The nonintegrability of the classical Yang-Mills equations

E. S. Nikolaevskiĭ

Institute for Theoretical and Experimental Physics

L. N. Shchur

L. D. Landau Institute for Theoretical Physics of the USSR Academy of Sciences

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A special case of the Yang-Mills equations is investigated, for which a transverse intersection of separatrices of unstable closed trajectories occurs. This implies the nonexistence of real-analytic first integrals of the motion other than the Hamiltonian. A consequence of this fact is the absence of a complete set of integrals for a system describing classical Yang-Mills fields. A qualitative proof is given for the absence of integrals other than the Hamiltonian for the case of transverse intersection of the separatrices. Numerical methods are described for the construction of separatrices and doubly asymptotic solutions, and for the determination of the angles between separatrices.

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§1. INTRODUCTION

The problem of integrability of the classical Yang-Mills equations is extremely important both for classical and for quantum field theory. The interest in this problem has increased after the appearance of the Refs. 1 and 2. In Ref. 1 it was shown that the self-duality equations determine a special integrable case for the classical Yang-Mills equations, and on the basis of this result multi-instanton solutions were constructed in Ref. 3. In Ref. 2 it was noted that there is no stochastization of the initial condition for the Klein-Gordon equation with a cubic nonlinearity—this is the case of a two-dimensional, one-component Yang-Mills field. Another interesting fact is the nontrivial analogy between a Yang-Mills field and an n-field. Both theories are renormalizable, both exhibit asymptotic freedom, and both have instanton solutions.[4] At the same time the equations of the n-field are a completely integrable system.[5] All these considerations led to the hope that the Yang-Mills equations are also completely integrable. The present paper proves the opposite, namely that a system of classical Yang-Mills equations is nonintegrable in the sense of not having a complete set of real-analytic integrals of the motion.

In the classical case the integrability of a system with N degrees of freedom means the existence of a Hamiltonian and $N-1$ constants of the motion whose Poisson brackets vanish pairwise. In principle, this allows one to find solutions. The integrability of the classical system implies, as a rule, the integrability of the corresponding quantum system[6a] in the sense of the existence of a complete set of commuting operators, although this is not always the case.[6b]

Thus, we are facing the question of integrability of the Yang-Mills equations[7]

$$\partial_\mu F_{\mu\nu}{}^a + g\varepsilon^{abc} A_\mu{}^b F_{\mu\nu}{}^c = 0, \tag{1.1}$$

where

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g\varepsilon^{abc} A_\mu{}^b A_\nu{}^c$$

($\mu,\nu, = 0,1,2,3; a,b,c, = 1,2,3$). Here A_μ^a are elements of an arbitrary Lie algebra. In the sequel we consider the case of the $SU(2)$ algebra (for which the notation is used above) such that $\mathbf{A}_\mu$ can be identified with vectors in the three-dimensional isospin space.

We shall reason in the following manner. Assume that we know a substitution satisfying the equations (1.1) and reducing them to a simpler system amenable to a detailed investigation. If the system (1.1) has a complete set of integrals of the motion, any subsystem with N degrees of freedom will also have N constants of the motion. If the subsystem does not have a complete set of integrals of the motion, then the original system will have fewer integrals smaller than degrees of freedom. Thus, the nonintegrability of the subsystem implies the nonintegrability of the whole system. We note that the integrability of a subsystem does not imply the integrability of the whole system.

To simplify the system (1.1) we make the substitution[8]

$$A_0{}^a = 0, \quad \partial_i A_j{}^a = 0, \quad A_i{}^a = O_i{}^a f^a, \quad x = f^1, \quad y = f^2, \quad z = f^3 = 0 \tag{1.2}$$

(here O_i^a are orthogonal constant matrices satisfying $O_i^a O_i^b = \delta^{ab}/g^2$, no summation over a), which reduces (1.1) to the system of ordinary differential equations

$$\ddot{x} + xy^2 = 0, \quad \ddot{y} + x^2 y = 0. \tag{1.3}$$

The dot denotes differentiation with respect to time. This system has the obvious integral of the motion

$$H = {}^1/_2 (\dot{x}^2 + \dot{y}^2 + x^2 y^2). \tag{1.4}$$

The equations (1.3) and the integral (1.4) are invariant with respect to the scale transformation

$$x \to \alpha x, \quad y \to \alpha y, \quad t \to \alpha^{-1} t, \quad E \to \alpha^4 E, \tag{1.5}$$

where E is the energy.

