Periodic motions of the magnetization in the B phase of helium-3

I. A. Fomin

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR
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Solutions to the spin dynamics equations for the B phase of helium-3 are found which describe the periodic motions of the magnetization in a constant magnetic field of arbitrary strength. The stability of the solutions found is investigated, and the effect exerted by dissipation on them is considered.

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1. INTRODUCTION

Periodic motions of magnetization are of interest for two reasons. First, they are important for the experimental investigations of the properties of magnetic materials, since resonance methods allow us to measure the periods of such motions with great accuracy. Second, the periodic solutions play a distinct role in the mathematical investigation of the equations of magnetodynamics.

The motion of the magnetization in superfluid helium-3 is described by the Leggett equations[1] in the absence of relaxation, and by the Leggett-Takagi equations[2] when the relaxation is taken into account. According to these equations, the motion of the magnetization in superfluid helium-3 occurs under the action of two torques: the Zeeman torque produced by the external magnetic field **H** and the spin-orbit torque produced by the dipole-dipole interaction with energy U_D. The magnitudes of the torques are characterized by the Larmor frequency $\omega_L = -gH$ (g is the gyromagnetic ratio for the He^3 nuclei) and by the frequency Ω of the longitudinal oscillations. Two asymptotic regions naturally arise: $\omega_L \ll \Omega$ and $\Omega \ll \omega_L$. Until recently, the spin-dynamics equations for superfluid helium-3 were investigated mainly in these regions, the equations possessing periodic solutions in both limits. In the strong-field limit (i.e., for $\omega_L \gg \Omega$) the periodic solutions describe a Larmor precession perturbed by the action of the moment of the dipole–dipole forces. The resulting corrections to the precession frequency[3,4] contain nontrivial information about the properties of the superfluid phases of helium-3, and they are used in the interpretation of pulsed NMR experiments performed on these phases.[5] In the opposite limiting case, i.e., $\omega_L \ll \Omega$, the motion of the magnetization is primarily determined by the moment of the dipole forces, and depends essentially on the specific form of U_D. The Leggett equations for the B phase in the case $\omega_L = 0$ have been solved exactly,[6] and the motion of the magnetization has been investigated in detail.[7] In particular, in this case the periodic solution to the Leggett equations, which is called the WP mode (wall-pinned mode), has been known for a long time,[8,6] This solution describes the periodic magnetization motion experimentally observed earlier.[9]

The magnetization motion in fields of intermediate strength (i.e., in fields for which $\omega_L \sim \Omega$) is also of interest, but only one of the earlier investigated periodic solutions of the Leggett equations, namely, the Brinkman–Smith solution[3] for the case in which the angle between the magnetization and the magnetic field is smaller than $\theta_0 = \arccos(-\frac{1}{4})$, remains valid for such fields. Recently, Novikov[10] showed that the Leggett equations for the B phase possess a number of periodic solution in the case when ω_L and Ω are arbitrarily related. The properties of the periodic solutions in the region of intermediate fields, were, however, not investigated. In the present paper we explicitly write out these solutions, investigate their stability against weak perturbations, and find out how they are affected when allowance is made for dissipation caused by an "internal" mechanism in the hydrodynamic approximation. Furthermore, to establish a correspondence with the known asymptotic solutions, we follow the passage to the limits $\omega_L \to 0$ and $\Omega/\omega_L \to 0$. A brief account of some of the results obtained can be found in Ref. 11.

2. CHOICE OF THE VARIABLES

To be in a position to subsequently take into account the effect of dissipation on the periodic solutions, we shall proceed from the equations of spin dynamics with dissipative terms. But we shall, recognizing that for intermediate magnetic fields the hydrodynamic approximation is applicable in a broad range of temperatures, limit ourselves to the consideration of only this case, and write the Leggett-Takagi equations in the hydrodynamic limit:

$$\dot{\mathbf{S}} = g[\mathbf{S}\times\mathbf{H}] + \mathbf{N}_D, \tag{1}$$

$$\dot{\mathbf{d}} = [\mathbf{v}\times\mathbf{d}], \tag{2}$$

$$\mathbf{S} = g\mathbf{H} + \mathbf{v} - \varkappa\mathbf{N}_D. \tag{3}$$

Here **S** denotes the spin of a unit volume of helium-3 (we use a system of units such that **S** is measured in $\sec^{-1}$ and the magnetic susceptibility $\chi = g^2$); **d** is the order parameter in the vector representation and is a function of the direction in momentum ($\hat{\mathbf{p}}$) space; the correspondence for the **B** phase is realized by the matrix $\hat{R}(\mathbf{n}, \theta)$ for rotation through the angle θ about the direction **n**:

$$d_i(\hat{\mathbf{p}}) = R_{ik}(\mathbf{n}, \theta)\, p_k;$$

v is the angular velocity of the order parameter, it being given by Eq. (3); $\mathbf{N}_D$ is the torque produced by the dipole-dipole forces and is connected by the relation

$$\mathbf{N}_D = -\mathbf{n}\,\partial U_D/\partial\theta,$$

with the dipole energy U_D, which, in the B phase, depends

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only on the angle θ:

$$U_D = \frac{8}{15}\Omega^2\left(\cos\theta + \frac{1}{4}\right)^2; \tag{4}$$

and $\varkappa$ is that combination of the constants of the Leggett-Takagi theory and characterizes, in accordance with (1)–(3), the energy dissipation rate:

$$dE/dt = -\varkappa N_D^2.$$

Equations (1)–(3) go over into the Leggett equations when $\varkappa = 0$. The Leggett equations have periodic solutions for the B phase because these equations have another integral of the motion besides the energy. To find the solutions, we should rewrite the system of equations in such variables that this integral would be explicitly separated out. One possible set of such variables has been suggested by Novikov.[10] The variables in Ref. 10 were chosen for reasons of convenience in the general analysis of the system of Leggett equations. Of greater importance in the present paper is the possibility of a direct physical interpretation of the solutions obtained and of a comparison with the previously obtained asymptotic solutions. For this purpose, it is more convenient to describe the motion of the order parameter with the aid of the Euler angles α, β, and γ. The integral of the motion is then also easy to find.

