increase in reliability will be determined here only by an increase of the duration of the signal (reduction of the bandwidth). In particular, in the transmission of binary messages the transmission of a single quantum may correspond to the message "one" and the unperturbed state of the line corresponds to the message "zero." To increase the reliability of information transfer without an increase in signal energy, one can also make use of so-called squeezed (or two-photon coherent) quantum states.[12] These states ae not orthogonal, but the degree of their mutual overlap decreases with the increase of mean energy much faster than for the case of coherent states. However, squeezed states yield a substantial gain in reliability only for sufficiently large $\bar{n}$; for $\bar{n}\approx 1$ the reliability of information transfer is approximately the same as for coherent states.[12]

3. It is convenient to characterize the reliability of communications quantitatively by means of the difference between the Shannon amount of information at the input port of the communications channel (including the receiver) and the amount of information about the input signal existing at the output port,[13] i.e., by the information loss ΔI. In the simplest case of the transmission of one bit of information, under the condition that the error probability is sufficiently small, we have

$$\Delta I=\frac{1}{2\ln 2}\left[\alpha\left(1+\ln\frac{1}{\alpha}\right)+\beta\left(1+\ln\frac{1}{\beta}\right)\right], \qquad (1)$$

where α and β are the statistical errors of the first and second kind.

If one codes by means of coherent states, the optimal coding from the point of view of minimizing the statistical errors for a given mean signal energy is phase coding, wherein the unit signal is represented by the coherent state $|A\rangle$ and the zero state is represented by the state $|-A\rangle$ ($|A|^2=\bar{n}$ is the mean number of quanta in the signal).[11] If the receiver is also coherent, it is easy to show that the loss of information will be (see Appendix I)

$$\Delta I=\frac{1}{\ln 2}\Phi(-(2\bar{n})^{1/2})\left[1-\ln\Phi(-(2\bar{n})^{1/2})\right], \qquad (2)$$

where

$$\Phi(x)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{x}e^{-t^2/2}\,dt$$

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is the error function. For $\bar{n}\gg 1$ the expression (2) becomes somewhat simpler:

$$\Delta I=\frac{1}{2\ln 2}\left(\frac{\bar{n}}{\pi}\right)^{1/2}e^{-\bar{n}}.$$

4. To realize a quantum nondemolition measurement of the energy of a travelling electromagnetic wave it is necessary that a portion of the transmission line be coupled to the measuring device in such a manner that the interaction Hamiltonian should be quadratic in the field. As an example one may consider a system analogous to the one discussed in Ref. 10: a waveguide segment is filled with a nonlinear medium and is also part of a lower frequency test resonant cavity. The presence of the signal wave leads to a modification of the permittivity of the medium, and consequently to a change in the frequency of the normal modes of the test cavity, a frequency detected by usual (linear) methods.

For the measurement error to be small it is necessary that the variable indirectly affected by a change in the signal (the generalized momentum of the measuring device) be well-defined. According to the Heisenberg uncertainty principle, the variance of the canonically conjugate variable (the generalized coordinate of the device) will be large, leading to an indeterminacy in the parameters of the line (the signal propagation speed) in the interaction segment. The presence of such an indeterminacy causes in turn a random phase shift in the transmitted wave. One can show that on account of this shift the following uncertainty relation holds:

$$\Delta n\Delta\varphi\geqslant 1/2,$$

where Δn is the error in the measurement of the number of quanta. In principle the quantity Δn is bounded from below only by the Bohr uncertainty principle

$$\Delta n\geqslant(\Delta n)_{\min}=1/2\omega\tilde{\tau} \qquad (3)$$

(ω is the signal frequency, $\tilde{\tau}$ is the duration of the measurement). However on account of the parametric character of the interaction with the measuring device there appears an additional source of loss of information, because the signal quantum can be reflected from the transmission-line inhomogeneity which arises in the measuring process, and thus will not be recorded by the receiving device. The degree of inhomogeneity, and consequently the reflection coefficient, increases as the accuracy of the measurements increases. At the same time, the reflection coefficient decreases as the length of the interaction section is increased. Consequently, for a given length of this segment there exists an optimal measurement-error value that exceeds the limit (3), and accordingly, a minimal value of the information loss. We note that there always exists a probability that the photon will be reflected, but will nevertheless be recorded by the receiving device. In the present paper (to simplify the calculations) we have assumed that this probability is zero. Thus, the information-loss estimates given below are upper limits.