On account of this the system (1.3) has no parameters which allows us, in particular, to set $E = 1/2$ hereafter without loss of generality.

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Reference 9 proves that the system (1.3) has no other integral of the motion under the assumption that it can be analytically continued into a strip of finite width in the complex plane. This assumption imposes considerable mathematical constraints on the form of the integral we are looking for.

In Ref. 10 approximately twenty periodic trajectories for the system (1.3) were found numerically, and it was discovered that in the numerical integration they are unstable with respect to a small change in the initial data for the trajectories. In Ref. 11 numerical calculations of long trajectories were the basis of the conclusion that the system (5.1) is stochastic. The results of Ref. 11 are an indication that the system (1.1) is nonintegrable.

An unambiguous criterion of nonintegrability is the intersection of separatrices. In our brief paper we stated that the separatrices of the system (1.3) intersect at a nonvanishing angle, i.e., transversally. This immediately implies the absence of a real-analytic integral of the motion other than the Hamiltonian. Thus, the system (1.3) is nonintegrable in the same sense as the three–body problem of celestial mechanics.[13] The nonintegrability of the system in the indicated sense definitely implies the absence of a complete set of integrals for the system (1.1), i.e., the nonintegrability of the classical Yang-Mills field.

In §2 we discuss a model exemplifying the construction of the Poincaré separatrices, and of the doubly asymptotic solutions with a computer.

In §3 we give a qualitative proof of the absence of any real-analytic integral of the motion other than the energy if the separatrices intersect transversally.

In §4 we discuss the behavior of the integral curves of the system (1.3) and determine the intersection points of the separatrices in the Poincaré section, as well as the angle between them.

In §5 the doubly asymptotic solutions are constructed, and the angles between the separatrices in phase space are calculated.

§2. BEHAVIOR OF SEPARATRICES IN THE POINCARÉ SECTION

In this section we demonstrate with a simple example how the separatrices of an unstable solution behave in the absence of integrals of the motion other than the Hamiltonian. We construct the Hamiltonian to be investigated, of a system with two degrees of freedom, in the following manner. In the y coordinate we introduce a harmonic oscillator with the potential energy $U(y) = y^2$ and in the x coordinate we consider the potential energy $U(x) = (x^2-1)^2$ with the manifestly unstable solution at $x = 0$. Moreover, we add a positive-definite nonlinear term of the simplest form $U(x,y) = \varepsilon x^2y^2$, where $\varepsilon \geqslant 0$. As a result we obtain a mechanical system with the Hamiltonian

$$H={}^1/_2[\dot{x}^2+\dot{y}^2+(x^2-1)^2+y^2+\varepsilon x^2y^2], \qquad (2.1)$$

with the assumption that $E > 1/2$. The equations of motion have the form

$$\ddot{x}+2x(x^2-1)+\varepsilon xy^2=0, \qquad \ddot{y}+y+\varepsilon x^2y=0. \qquad (2.2)$$

The phase space of the system is four-dimensional with coordinates $(x,\dot{x},y,\dot{y})$. The solutions of the system (2.2) are curves in this space. We fix the value of the energy. Each definite energy defines in the four-dimensional phase space a three-dimensional hypersurface on which the trajectories of the system are located. We intersect the constant energy surface with the half-plane defined by $y = 0, \dot{y} > 0$ (the Poincaré section or map[13]). We choose x and $\dot{x}$ as coordinates on this half-plane.

An arbitrary trajectory "pierces" the half-plane in an arbitrary number of points. A closed trajectory pierces the half-plane in a finite number of points N. After N intersections with the half-plane (or, what amounts to the same, after the period T of the trajectory) such a trajectory will intersect the plane at its original point, i.e., will close on itself. Thus, such a point A_0 is a fixed point of the N-th iterated Poincaré mapping of the half-plane into itself.