Let us, following the usual definition of the Euler angles, set

$$\hat{R}(\mathbf{n}, \theta) = \hat{R}(\alpha, \beta, \gamma) = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma), \tag{5}$$

where $\hat{R}_z(\alpha)$ is the matrix for the rotation through the angle α about the z axis, etc. The z axis is oriented in the direction opposite to the direction of the field $\mathbf{H}$, in accordance with the fact that the gyromagnetic ratio for He^3 is negative. Then at equilibrium $\mathbf{S}$ is oriented along the z axis. Let us introduce, besides the fixed coordinate system (x,y,z), a moving system (ξ, η, ζ) rotating together with the $\mathbf{d}$ vectors, i.e., let $\zeta = \hat{R}(\mathbf{n}, \theta)z$, etc. To go over to the Euler angles in the equations of motion, we project Eq. (1) onto the unit vectors $\hat{z}$ and $\hat{\zeta}$ and the perpendicular to the $(\hat{z}, \hat{\zeta})$ plane (let us denote the unit vector in this direction by $\hat{\beta}$). We should, in doing this, bear in mind that the unit vectors $\hat{\zeta}$ and $\hat{\beta}$ are moving (i.e., rotating) vectors, and to the time derivatives should be added terms taking account of this rotation. The components of the rotation axis $\mathbf{n}$, which are the same in the moving and fixed bases, can be expressed in terms of the Euler angles as follows:

$$n_i = -\frac{1}{2\sin\theta} e_{ikl} R_{kl}(\alpha, \beta, \gamma), \tag{6}$$

with the angle θ given by the relation

$$1 + 2\cos\theta = \cos\beta + \cos\Phi + \cos\beta\cos\Phi, \tag{6'}$$

where $\Phi = \alpha + \gamma$. As a result of the indicated operations, we obtain equations expressing the time derivatives of S_z, S_ζ, and S_β in terms of these quantities themselves and the angles α, β, and γ. Three other equations are obtained by projecting Eq. (3) onto $\hat{\mathbf{z}}$, $\hat{\boldsymbol{\zeta}}$, and $\hat{\boldsymbol{\beta}}$, and by substituting into the resulting equations the components of the angular velocity $\boldsymbol{\nu}$ expressed in terms of the derivatives of the Euler angles (see, for example, Ref. 12):

$$\nu_z = \dot{\alpha} + \dot{\gamma}\cos\beta, \quad \nu_\zeta = \dot{\alpha}\cos\beta + \dot{\gamma}, \quad \nu_\beta = \dot{\beta}.$$

As a result, we obtain a closed system of equations for the six variables α, β, γ and S_z, S_β, S_ζ, which, in the absence of dissipation, are respectively canonically conjugate variables. Let us note further that, according to (4) and (6'), the dipole energy depends on α and γ only through the combination $\Phi = \alpha + \gamma$:

$$U_D(\alpha, \beta, \gamma) = {}^2/_{15}\Omega^2[\cos\beta - {}^1/_2 + (1+\cos\beta)\cos\Phi]^2. \tag{7}$$

As a result, the quantity $P = S_z - S_\zeta$ is conserved in the absence of dissipation. Let us therefore incorporate Φ and P into the set of independent variables in place of γ and S_ζ. Solving the resulting equations for the derivatives, we finally obtain the following system of equations:

$$dS_z/dt = -\partial U/\partial\Phi, \tag{8}$$

$$\frac{dS_\beta}{dt} = \frac{\sin\beta}{(1+\cos\beta)^2}\left(\frac{\cos\beta}{1-\cos\beta}P + S_z\right)\left(\frac{P}{1-\cos\beta} - S_z\right) - \frac{\partial U}{\partial\beta} + \frac{\varkappa}{2}\frac{\partial U}{\partial(\cos\theta)}\frac{\sin\Phi}{\sin\beta}[P - S_z(1-\cos\beta)], \tag{9}$$

$$\frac{dP}{dt} = \frac{\varkappa}{2}\frac{\partial U}{\partial(\cos\theta)}\{(1+\cos\Phi)[P\cos\beta + (1-\cos\beta)S_z] - S_\beta\sin\Phi\sin\beta\}, \tag{10}$$

$$\frac{d\alpha}{dt} = -\omega_L + \frac{1}{\sin^2\beta}[P\cos\beta + S_z(1-\cos\beta)] + \frac{\varkappa}{2}\frac{\partial U}{\partial(\cos\theta)}\sin\Phi, \tag{11}$$

$$\frac{d\beta}{dt} = S_\beta + \frac{\varkappa}{2}\sin\beta(1+\cos\Phi)\frac{\partial U}{\partial(\cos\theta)}, \tag{12}$$

$$\frac{d\Phi}{dt} = -\omega_L + \frac{1}{1+\cos\beta}(2S_z - P) + \varkappa\sin\Phi\frac{\partial U}{\partial(\cos\theta)}. \tag{13}$$

In certain cases, instead of the set of angular momenta S_z, S_β, and P, it turns out to be more convenient to use S_ζ, S_β, and P. The transition to these variables is accomplished directly with the aid of the definition of P.