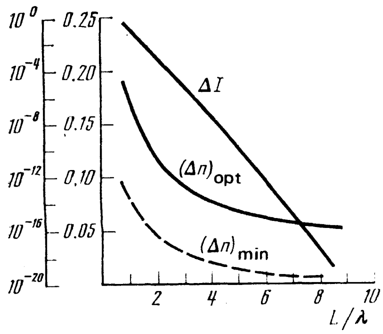

FIG. 1.

We list the results of calculations based on the described recording scheme (see Appendix II). The probabilities of "false alarm" α and of "loss of the target" β are equal to

$$\alpha=\frac{\Delta n}{(2\pi)^{1/2}\xi}\exp\left\{-\frac{\xi^2}{2(\Delta n)^2}\right\},$$

$$\beta=\frac{1}{4(\Delta n)^2}\exp\left\{-\frac{2\pi L}{\lambda}+1\right\}+\frac{\Delta n}{(2\pi)^{1/2}(1-\xi)}\exp\left\{-\frac{(1-\xi)^2}{2(\Delta n)^2}\right\}, \qquad (4)$$

where L is the length of the interaction segment, λ is the mean wavelength of the signal, ξ is a parameter of the data-reduction procedure: the decision "one" is adopted if the reading of the instrument exceeds ξ, and "zero" is chosen in the opposite case. Substituting Eq. (4) into Eq. (2) and minimizing the expression so obtained with respect to Δn and ξ one can determine the optimal values $(\Delta n)_{\rm opt}$ and $\xi_{\rm opt}$ and the corresponding minimal value of information loss $\Delta I_{\min}$. The figure shows the graphs of $\Delta I_{\min}$ and $(\Delta n)_{\rm opt}$ as a function of L/λ, as obtained from numerical calculations. The same figure also shows the minimal values $(\Delta n)_{\min}$ admissible from the point of view of the Bohr relation (3), as a function of the same variable. For sufficiently large vaues of L/λ one may use the following asymptotic estimate for the quantity ΔI:

$$\Delta I=2^{1/2}\left(2\pi\frac{L}{\lambda}-1\right)^2\exp\left\{-\frac{2\pi L}{\lambda}+1\right\}. \qquad (5)$$

It is clear from the figure that a high reliability of information transfer is attained already for $L/\lambda\simeq 3$ to 10, $\Delta n\simeq 0.1$. At the same time, the use of coherent states would require an increase by an order of magnitude and a half of the power in the communication channel, to attain the same reliability. Indeed, for a coherent signal and $\Delta I\simeq 10^{-10}$–10^{-12} it is necessary to have $\bar{n}\simeq 22$–27; at the same time, for a pure energy signal $\bar{n}=\frac{1}{2}$ (if the transmission of zeros and ones are equally probable), and the same reliability of transmission is attained for $L/\lambda\simeq 5-6$ and $\Delta n\simeq 0.07$.

We note that the increase in reliability due to the use of pure energy signals entails a narrowing of the bandwidth $\Delta f/f$ of the information channel (since $\lambda/L\simeq\Delta f/f$). At the same time the reduction in energy necessary for the transmission of one bit signifies a proportional increase of the dynamic range, and consequently of the transmission of the communication channel.

5. We analyze the influence of damping in the communication channel for the case $kT<\hbar\omega$ (here k is the Boltzmann constant). If transmission of unity corresponds to a pulse containing exactly n quanta and the transmission of zero corresponds to inactivity in the line, then the additional probability of signal loss due to possible absorption of all n quanta in the transmission line equals

$$\beta^* = [1-\exp(-Z/Z^*)]^n, \qquad (6)$$

where Z is the length of the line, Z^* is the distance of e-fold decrease of the energy in the pulse. From Eq. (6) follows the following estimate from above for the quantity β^*:

$$\beta^* \leqslant \exp(-2n^*), \qquad (7)$$

where $n^* = \frac{1}{2} n \exp(-Z/Z^*)$ is the mean number of quanta which arrive at the receiving end (assuming that the transmission of zeros and ones is equally probable).