By means of the Poincaré section the study of the behavior of trajectories which are close to closed trajectories is replaced by an analysis of the behavior of the points in the section. We place the coordinate origin in the fixed point A_0 of the ($y = 0, \dot{y} > 0$) half-plane. The coordinates in the vicinity of this point will be denoted by $(\xi,\dot{\xi})$. A trajectory of the system (2.2) with initial point $(\xi_1,\dot{\xi}_1)$ will intersect a neighborhood of the origin at the point $(\xi_2,\dot{\xi}_2)$. Considering the deviations $(\xi_i,\dot{\xi}_i)$ small, we introduce the matrix A^T:

$$\begin{pmatrix} \xi_{i+1} \\ \dot{\xi}_{i+1} \end{pmatrix} = A^T \begin{pmatrix} \xi_i \\ \dot{\xi}_i \end{pmatrix}.$$

Thus, the matrix A^T realizes the mapping of the half-plane into itself in the neighborhood of the fixed point (the so-called monodromy matrix of the Poincaré map). The monodromy matrix is constructed in the following manner. We linearize the system (2.2) near the periodic solution $x = x_0(t), y = y_0(t)$ by means of the substitution $x = x_0 + \xi$, $y = y_0 + \eta$, considering ξ and η as small:

$$\ddot{\xi}+2(3x_0^2-1)\xi+\varepsilon(y_0^2\xi+2x_0y_0\eta)=0,$$
$$\ddot{\eta}+\eta+\varepsilon(x_0^2\eta+2x_0y_0\xi)=0. \qquad (2.3)$$

The simplest periodic solution of the system (2.2) with period $T = 2\pi$ is:

$$x_0=0, \quad y_0=\sin t, \quad \dot{x}_0=0, \quad \dot{y}_0=\cos t \qquad (2.4)$$

(for definiteness we have set $E = 1$) and determines a closed trajectory in phase space. This trajectory intersects the half-plane at the point $x = 0, \dot{x} = 0$, i.e., the point $A_0(0,0)$ is a fixed point of the mapping of the half-plane into itself.

For the solution (2.4) the system (2.3) takes on the form

$$\ddot{\xi}-2\xi+\varepsilon y_0^2\xi=0, \qquad \ddot{\eta}+\eta=0. \qquad (2.5)$$

It is obvious that the first equation in (2.5) has growing solutions, i.e., the periodic solution is unstable. The variations η and $\dot{\eta}$ correspond to a shift along the trajectory and a change of the time scale (or, what amounts to the same, the energy scale), and are unimportant in the case under discussion.

We first set $\varepsilon = 0$, when the solutions of the system (2.5) have the form

$$\xi=\xi_1 \exp(\sqrt{2}t)+\xi_2 \exp(-\sqrt{2}t),$$
$$\eta=\eta_1 \exp(it)+\eta_2 \exp(-it). \qquad (2.6)$$

In the basis[14] $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ the matrix A^T has the form

$$A^T=\begin{pmatrix} \operatorname{ch}\tau & \operatorname{sh}\tau/\sqrt{2} \\ \sqrt{2}\operatorname{sh}\tau & \operatorname{ch}\tau \end{pmatrix} \qquad (2.7)$$

where $\tau = 2\sqrt{2}\pi$.

Owing to the fact that the system (2.2) is Hamiltonian, the Liouville theorem implies $\det A^T = 1$ and the Poincaré map is area-preserving. The eigenvalues of the matrix A^T are determined[14] from the equation (Sp = trace)

$$\lambda^2-\operatorname{Sp} A^T\lambda+1=0, \qquad (2.8)$$

with $\lambda_1\lambda_2 = 1$. It is obvious that in our case λ_1 and λ_2 are real for simplicity we shall denote the larger of the eigenvalues by λ. In the case $\varepsilon = 0$

$$\lambda=\exp(2\sqrt{2}\pi)\approx 7228.34857\ldots .$$

The eigenvectors of the matrix A^T define a pair of straight lines which are invariant under the Poincaré mapping of the neighborhood of A_0 into itself. For the matrix (2.7) the invariant straight lines are given by

$$\dot{\xi}=\pm\sqrt{2}\xi. \qquad (2.9)$$

In a neighborhood of the point $A_0(0,0)$ these lines are asymptotes of the separatrices (of the invariant curves passing through the fixed point A_0).

We note that for $\varepsilon = 0$ the system (2.2) has, in addition to the Hamiltonian (2.1) the integral

$$I_1={}^1/_2[\dot{x}^2+(x^2-1)^2], \qquad (2.10)$$

which determines the energy of the oscillations in the x direction. For a fixed value of the Hamiltonian (2.1) it is obvious that to different values of the integral I_1 will correspond different invariant curves in the Poincaré section (see Fig. 1). For $I_1 = 1/2$, (2.10) is the equation of separatrices of the point $A_0(0,0)$.[1)] These separatrices intersect at the points $x = \pm\sqrt{2}$ under a zero angle, i.e., they coincide there, forming a closed curve. Thus, we note that if there exists a complete set of integrals (the number of integrals equals the number of degrees of freedom) the invariant curves in the Poincaré section are closed.