3. THE PERIODIC SOLUTIONS

The right-hand sides of Eqs. (8)–(13) do not contain the angle α; therefore, the system admits of solutions of the form

$$\dot{\alpha} = -\omega_L + \Delta(\beta^{(0)}, P^{(0)}, S_z^{(0)}, \Phi^{(0)}, S_\beta^{(0)}) = \text{const}, \tag{14}$$

where $\beta^{(0)}$, $\Phi^{(0)}$,... are those values of the variables at which the right-hand sides of Eqs. (8)–(10), (12), and (13) vanish, i.e., at which

$$\frac{dS_z}{dt} = \frac{dS_\beta}{dt} = \frac{dP}{dt} = \frac{d\beta}{dt} = \frac{d\Phi}{dt} = 0. \tag{15}$$

The angle α for such solutions increases linearly in time, which, according to the definition (5), corresponds to a periodic motion: rotation of the order parameter about the z axis. On account of the condition $\dot{\Phi} = 0$, the angle γ also varies in this case, its rate of variation being equal in magnitude but opposite in sign to the rate of variation of the angle α. We shall at first neglect the dissipation (i.e., set $\varkappa = 0$). Then the condition $dP/dt = 0$ is automatically fulfilled. Thus, P is that integral of the motion which is discussed in Sec. 2; it corresponds to the integral P_1 in the Maki-Ebisawa notation[6] and the integral A_1 in the Novikov notation.[10] There remain four conditions on five variables, i.e., the periodic solutions form a one-parameter family.

The condition for the vanishing of the right-hand side of Eq. (8) has the form

$$\partial U/\partial\Phi = -{}^8/_{15}\Omega^2(\cos\theta + {}^1/_4)\sin\Phi(1+\cos\beta) = 0. \qquad (16)$$

It is fulfilled in the following four cases:

I) $\cos\theta = -{}^1/_4$, II) $\Phi = 0$,

III) $\Phi = \pi$, IV) $\cos\beta = -1$.

The cases II) and III) correspond to the condition $\sin\Phi = 0$, but they should not be combined, since the solutions corresponding to them differ greatly in their properties.

For all the four cases we find from (12) that

$$S_\beta = 0, \qquad (17)$$

i.e., that S lies in the plane of the unit vectors $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\zeta}}$. The vanishing of the right-hand sides of Eqs. (9) and (13) establishes a relation connecting β, P, and S_z, whereupon Eq. (11) determines the dependence of the angular frequency of the periodic motion on the integral P. In certain cases it is more convenient to use the angle β as the parameter specifying the solution.

Let us now consider in greater detail the solutions corresponding to the above cases.

The case I), as can be seen from the formula (4), corresponds to the minimum of the dipole energy, and in this case not only is $\partial U/\partial\Phi = 0$, but also

$$\frac{\partial U}{\partial\beta} = \frac{1+\cos\Phi}{2}\frac{\partial U}{\partial(\cos\theta)} = 0.$$

In this case the moment of the dipole forces is equal to zero, and the motion of the magnetization occurs under the action of the magnetic field only. Let us write the conditions resulting from Eqs. (9) and (13) in terms of the variables S_z and S_ζ, with respect to which these conditions are symmetric:

$$(S_z - S_\zeta\cos\beta)(S_\zeta - S_z\cos\beta) = 0, \qquad (18)$$

$$S_z + S_\zeta = \omega_L(1+\cos\beta). \qquad (19)$$

This system of equations has two solutions:

Ia) $S_z = \omega_L$, $S_\zeta = \omega_L\cos\beta$,

Ib) $S_z = \omega_L\cos\beta$, $S_\zeta = \omega_L$.

For the solution Ia) the spin has the equilibrium value, the torque produced by the external field is also equal to zero in this case, and the solution Ia) is thus a steady-state solution. The substitution of Ia) into the right-hand side of (11) gives, as was to be expected, $d\alpha/dt = 0$. The angle β, according to Eq. (6) and the condition $\cos\theta = -\frac{1}{4}$, can vary within the limits $0 \leqslant \beta \leqslant \theta_0 \equiv \arccos(-\frac{1}{4})$, thereby defining an entire branch of steady-state solutions. The existence of such a branch is a consequence of the independence of the dipole energy of the orientation of the axis $\mathbf{n}$ of rotation. Indeed, we can, by computing with the aid of the formula (6) the third component of $\mathbf{n}$ for the solution Ia):

$$n_z = n_\zeta = \frac{\sin\Phi}{2\sin\theta}(1+\cos\beta) = \left[\frac{4}{5}\left(\frac{1}{4}+\cos\beta\right)\right]^{1/2}, \qquad (20)$$

verify that it varies from one to zero as β varies from zero to θ_0. The specific choice of the equilibrium orientation of $\mathbf{n}$ under given external conditions is dictated by interactions, not considered here, that are weaker than the dipole interaction.[13]

In the case of Ib) we obtain from (11) the relation $d\alpha/dt = -\omega_L$. This solution corresponds to a steady procession of the spin with the Larmor frequency. The spin is oriented along the ζ axis, i.e., it forms an angle β with the direction of the magnetic field. The solution Ib) also exists only when $0 \leqslant \beta \leqslant \theta_0$; it coincides with the Brinkman-Smith solution,[3] and describes the magnetization precession in pulsed NMR experiments under conditions when the initial angle of deviation of the magnetization is smaller than θ_0 (Ref. 5). As $\omega_L \to 0$, the solutions Ia) and Ib) go over into the trivial solution $\mathbf{S} = 0$, $\partial U_D/\partial\theta = 0$, which corresponds to the equilibrium state.