In the process of relaxation, a coherent state conserves its coherence, only its mean energy decreases. Therefore the statistical errors of encoding by coherent states will be equal to (see Appendix I)

$$\alpha = \beta = \Phi(-(2n^*)^{1/2}) \approx \frac{1}{2(\pi n^*)^{1/2}} \exp(-n^*). \qquad (8)$$

A comparison of (7) and (8) shows directly that for the same value of n^* the probability of encoding errors is considerably smaller for pure energy states.

6. The calculations of the quantity ΔI carried out for the case of sending and recording a single quantum in an energy state (see Eqs. (4) and (5)) yield essentially an exact value of the parameter R, usually designated as the quantum yield of the detector; $\Delta I = 1 - R$ for a quantum nondemolition measurement. In our opinion our results are useful not only for problems of reliable information transfer, but also for many cases of indirect quantum measurements.

APPENDIX I

We assume that the transmission of ones and zeros correspond to coherent states with parameters A_1 and A_0, respectively. A receiver which realizes a coherent-state measurement is described by a spectral decomposition of unity of the form

$$\hat{\Pi}_0 = \frac{1}{\pi}\int \Omega_0(A)|A\rangle\langle A| d^2A, \quad \hat{\Pi}_1 = \frac{1}{\pi}\int \Omega_1(A)|A\rangle\langle A| d^2A, \qquad (\text{A.I.1})$$

with

$$\Omega_0(A) + \Omega_1(A) = 1, \qquad (\text{A.I.2})$$

where Ω_0 and Ω_1 are functions describing the data reduction procedure: if the measuring device reads A, then a decision "0" is adopted with probability $\Omega_0(A)$ and a decision "1" with probability $\Omega_1(A)$. The statistical errors are

$$\alpha = \mathrm{Sp}(\hat{\Pi}_1|A_0\rangle\langle A_0|) = \frac{1}{\pi}\int \Omega_1(A)\exp(-|A-A_0|^2)\, d^2A, \qquad (\text{A.I.3})$$

$$\beta = \mathrm{Sp}(\hat{\Pi}_0|A_1\rangle\langle A_1|) = \frac{1}{\pi}\int \Omega_0(A)\exp(-|A-A_1|^2)\, d^2A.$$

Minimizing the functionals (A.I.3) under the condition (A.I.2) we find that the optimal strategy is the following:

$$\Omega_0(A) = \begin{cases} 1, & \text{if } |A-A_1| > |A-A_0| \\ 0, & \text{if } |A-A_1| < |A-A_0| \end{cases}$$

i.e., the value "1" is adopted if A lies closer to A_1 than to A_0, and "0" is picked in the opposite case. Then

$$\alpha = \beta = \Phi\left(-\frac{|A_1-A_0|}{\sqrt{2}}\right), \qquad (\text{A.I.4})$$

where $\Phi(x)$ is the error function.

If one imposes the condition that the mean energy of the message be a minimum:

$$\bar{n} = (|A_0|^2 + |A_1|^2)/2, \qquad (\text{A.I.5})$$

the values A_0 and A_1 must be symmetric with respect to the origin:

$$A_1 = -A_0, \qquad (\text{A.I.6})$$

this corresponds to statistical errors equal to

$$\alpha = \beta = \Phi(-(2\bar{n})^{1/2}). \qquad (\text{A.I.7})$$

APPENDIX II

A measuring instrument is characterized by a probability distribution $w_n(\tilde{n})$ for obtaining the result $\tilde{n}$ of the measurement under the condition that the received pulse contains n quanta. One may consider that the distribution is normal:

$$w_n(\tilde{n}) = \frac{1}{(2\pi(\Delta n)^2)^{1/2}} \exp\left(-\frac{(\tilde{n}-n)^2}{2(\Delta n)^2}\right) \qquad (\text{A.II.1})$$