What happens for $\varepsilon\neq 0$? To answer this question it is necessary to construct the separatrices. For definiteness we set $\varepsilon = 1$. In this case the system (2.2) has again the solutions (2.4) with $T= 2\pi$. The monodromy matrix is constructred by numerical integration of the systems (2.2) and (2.5). Its eigenvalues are real and $\lambda = 2128.14068\ldots$, while $\mu\approx(1/T)\ln\lambda = 1.21960\ldots$. The invariant straight lines have the form

$$\dot{\xi}=\pm 1.34402696\xi. \qquad (2.11)$$

We note that knowing the Lyapunov exponent μ, the period T of the trajectory, and the error per step ρ one can estimate the number $N_{\max}$ of mappings for which the counting error will not manifestly influence the results. In our case $\rho\sim 10^{-12}$ and

$$N_{max}\approx(1/\mu)T\ln\rho^{-1}\approx\ln\rho^{-1}/\ln\lambda<4. \qquad (2.12)$$

It should be noted that this error is underestimated, since the counting error in the neighborhood of the point A_0 grows, roughly speaking, "along" the separatrix according to Eq. (2.12) and decays in the "transverse" direction. This is a usual property of hyperbolic mappings (cf., e.g., Ref. 15).

Starting in the neighborhood of the point A_0 with the invariant line (2.11, +), we shall mark the intersection points of the trajectory with the Poincaré section. As the initial data traverse a portion of this line the intersection points will trace out the separatrix in the half-plane. (We note that one can increase the numerical accuracy considerably by choosing in the vicinity of the line (2.11, +) such initial data that for $t\to-\infty$ the intersection points should not go off the line (2.11, +). This significantly strengthens the guarantee given by the estimate (2.12), of a correct result.

The separatrices of the unsable point $A_0(0,0)$ for $\varepsilon = 1$ are shown in Fig. 2. The angle at which the separatrices intersect is clearly not zero. We remind the reader that a separatrix is called unstable if the trajectory which intersects the half-plane along this separatrix, approaches the point A_0 as $t\to+\infty$.

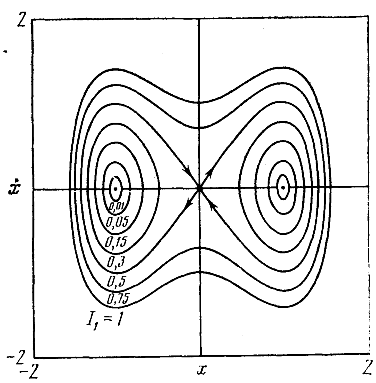

FIG. 1. The invariant curves in the Poincaré section $y = 0, \dot{y} > 0$ for the system (2.2) at $\varepsilon = 0$ and $E = 1$. The values of the integral I_1 (2.10) are marked under the curves. The curve with $I_1 = 1/2$ represents the separatrices of the unstable fixed point $A_0(0,0)$.

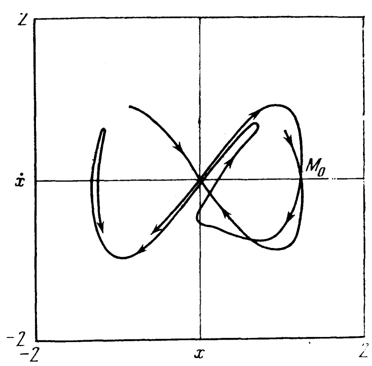

FIG. 2. The separatrices of the unstable fixed point $A_0(0,0)$ in the section $y = 0, \dot{y} > 0$, for the system (2.2) at $\varepsilon = 1$ and $E = 1$. It can be seen that the separatrices intersect transversally at the point M_0.

Since the equations (2.1) and the integral (2.2) are invariant with respect to the substitutions $t \mapsto -t$ and $y \mapsto -y$, the stable and unstable separatrices are symmetric about the x axis. Similarly, on account of invariance with respect to the transformations $t \mapsto -t$ and $x \mapsto -x$ the separatrices are symmetric about the $\dot{x}$ axis.