In the case II) the dipole energy is stationary only with respect to the variable Φ. Setting $\Phi = 0$ in (6′), we obtain the relation $\cos\theta = \cos\beta$. Further, setting

$$d\alpha/dt = {}^1/_2(\delta - \omega_L), \qquad (21)$$

we obtain from (9) and (11) the relation

$$\omega_L^2 - \delta^2 = 4\partial U/\partial(\cos\beta) = {}^{16}/_{15}\Omega^2(1+4\cos\beta). \qquad (22)$$

For $\frac{16}{15}\Omega^2(1+4\cos\beta) < \omega_L^2$ Eq. (22) furnishes two real values of δ that differ in sign:

$$\delta^{(\pm)} = \pm[\omega_L^2 - {}^{16}/_{15}\Omega^2(1+4\cos\beta)]^{1/2}. \qquad (23)$$

The values of the S-vector components corresponding to these values of δ are found from Eqs. (11) and (13):

$$S_z^{(\pm)} = \omega_L\cos^2\frac{\beta}{2} \pm |\delta|\sin^2\frac{\beta}{2}, \qquad (24)$$

$$S_\zeta^{(\pm)} = \omega_L\cos^2\frac{\beta}{2} \mp |\delta|\sin^2\frac{\beta}{2}. \qquad (25)$$

The disposition of the vectors $\mathbf{S}^{(+)}$ and $\mathbf{S}^{(-)}$ in the ($\hat{\mathbf{z}}$, $\hat{\boldsymbol{\zeta}}$) plane for $\cos\beta < -\frac{1}{4}$ is shown in Fig. 1, where we have introduced,

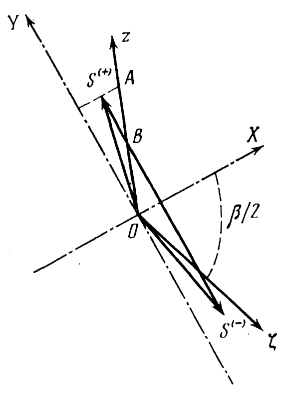

FIG. 1. Disposition of the vectors $\mathbf{S}^{(+)}$ and $\mathbf{S}^{(-)}$ for the periodic solutions IIa) and IIb) respectively. The distance $OA = \delta$ and $OB = \omega_L$.

besides the axes $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\zeta}}$, the orthogonal axes X and Y oriented respectively along $\hat{\mathbf{z}}+\hat{\boldsymbol{\zeta}}$ and $\hat{\mathbf{z}}-\hat{\boldsymbol{\zeta}}$. The components of $\mathbf{S}$ along these axes are found from the formulas (22) and (23), and are equal to

$$S_X^{(+)}=S_X^{(-)}=\omega_L\cos\beta/2,\qquad S_Y^{(+)}=-S_Y^{(-)}=|\delta|\sin\beta/2,$$

i.e., $S^{(+)}$ and $S^{(-)}$ are symmetric about the X axis. The angular frequencies of the corresponding motions of the order parameter are, according to (21), equal to

$$\dot\alpha^{(\pm)}=-\frac{1}{2}\{\omega_L\pm[\omega_L^2-{}^{16}/_{15}\Omega^2(1+4\cos\beta)]^{1/2}\}.\qquad(26)$$

Let us now investigate how the solutions found behave in the limiting cases $\omega_L\to0$ and $\Omega/\omega_L\to0$. When $\omega_L=0$, the two frequencies coincide, and are equal to the frequency of the WP mode[8]; $\mathbf{S}^{(+)}$ and $\mathbf{S}^{(-)}$ are then parallel and antiparallel respectively to the Y axis. Below we shall designate the solution corresponding to the upper signs in the formulas (23)–(26) as IIa) and the solution corresponding to the lower signs by IIb), and consider these solutions separately.

In the case IIa) the direction of the spin $\mathbf{S}$ approaches that of the z axis as the field intensity increases, and the angle between them is $\sim(\Omega/\omega_L)^2$ when $\omega_L\gg\Omega$. The S-precession frequency for $(\Omega/\omega_L)\ll1$ is obtained by expanding the expression (26); it is equal to

$$\dot\alpha^{(+)}=-\frac{4}{15}\frac{\Omega^2}{\omega_L}(1+4\cos\beta).\qquad(27)$$

Notice that the precession frequency is much lower than the Larmor frequency. This is due to the cancellation of the magnetic-field-induced torque by the torque $\mathbf{N}_D$ produced by the dipole-dipole forces. The torque produced by the field is proportional to the $\mathbf{S}$ component perpendicular to the field, and tends to zero as the angle between $\mathbf{S}$ and $\mathbf{H}$ tends to zero. The torque $\mathbf{N}_D$ is determined by the orientation of the order parameter and does not depend on the angle between $\mathbf{S}$ and $\mathbf{H}$; therefore this angle can be chosen such that the two torques are close in magnitude. The direction of $\mathbf{N}_D$ coincides with that of $\mathbf{n}$. For $\Phi=0$ the vector $\mathbf{n}$ is perpendicular to the $(\mathbf{z},\boldsymbol{\zeta})$ plane, and, consequently, $\mathbf{N}_D\perp\mathbf{S}$, as is the torque produced by the field. As can be seen from the answer, the terms or order $(\Omega/\omega_L)^2$ cancel out completely. Low-frequency precessions of the type under consideration have thus far not been observed. They can apparently be excited by a variable field, starting from the configuration in which $\mathbf{n}\perp\mathbf{S}$ and $\beta=\theta_0$ and smoothly varying the frequency of the exciting field from zero to the value corresponding to the requisite motion.