(the variance $(\Delta n)^2$ describes the accuracy of measurement). The statistical errors will be in this case

$$\alpha = \int_\xi^\infty w_0(\tilde{n})\, d\tilde{n} = \Phi\left(-\frac{\xi}{\Delta n}\right),$$

$$\beta = \int_{-\infty}^{\xi} w_1(\tilde{n})\, d\tilde{n} + P = \Phi\left(\frac{1-\xi}{\Delta n}\right) + P, \qquad (\text{A.II.2})$$

where P is the reflection coefficient of the photon from the instrument. To simplify the calculations, we assume in the computation of P that the generalized mass of the measuring instrument is sufficiently large so that one may neglect the change of its coordinate over the duration of the measurement. In this case it suffices for the determination of P to solve the simple static problem of reflection of a wave from an inhomogeneity of the transmission line:

$$\frac{d^2\Psi(z)}{dz^2} + K^2(z)\Psi(z) = 0, \qquad (\text{A.II.3})$$

$$K(z) = K_0 = 2\pi/\lambda \quad \text{for} \quad z<0, \quad z>L, \qquad (\text{A.II.4})$$

$$\Psi(z) = \begin{cases} \exp(iK_0z) + r\exp(-iK_0z) & \text{for } z \leqslant 0, \\ D\exp(iK_0z) & \text{for } z \geqslant L, \end{cases} \qquad (\text{A.II.5})$$

where λ is the mean wavelength of the signal, r is the reflected amplitude, and $(0, L)$ the interval of interaction with the instrument.

We make use of a piecewise linear approximation for $K^2(z)$, partitioning the interaction interval into portions of length l. On the m-th portion $l(m-1) < z < lm$ we have

$$K^2(z)=K_0^2\left\{1+l\sum_{j=0}^{m-1}\varkappa_j+\varkappa_m[z-l(m-1)]\right\}. \qquad \text{(A.II.6)}$$

Taking Eq. (A.II.6) into account, we obtain an exact solution of Eq. (A.II.3):

$$\Psi(z)=\frac{K(z)}{(K_0^2|\varkappa_m|)^{1/3}}\left[C_m^1H_{1/3}^{(1)}\left(\frac{2}{3}\frac{K^3(z)}{K_0^2|\varkappa_m|}\right)+C_m^2H_{1/3}^{(2)}\left(\frac{2}{3}\frac{K^3(z)}{K_0^2|\varkappa_m|}\right)\right] \quad \text{for } l(m-1)<z<lm, \qquad \text{(A.II.7)}$$

where $C_m^{(1,2)}$ are coefficients, and $H_{1/3}^{(1,2)}$ are the Hankel functions of order $\frac{1}{3}$ of the first and second kind, respectively.

We consider the case of greatest practical importance, when $K(z)$ is slowly varying over distances of the order λ, and the deviation of $K(z)$ from the value K_0 is small:

$$|dK(z)/dz|\ll K^2(z), \qquad |K(z)-K_0|\ll K_0. \qquad \text{(A.II.8)}$$

In this case the argument of the Hankel functions is much larger than unity in Eq. (A.II.7), and one may make use of the asymptotic expression for cylinder functions. Joining the solutions obtained in this manner at the points lm, $m- =0,1,...,L/l$ with allowance for (A.II.5), and taking the limit $l\to 0$ with (A.II.8) taken into account, we obtain:

$$r=\frac{i}{4K_0^2}\int_0^L\exp\left(-2i\int_0^z K(x)\,dx\right)\frac{d^2K(z)}{dz^2}\,dz. \qquad \text{(A.II.9)}$$

Minimizing the functional (A.II.9) with respect to $K(z)$ with the boundary conditions (A.II.4) taken into account, we obtain:

$$|r|_{\min}^2=(\Delta\varphi)^2(1-\cos 2\Delta\varphi)\exp(-K_0L+1), \qquad \text{(A.II.10)}$$

where $\Delta\varphi=\int_0^L(K(z)-K_0)\,dz$ is the phase shift of the wave as it passes through the instrument. The perturbation of the phase is connected to the error in the measurement by the uncertainty relation:

$$\langle(\Delta\varphi)^2\rangle(\Delta n)^2\geqslant 1/4.$$

Consequently

$$P=\langle|r|_{\min}^2\rangle=\frac{\exp(-K_0L+1)}{4(\Delta n)^2}. \qquad \text{(A.II.11)}$$

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