Thus, for $\varepsilon \neq 0$ the separatrices do not coincide, but intersect at a nonvanishing angle (i.e., transversally) and do not close. We note that the separatrices intersect for arbitrary nonvanishing values of ε and that at $\varepsilon < 1$ the angle of the first intersection of the separatrices is small: $\varphi \sim \varepsilon$,[16] but in the subsequent intersections the angle can become arbitrarily large. For $\varepsilon \ll 1$ we would see by constructing the separatrices that they practically coincide, but that in subsequent intersections in a small neighborhood of the point A_0 the angle between the separatrices is no longer small and the oscillations of the separatrix along itself become arbitrarily large. The transversal intersection of the separatrices bears witness to the fact that at $\varepsilon \neq 0$ the system (2.2) does not have in the general case a first integral other than the Hamiltonian.[13] This does not, however, say that as $\varepsilon \to \infty$, i.e., when the system (2.2) goes over into (1.3), one will observe an intersection of the separatrices. A simple example is the system with the Hamiltonian

$$H = {}^1/_2[\dot{x}^2+\dot{y}^2+(x^2-1)^2+y^2+\varepsilon(x^2+y^2)^2],$$

which has obvious integrals and no intersection of separatrices in both limiting cases ($\varepsilon \to 0$ and $\varepsilon \to \infty$) and at $\varepsilon \neq 0$ the separatrices intersect with unity probability and the system is nonintegrable.

§3. INTERSECTION OF SEPARATRICES AND THE ABSENCE OF AN ADDITIONAL INTEGRAL

The phenomenon of intersection of separatrices was discovered by Poincaré[13] towards the end of the last century, during his investigation of the three-body problem. In the same paper he has shown that in the case of intersection of the separatrices there exists no additional integral. However, a rigorous proof appeared only recently in Ref. 17. Here we give a qualitative proof.[2)]

Figure 2 shows the picture of intersection of separatrices of the system (2.2) with $\varepsilon = 1$. Figure 3 was obtained from Fig. 2 by a change of scale, to increase clarity. Qualitatively the two pictures are equivalent.

We assume that the system has an additional integral I_1. Then M_0 is the "first" intersection point of the separatrices. By definition, a point belonging to a separatrix goes over into a point belonging, to the same separatrix (i.e., one and the same phase-space trajectory passes through these points). The point M_0 belongs to both separatrices, which means that under the Poincaré mapping it will go over into a point of intersection of the separatrices (in Fig. 3, e.g., this will be the point M_1). The point N_1 is mapped into the point N_2. It is obvious that in our case the whole region L_0 is mapped into L_1 the region L_1 is mapped into L_2, etc. In Fig. 3 we have $M_i \mapsto M_{i+1}$, $N_i \mapsto N_{i+1}$, $L_i \mapsto L_{i+1}$, and $R_i \mapsto R_{i+1}$, for arbitrary $i = ..., -2, -1, 0, 1, 2, ...$ and $t > 0$.

Since the system is Hamiltonian, the mapping is area-preserving, so that the areas of the regions are equal to each other. Thus, the length of the regions L_i (i.e., the length of the stable separatrix between the points M_i and N_i) will tend to zero as $t \to +\infty$, and the width (the length of the unstable separatrix between the same points) will tend to infinity.

If the system has an integral I_1 in addition to the Hamiltonian, it has a fixed value on each invariant curve, and to different invariant curves will correspond different values of the integral (see §2). Thus, the integral I_1 has a fixed value only on the curve. We denote by c_1 its value on the separatrices, i.e., $I_1 = c_1$. Then its value at the boundaries of an infinitely long and infinitely narrow region (L_k or R_k, where $k \gg 1$) is equal to the same constant.

Two possibilities arise here[19]: either the integral I_1 is equal to a constant in the whole region, i.e., $I_1 = c_1$, or the integral has a real singularity inside the region. If the second alternative is assumed, then what was said above about the mapping of regions implies that the integral has singularities in all the regions L_i and R_i.

It is known[20] that the intersection points of the separatrices form a Cantor set, and thus the regions between them have a quite complicated structure. An integral having real singularities on such a set is quite an exotic object. Therefore the existence of such an integral seems unlikely. If one assumes that the integral has no real singularities, then it is equal to a constant in the region L_k ($k \gg 1$) and in all the regions L_i and R_i. Thus, the fixed value of the integral