In the case IIb) the direction of $\mathbf{S}$ approaches that of $\boldsymbol{\zeta}$ as the field intensity increases, and the precession frequency tends, in the limit of high ω_L, to the Larmor frequency and is given up to terms $\sim(\Omega/\omega_L)^2$ by the formula

$$\dot\alpha^{(-)}=-\omega_L\left[1-\frac{16}{15}\frac{\Omega^2}{\omega_L^2}\left(\frac{1}{4}+\cos\beta\right)\right],\qquad(28)$$

which was first obtained by Brinkman and Smith.[3] Thus, the WP mode and the steady-state precession of the magnetization in pulsed NMR experiments[5] for $\beta>\theta_0$ are limiting cases of one and the same periodic solution to the Leggett equations.

For $\cos\beta>-\frac{1}{4}$, periodic solutions of the type II) exist only when $\omega_L^2>\frac{16}{15}\Omega^2(1+4\cos\beta)$; the $\mathbf{S}^{(\pm)}$ are in this case located inside the angle formed by the positive directions of the z and ζ axes. These solutions are, however, unstable (see Sec. 5 of the present paper), and will not be considered further here.

In the case III) $\cos\Phi=-1$ and, according to (6′), $\cos\theta=-1$. In this case U_D does not depend on β, and, taking (21) into account, we have $\delta=\pm\omega_L$. Equations (22) and (23) remain valid, and we have, in accordance with two possible signs of δ, two solutions: the steady-state solution

$$\dot\alpha=0,\qquad S_z^{(+)}=\omega_L,\qquad S_\zeta^{(+)}=\omega_L\cos\beta$$

and the Larmor precession:

$$\dot\alpha=-\omega_L,\qquad S_z^{(-)}=\omega_L\cos\beta,\qquad S_\zeta^{(-)}=\omega_L.$$

Both of these solutions are also unstable.

The case IV) corresponds to the singular—for the chosen coordinates—line $\cos\beta=-1$, on which the angles α and γ are not independent coordinates. There arise ambiguous expressions in Eqs. (8)–(13) when $\cos\beta=-1$, and it is convenient in this case to proceed directly from Eqs. (1)–(3). When $\cos\beta=-1$, $\cos\theta=-1$ also. As a result, $\partial U_D/\partial\theta=0$ and, consequently, $\mathbf{N}_D=0$. The system (1)–(3) possesses in this case an obvious periodic solution corresponding to the rotation of the order parameter with angular velocity parallel to the z axis and equal to $\nu_z=S_z-gH$. In this case $\mathbf{S}$ is also parallel to the z axis. Notice, however, that the value $\cos\theta=-1$ corresponds to the maximum of U_D as a function of θ, and it is to be expected that solutions of the type IV) will be unstable.

4. EFFECT OF DISSIPATION

Let us now find out how the solutions found will be affected when allowance is made for the dissipative terms dropped earlier from the equations of motion. Notice that the condition (16) for the existence of periodic solutions is satisfied by all the extrema of U_D as a function of θ: to them correspond the solutions I), III), and IV). At the same time, all the dissipative terms are proportional to $\mathbf{N}_D$, and go to zero together with $\partial U_D/\partial\theta$. Thus, the switching on of the dissipation in no way affects the solutions I), III), and IV), and only the solutions of the type II) need to be analyzed further. In this case the relaxation terms in Eqs. (10) and (12) are nonzero. The right-hand side of (12) can be made to vanish by choosing the appropriate S_β; this will not affect the remaining stationarity conditions. On the other hand, the right-hand side of Eq. (10) cannot be made to vanish by making small changes in the solutions II); as a result, for these solutions $dP/dt\neq0$, and they cease to be periodic in the strict sense of the word. If, however, the dissipation is so weak that the characteristic relaxation time is long compared to the periods of the nonrelaxational motions, then we can assume that P varies slowly, and that the remaining variables are the same functions of the instantaneous value of P that we have in the absence of dissipation. The motion of the order parameter will then be almost periodic, but with a slowly varying angular frequency. Since it is precisely the angular frequency that is a directly measurable quantity, it is useful to find the law according to which it varies in the course of the relaxation.

To do this, let us substitute into the right-hand side of Eq. (10) the solution II), i.e., $\Phi = 0$, and the quantities S_z and S_ξ as given by the formulas (24) and (25). Allowing also for the fact that $P = \delta\,(1 - \cos\beta)$, and that δ and β are connected by Eq. (23), we obtain an equation describing the variation of δ and, consequently, the angular frequency $\dot\alpha$ of the precession in the course of the relaxation:

$$(5\lambda^2+3\delta^2-\omega_L^2)\frac{d\delta}{dt}$$
$$=\frac{\varkappa}{32\lambda^2}(\delta^2-\omega_L^2)(\delta^2-\omega_L^2+5\lambda^2)(\delta^2-3\lambda^2-\omega_L^2)(\delta+\omega_L), \quad (29)$$

where, to abbreviate the notation, we have set $\lambda^2 = 16/15 \times \Omega^2$. Equation (29) can be integrated, and determines implicitly the dependence of δ on the time:

$$\Psi(\delta/\lambda)-\Psi(\delta_0/\lambda)=\varkappa\Omega^2 t, \quad (30)$$

where

$$\Psi(y)=-\frac{5+2\tilde\omega^2}{\tilde\omega}\frac{1}{y+\tilde\omega}-\left[\frac{19}{3}-\frac{5}{2\tilde\omega^2}+\frac{8\tilde\omega^2}{15}\right]\ln(y+\tilde\omega)$$
$$-\frac{5+2\tilde\omega^2}{2\tilde\omega^2}\ln(y-\tilde\omega)+\frac{3}{20}a\left[\tilde\omega\ln\frac{y-a}{y+a}-a\ln(y^2-a^2)\right]$$
$$-\frac{5}{12b}(4+b^2)\left[\tilde\omega\ln\frac{y-b}{y+b}-b\ln(y^2-b^2)\right]. \quad (31)$$

Here we have introduced the notation $a^2 = \tilde\omega^2 - 5$, $b^2 = \tilde\omega^2 + 3$, $\tilde\omega = \omega_L/\lambda$. The function $\Psi(y)$ has singularities at the points $y = \pm\tilde\omega$ ($\cos\beta = -\frac{1}{4}$) and $y = \pm b$ ($\cos\beta = -1$). The difference $y^2 - a^2$ does not vanish when $-1 \leqslant \cos\beta \leqslant -\frac{1}{4}$. For the δ/λ values lying in the interval $-(\tilde\omega^2+3)^{1/2} < \delta/\lambda < -\tilde\omega$, the formulas (30) and (31) describe the relaxation of the solutions IIb); correspondingly, in the interval $\tilde\omega < \delta/\lambda < (\tilde\omega^2+3)^{1/2}$ the formulas describe the relaxation of the solutions IIa). The direction of the relaxation in both cases corresponds to the decrease of the absolute value of δ/λ, i.e., to $\cos\beta \to -\frac{1}{4}$, in view of which the relaxation should be considered in the neighborhoods of the singularities $\delta/\lambda = \pm\tilde\omega$. The analysis of the indicated singularities is not complicated. Here we shall discuss only the case of small $\tilde\omega$; the singularity then becomes less trivial because of the presence of $\tilde\omega$ in the denominators of the first three terms in the formula (31). The investigation of this case is also interesting in connection with the experiments that have been performed to study the relaxation of the WP mode.[9]

Let us retain the leading—in $\tilde\omega \ll 1$ and $y \ll 1$—terms in the formula (31). Setting $\xi = \omega_L/\delta$ ($|\xi| < 1$), we obtain from (30) and (31) the formula

$$\left[\frac{1}{2}\ln\frac{1+\xi}{1-\xi}-\frac{\xi}{1+\xi}\right]_{\xi_0}^{\xi}=\frac{3}{16}\varkappa\omega_L^2 t, \quad (32)$$

where ξ_0 is the value of ξ at $t = 0$. For $\xi \ll 1$ the formula (32) gives the following well-known asymptotic law of relaxation of the WP mode[2] for both of the solutions IIa) and IIb):

$$1/\delta^2 \approx {}^3/_{16}\varkappa t, \quad (33)$$

But when $\omega_L \neq 0$ this law does not describe the relaxation of δ at $t \to \infty$, since in this case $|\xi|$ should tend to unity, and the condition $|\xi| \ll 1$ is not fulfilled. The complete formula (32) should be used when $\xi \sim 1$. And at the final stage of the relaxation we have in the case of the solution IIb) $\xi \to -1$, i.e., $\delta \to (-\omega_L)$ according to the law

$$1/(\delta+\omega_L) \sim {}^3/_{16}\varkappa\omega_L t.$$

For the solution IIa) $\xi \to 1$, and from (32) we obtain the relation

$$\delta-\omega_L \sim \exp\left[-{}^3/_8\varkappa\omega_L^2 t\right].$$

The frequency relaxation law can be simplified in the limiting cases $\omega_L = 0$ and $\Omega/\omega_L \to 0$ as well. For $\omega_L = 0$ the formulas (30) and (31) go over into the well-known expression for the relaxation of the frequency of the WP mode.[14] To obtain the asymptotic law of relaxation for $\Omega/\omega_L \to 0$, it is convenient to proceed directly from Eq. (29). In the case of IIa) this equation in the limit in question has the form

$$\frac{d\delta}{dt}=\frac{\varkappa\omega_L^2}{4\lambda^2}(\delta-\omega_L)(\delta-\lambda a)(\delta-\lambda b). \quad (34)$$

After integrating and substituting the precession frequency $\dot\alpha$ as given by (21) into the result, we obtain

$$\left[\frac{\dot\alpha(t)}{\dot\alpha(0)}\right]^8\left[\frac{\dot\alpha(0)+4\Omega^2/3\omega_L}{\dot\alpha(t)+4\Omega^2/3\omega_L}\right]^3\left[\frac{\dot\alpha(0)-4\Omega^2/5\omega_L}{\dot\alpha(t)-4\Omega^2/5\omega_L}\right]^5$$
$$=\exp(-8\varkappa\Omega^2 t), \quad (35)$$

i.e., the relaxation occurs with characteristic time $\sim 1/\varkappa\Omega^2$. The relaxation of the solution IIb) in the case $\Omega/\omega_L \ll 1$ has already been investigated in connection with pulsed NMR experiments in strong magnetic fields.[15–17] The characteristic relaxation time in this case is $\sim\omega_L^2/\varkappa\Omega^4$, i.e., it is longer than the characteristic time in the case of IIa) by a factor of $(\omega_L/\Omega)^2$.