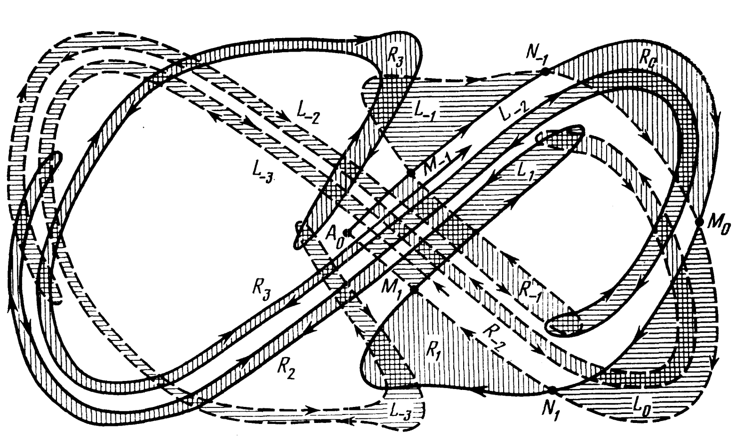

FIG. 3. The same as Fig. 2. For clarity it is not to scale. The solid line represents the unstable separatrix and the dashed line represents the stable separatrix. Only one branch of each separatrix is shown.

$I_1 = c_1$ extends from the curves into a region of nonvanishing area. However, the integral which has a fixed value in the whole region is just the Hamiltonian.

Thus, the fact of transversal intersection of separatrices implies the absence of an integral other than the Hamiltonian, and having no singularities on the real axis. This is the strongest of the known cases of nonintegrability: the Poincaré case. The transversal intersection of separatrices is responsible for the nonintegrability of the three-body problem of celestial mechanics.[13] In the same sense some particular classical equations obtained by means of the substitution (1.2) are nonintegrable, as will be shown below.

§4. INTERSECTION OF THE SEPARATRICES IN THE CASE OF THE YANG-MILLS EQUATIONS

The system of equations (1.2) has the obvious periodic solutions[8]

$$x=\pm y=\pm F, \tag{4.1}$$

where $F = \mathrm{cn}\,(t,1/\sqrt{2})$ is Jacobi's elliptic cosine.

We reason by analogy with §2. We linearize the system (1.3) near the solutions (4.1). Introducing the variables $x = F + \xi + \eta$ and $y = F + \xi - \eta$ we obtain

$$\ddot{\xi}+3F^2\xi=0, \qquad \ddot{\eta}-F^2\eta=0. \tag{4.2}$$

It is obvious that the solutions (4.1) are unstable in the "transverse" direction. We construct numerically the monodromy matrix A^T for these solutions and determine its eigenvalues. A numerical calculation yields

$$\lambda_1=129.647014\ldots, \qquad \lambda_2=1/\lambda_1. \tag{4.3}$$

In the half-plane ($y = 0, \dot{y} > 0$) the fixed points corresponding to the solutions (4.1) are $A_1(0,1/\sqrt{2})$ and $A_2(0, -1/\sqrt{2})$.

The procedure for constructing the separatrices is described in §2. The separatrices are represented in Fig. 4. Under the transformations $t\mapsto -t$, $y\mapsto -y$, and $t\mapsto -t$, $x\mapsto -x$ the unstable separatrices become stable. It can be clearly seen that at the "first" intersection point $K_1(1.424932...,0)$ the separatrices intersect transversally and their angle of intersection is approximately 72°. If one intersection point exists it follows that there is an infinity of them. Figure 4 illustrates this fact clearly. The oscillating tongues of the trajectories are long, and with each oscillation their length increases. This can be seen from the figure, where the tongues execute in each rotation a large but finite number of rotations around the point (0,0), after which the separatrices squeeze closer to one another.

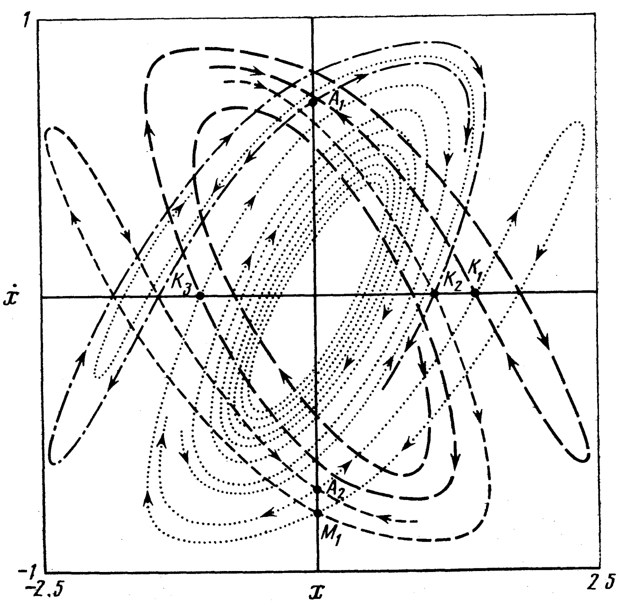

FIG. 4. The separatrices of the fixed points $A_1(0,1/\sqrt{2})$ and $A_2(0, -1/\sqrt{2})$ in the section $y = 0$, $\dot{y} > 0$ for the system (1.2): the unstable (dash-dot) and stable (long dashes) separatrices of the point A_1; the unstable (dotted) and stable (short dashes) separatrices of the point A_2.