5. STABILITY OF THE SOLUTIONS

For applications, it is important to know whether the solutions found are stable. In this section we shall investigate the stability of the periodic solutions against weak perturbations. To simplify the analysis, let us initially drop the dissipative terms from Eqs. (8)–(13). Let us, following the usual procedure, set $S_z = S_z^{(0)} + \zeta$, $S_\beta = \eta$, $\Phi = \Phi^{(0)} + \varphi$, $\beta = \beta^{(0)} + \psi$, where the symbols with the index 0 denote those values of the variables which correspond to the periodic solution in question and ζ, η, φ, and ψ are small perturbations. According to Eq. (10), P is an exact integral of the motion, and it is not necessary to perturb it. For each of the periodic solutions considered in Sec. 3, the linearization of the system (8), (9), (12), and (13) with respect to the small perturbations and the substitution of solutions of the form ζ, η, φ, $\psi \sim e^{-i\omega t}$ lead to an equation for ω^2. If all the ω^2 are real and positive, then the perturbations remain small in time. In the opposite case we have growing perturbations, and the original solution is unstable. For stable solutions, not only the signs but also the magnitudes of ω^2 are of interest, since they determine the frequencies of small oscillations about the periodic solutions. These frequencies characterize the particular original solution, and can be used to identify it experimentally. Let us, omitting long, but simple calculations, give only the final results of the investigation of the stability of the solutions found.

In the case I) we obtain for both of the two branches Ia) and Ib) one and the same equation for the frequencies:

$$\omega^4-(\omega_L^2+\Omega^2)\omega^2+\frac{1+4\cos\beta}{5}\omega_L^2\Omega^2=0 \qquad (36)$$

The roots of this equation are

$$\omega_\pm^2={}^1/_2(\omega_L^2+\Omega^2)\pm[{}^1/_4(\omega_L^2+\Omega^2)^2-{}^1/_5\omega_L^2\Omega^2(1+4\cos\beta)]^{1/2}. \qquad (37)$$

In the entire domain of existence of the solution I) (i.e., for $\cos\beta>-\frac{1}{4}$), both roots are positive, and the solutions Ia) and Ib) are stable. This was to be expected, since the solutions I) correspond to the minimum of the dipole potential U_D. The formula (37) gives the frequencies of the two oscillation branches. When $\cos\beta=1$, one of the oscillations goes over into longitudinal oscillations with frequency Ω, while the other goes over into the Larmor precession.

The second term of the radicand in the formula (37) can be rewritten in the form $-\omega_L^2\Omega^2n_z^2$ with the aid of (20), i.e., the frequencies of the small oscillations give direct information about the orientation of the vector **n**. This property of the oscillations in question has been used in experimental investigations[13] of the magnetic textures in the B phase of helium-3. The angle between **n** and the magnetic field was determined in these experiments from the transverse-NMR resonance frequency given by the formula (37). In the cited paper the discussion is about the frequencies of small oscillations about the static solution Ia). The formula (37) also gives the frequencies of the oscillations about the periodic solution Ib), and this allows us to use it to determine the orientation of **n** in the case when the deviation of **n** from the equilibrium direction is produced by dynamical means, as is done in pulsed NMR experiments. Such measurements could be useful in, for example, the investigation of the magnetic relaxation in the B phase. In this case it may turn out to be more convenient to excite such oscillations with the aid of a longitudinal variable magnetic field.

In the case II) the equation for the frequencies of the perturbations has the form

$$[\omega^2+\lambda^2({}^1/_4+\cos\beta)][\omega^2-\omega_L^2-{}^1/_2\lambda^2(1+\cos\beta)(1-6\cos\beta)] +{}^1/_2\lambda^2\omega_L^2(1-\cos\beta)({}^1/_4+\cos\beta)=0. \qquad (38)$$

Elementary analysis shows that the condition for both of the roots of Eq. (38) to be positive is $\cos\beta<-\frac{1}{4}$, i.e., the solutions of the type II) are stable only when $\beta>\theta_0$. The small-oscillation frequencies can easily be found from Eq. (38), but the resulting expressions are unwieldy, and we shall consider only the limiting cases.

For $\omega_L=0$ we obtain two pairs of frequencies from (38):

$$\omega_1^2={}^1/_2\lambda^2(1+\cos\beta)(1-6\cos\beta), \qquad (39)$$

$$\omega_2^2=-\lambda^2({}^1/_4+\cos\beta). \qquad (40)$$

Comparison of the frequencies of small oscillations about the solutions II) with the rate of relaxation of these solutions allows us to establish a criterion for the slowness of the relaxation and, consequently, for the applicability of the method used in Sec. 4 to describe the relaxation. Indeed, for the magnetization and the order parameter not to move away from the periodic solution II) in the course of the relaxation, the relaxation rate must be low compared to the small-oscillation frequencies $\omega_{1,2}$, i.e., we must have

$$\frac{d}{dt}\left(\frac{1}{\omega_{1,2}}\right)\ll 1. \qquad (41)$$

Let us, in particular, apply this criterion to the WP mode. For $\omega_L=0$, the frequency ω_2 is more "dangerous," since $\cos\beta\to-\frac{1}{4}$ in the course of the relaxation. For this frequency we obtain from (41) with the aid of (22) and (33) the condition

$$\frac{d}{dt}\left(\frac{1}{\omega_2}\right)\sim\frac{d(\varkappa t)^{1/2}}{dt}\sim\left(\frac{\varkappa}{t}\right)^{1/2}\ll 1. \qquad (42)$$

Since, at any rate, $t\gg 1/\delta$, $\delta\lesssim\Omega$, and $\varkappa\Omega\ll 1$ on account of the assumption that the hydrodynamic approximation is applicable, the condition (42) is always fulfilled, the fulfillment becoming better and better as t increases.

In the opposite limiting case, i.e., for $\Omega/\omega_L\ll 1$, the frequencies of small perturbations of the solutions II) are given by the following expressions:

$$\omega_1^2=\omega_L^2+{}^2/_{15}\Omega^2\{5(1+\cos\beta)(1-4\cos\beta)-2(1+4\cos\beta)\}, \qquad (43)$$

$$\omega_2^2=-{}^8/_{15}\Omega^2({}^1/_4+\cos\beta)(1+\cos\beta). \qquad (44)$$

The criterion (41) in this case also reduces to the condition $\varkappa\Omega\ll 1$, which is always fulfilled in the region of applicability of the system (8)–(13).

In the case III) the equation for the frequencies of the perturbations has the form

$$\omega^4-\omega^2(\omega_L^2-{}^2/_5\Omega^2)-{}^1/_5\omega_L^2\Omega^2(1+\cos\beta)=0. \qquad (45)$$

Elementary investigation shows that both roots ω_1^2 and ω_2^2 of this equation are real, and that their product is negative, i.e., solutions of the type III) are unstable.

The case IV) is a more complicated singularity, which it is natural to investigate with allowance for the relaxation terms. In the present work we did not investigate the stability of solutions of the type IV).

In the case of small oscillations about the stable periodic solutions, there naturally arises the question of the lifetime of such oscillations. Physically, the most interesting in this sense are the oscillations about the solutions Ib) and IIb), since these solutions describe states that arise in pulsed NMR experiments. In order to find the corresponding lifetimes, we should take the dissipative terms in Eqs. (8)–(13) into consideration as well in the stability analysis. For solutions of the type Ib) allowance for the dissipative terms leads to the replacement $\Omega^2\to\Omega^2(1-i\varkappa\omega)$ in Eqs. (36) and (37). Since it was assumed from the very beginning that $\omega\varkappa\ll 1$, we should limit ourselves in the solutions to the terms of first order in this quantity. For example, according to (37), there are two branches in the limit $\Omega/\omega_L \ll 1$:

$$\omega_+^2=\omega_L^2+{}^4/_5\Omega^2(1-\cos\beta)(1-i\varkappa\omega_+), \tag{46}$$

$$\omega_-^2={}^1/_5\Omega^2(1+4\cos\beta)(1-i\varkappa\omega_-). \tag{47}$$

The solutions IIb) relax; therefore, for them the question of attenuation of the perturbations makes sense only if by chance they attenuate over a time period that is short compared to the relaxation time of the solution itself. Such a situation obtains in the limit of strong fields, i.e., for $\omega_L \gg \Omega$. The relaxation time of the solutions IIb) in this case is $\sim\omega_L^2/\varkappa\Omega^2$, and for the frequencies of the perturbations when allowance is made for the dissipation we have

$$\omega_1^2=\omega_L^2-{}^{16}/_{15}\Omega^2(2+\cos\beta)({}^1/_4+\cos\beta)$$
$$+{}^2/_{15}\Omega^2(1+\cos\beta)(7-12\cos\beta)(1-i\omega_1\varkappa), \tag{48}$$

$$\omega_2^2=-{}^8/_{15}\Omega^2(1+\cos\beta)({}^1/_4+\cos\beta)(1-i\omega_2\varkappa), \tag{49}$$

i.e., the longest of the lifetimes $\sim 1/\varkappa\Omega^2 \ll \omega_L^2/\varkappa\Omega^4$. Both the frequencies and the lifetimes of the low-frequency branches have already been computed.[4,15] The frequencies were determined correctly, but there is an error in the dissipative terms, which leads to the appearance of a superfluous factor $(1+\cos\beta)^{-1}$ in front of $\omega\varkappa$. The error was detected by Schoepe and Schertler in the course of a numerical solution of the spin-dynamics equations for the B phase of helium-3 (see Ref. 11). The formulas (47) and (49) agree with the results of the numerical computation.

6. CONCLUSION

In the present paper we have considered those periodic solutions of the spin-dynamics equations for the B phase of helium-3 which arise because of the presence in these equations of another integral P of the motion besides the energy. This naturally does not exclude the possibility of the existence of other periodic solutions at some isolated values of the parameters. For periodic solutions of the type considered here to exist, the moment of the dipole-dipole forces and the Zeeman torque must actually be collinear; they they will together cause the spin to precess.

For applications it is important that there be periodic solutions for an arbitrary relation between Ω and ω_L; this will allow the interpretation of experiments performed in arbitrary magnetic fields, and not just in fields corresponding to the asymptotic regions. The stable periodic solutions are of interest in connection with experiment; there are four of them: Ia), Ib) and IIa), IIb). The solutions Ia) and Ib) are known.[3,13] The solutions IIa) and IIb) are new. In the limit $\omega_L=0$ they go over into the well-known solution corresponding to the WP mode. In the limit of strong fields the solution IIb) describes the precession of the magnetization in pulsed NMR experiments at magnetization tipping angles greater than 104°; the solution IIa) has thus far not been investigated in the limit $\Omega/\omega_L \to 0$.

When the stable periodic solutions are perturbed slightly, there arises small oscillations that are also periodic motions. There is, however, an important difference between these oscillations and the original periodic motions. It consists in the fact that the small oscillations are periodic only in the linear approximation: as their amplitude increases, the various modes begin to interact with each other, and the periodicity is, generally speaking, destroyed. Let us also note that all the small oscillations become damped when allowance is made for the dissipation.

The analysis performed here allows us to determine the behavior of the phase trajectories of the system of equations (8)–(13) only in the vicinity of the periodic solutions found. This, however, is not sufficient for the establishment of the general picture of the phase trajectories of the indicated system in its six-dimensional phase space. Further investigations are necessary for the elucidation of this important—for applications—equation.

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