We note that in distinction from the case considered in §2 the unstable separatrix of the point A_1 intersects both the stable separatrix of A_1 and the unstable separatrix of A_2. The intersection points of the separatrices of one fixed point are homoclynic (in Fig. 4 these are the points M_i), and those belonging to different fixed points are heteroclynic (the points K_i in Fig. 4).[13,18] The difference between these two kinds of points is the following. In the case when the separatrices close (Fig. 1) there is a doubly asymptotic trajectory passing through each point of the separatrix ("piercing" the Poincaré section). As $t\to +\infty$ it approaches the point A_1 intersecting the stable separatrix and winding itself monotonically to a periodic trajectory from one side. As $t\to -\infty$ it approaches the periodic trajectory from the other side, intersecting the unstable separatrix at the same point A_1. Doubly asymptotic trajectories exist not for all points of a separatrix when the separatrices intersect, but only for their intersection points. Through homoclynic points pass trajectories which are doubly asymptotic to the same periodic trajectory—homoclynic trajectory. Heteroclynic trajectories pass through heteroclynic points and are doubly asymptotic to two different trajectories.[13] This property of the homo- and heteroclynic trajectories can be used to verify whether or not the separatrices intersect, and to calculate the angle between them (see §5).

We note that the separatrices of an unstable periodic trajectory are two-dimensional surfaces on the three-dimensional hypersurface $E = \mathrm{const}$. The doubly periodic solutions are situated on these separatrices. The homo- and heteroclynic trajectories are the lines of intersection of a stable and an unstable separatrix, respectively. The separatrices in the Poincaré section are the intersections of the phase-space separatrices with a half-plane. The homo- and heteroclynic points are the intersections of a half-plane with the corresponding trajectories.

§5. DETERMINATION OF THE HOMO- AND HETEROCLYNIC TRAJECTORIES

One can use of the already mentioned symmetry of the equations (1.2) and Hamiltonian (1.3) to determine the homo- and heteroclynic trajectories.

Under the substitution $t\mapsto -t$, $x\mapsto -x$ a point of the phase space $(x,\dot{x},y,\dot{y})$ with coordinates $M(0,\dot{x},0,\dot{y})$ goes over into itself. Under this substitution the unstable separatrix goes into the related stable separatrix. If there exists a point $M(0,a,0,\dot{y})$ situated on the stable separatrix, then the separatrices intersect at this point. Thus, to determine the intersection point of the separatrices it suffices to determine a value $\dot{x} = a$ such that a doubly asymptotic trajectory passes through it as $t \to \pm \infty$ approaches a periodic solution. Here, as in the preceding constructions, the knowledge of the monodromy matrix allows one to monitor the accuracy of the computations. If the separatrices intersect at M under a nonzero angle, then the point M will be homoclynic. The points $K(x,0,0,\dot{y})$ are invariant with respect to the transformations $t \mapsto -t$ and $y \mapsto -y$. Since under this transformation the unstable separatrix of the solution (4.1, +) goes into the stable separatrix of the solution (4.1, −), the intersection of separatrices gives rise to heteroclynic solutions. In the table are listed the coordinates of the points K_i and M_i and the angles of intersection of the separatrices in a plane perpendicular to the trajectory.

In the vicinity of doubly asymptotic trajectories, singly asymptotic trajectories (i.e., trajectories which are asymptotic to a closed trajectory only as $t \to +\infty$ on the stable separatrix and as $t \to -\infty$ on the unstable separatrix) are situated on the stable and unstable separatrices. Knowing them allows one to determine the angle between the separatrices. These trajectories intersect the plane perpendicular to the doubly asymptotic trajectory in the points Z_1 and Z_2. The angle between the straight lines passing through these points and through the point of intersection of the doubly asymptotic trajectory with the plane orthogonal to it is the angle we are looking for (accurate to within a small neighborhood).

We have determined in a similar manner the angles of intersection of the separatrices for the system of equations

$$\ddot{x}+x(y^2+z^2)=0, \quad \ddot{y}+y(x^2+z^2)=0, \quad \ddot{z}+z(x^2+y^2)=0, \tag{5.1}$$

obtained by the same substitution (1.2) for $z = f^3$ (see Refs. 8,10,11). The integral of the system (5.1) has the form

$$H={}^1/_2(\dot{x}^2+\dot{y}^2+\dot{z}^2+x^2y^2+x^2z^2+y^2z^2). \tag{5.2}$$

The variational equations of the trajectories

$$x=\pm y=\pm F, \quad z=0, \quad F=\mathrm{cn}(t, 1/\sqrt{2}) \tag{5.3}$$

are

$$\ddot{\xi}+3F^2\xi=0, \quad \ddot{\eta}-F^2\eta=0, \quad \ddot{\zeta}+2F^2\zeta=0 \tag{5.4}$$

(here we have utilized the substitutions $x = F + \xi + \eta$ and $y = F + \xi - \eta$, $z = \zeta$). It is clear that ξ, η, and ζ in (5.4) are independent of each other. Therfore the monodromy matrix (a square 6×6 matrix) decomposes into three 2×2 matrices. The eigenvalues along the trajectory are equal to unity, as usual. The eigenvalues in the transverse direction in the $z = 0$ plane are the same as in the two-dimensional case (4.3). The other two eigenvalues are also real and have the values

$$\lambda_1=5.75158\ldots, \quad \lambda_2=1/\lambda_1. \tag{5.5}$$

Thus, along the direction z there is also instability of the trajectories. In this case the separatrices of the periodic solutions (5.3) are three-dimensional manifolds on the five-dimensional hyperplane $E =$ const. The intersection of separatrices is now determined by two angles, and at the point K_1

TABLE I. The angles of intersection of the separatrices of the system (1.2) at the homoclynic points M_i and at the heteroclynic points K_i; $y = 0$, $\dot{y} = (1 - \dot{x}^2)^{1/2}$.

	x	$\dot{x}$	φ
K_1	1.4249238	0.0	72°
K_2	1.0952220	0.0	86°
K_3	−0.9749564	0.0	82°
M_1	0.0	−0.8211368	135°
M_2	0.0	0.7901436	124°

(see Fig. 4) the angles are 72° and 106°. The fact that the intersection of separatrices is transversal is a proof of the absence of two additional integrals of the motion for the system (5.1). Thus, the system (5.1) has only one integral, the energy.

§6. CONCLUSIONS

The observed transversality of intersection of separatrices of the unstable periodic trajectories (4.1) proves the absence of one additional integral of the motion for the system (1.3) and of two additional integrals for the system (5.1). This confirms the conjecture[10] that the system (1.3) is stochastic and the conclusion[11] that the system (5.1) is stochastic.

In the present paper it was shown that there are no real-analytic integrals; this is the strongest case of nonintegrability, that in the sense of Poincaré. This result does not exclude the possibility that some special cases of the Yang-Mills equations may turn out to be completely integrable (see, e.g., Ref. 2), although a transition from them to a larger number of degrees of freedom will yield a nonintegrable system.

From the absence of an additional integral and the presence of intersection of the separatrices follows immediately the complicated behavior of the trajectories, i.e., stochasticity. However, this subject goes beyond the framework of the present paper, since it requires a more detailed consideration.

The method of numerical calculations described in the present paper rests on a knowledge of the characteristics of the trajectories (Lyapunov exponents, eigenvectors of the monodromy matrix, the period) and allows one to derive results with any arbitrarily prescribed and controllable accuracy. This indicates the high reliability of the results obtained.

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[1)]From the form (2.7) of the matrix A^T it is easy to see the following. When this matrix acts upon a point with coordinates $(\xi_1, \sqrt{2}\xi_1)$ of the straight line (2.9, +) this point goes over into the point $(\xi_2, \sqrt{2}\xi_2)$, of the same line, with $\xi_2 > \xi_1$. Thus, along the line (2.9, +) the points "move away" from the fixed point A_0. This is why one says that the separatrix with the asymptote (2.9, +) is unstable. Conversely, for the points of the line (2.9, −) $\xi_2 < \xi_1$, and this straight line defines a stable separatrix of the fixed point A_0.

[2)]A proof of this kind was communicated to us by Ya. G. Sinaĭ.

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