Effect of laser field on beta decays of nuclei

A. I. Nikishov and V. I. Ritus

P. N. Lebedev Physics Institute, USSR Academy of Sciences
(Submitted 7 February 1983)
Zh. Eksp. Teor. Fiz. **85**, 24–40 (July 1983)

We consider the effect of a laser field on the beta decay of nuclei when the field intensity F is low compared with the characteristic field F_0 determined by the decay energy I and the electron mass, $F \ll F_0 = 2I(2mI)^{1/2}/e\hbar$ and the field frequency satisfies the condition $\hbar\omega \lesssim I$. The decay probability is an asymptotic expansion in powers of $(F/F_0)^2$ and in small terms non-analytic at $F = 0$, similar to the expansion for the constant field, with coefficients that depend on the frequency like $a + b(/I)^2 + \ldots$, where $a,b \sim 1$. The principal probability correction terms, being analytic in the charge, are determined by perturbation theory. The obtained probability structure is characteristic of processes that take place also without a field, so long as the field frequency is low compared with the characteristic kinetic energy of the light charged particles and the field intensity is insufficient to change this energy significantly in the course of evolution of the process.

PACS numbers: 23.40.Bw

1. INTRODUCTION

Becker, Louisell, McCullen, Schlichter, and Scully[1–3] have recently found by numerical calculation that the β-decay probability of tritium is increased 10^2–10^4 times by a laser field if the parameter $\nu = ea/m = eF/m\omega$, which characterizes the amplitude F the field strength, varies in the range $0.3 < \nu < 1.0$, and the field frequency[1)] is $\omega = 1$ eV. These results drastically contradict our own results,[4–7] according to which, if the laser field F is weak compared with the characteristic field $F_0 = 2I(2mI)^{1/2}/e$ [I is the decay energy, see Eq. (8)], i.e., if the parameter

$$\chi = F/F_0, \quad F_0 = 2I(2mI)^{1/2}/e, \tag{1}$$

is small, $\chi \ll 1$, and the process can be regarded as multiphoton, i.e., the parameter

$$\xi = \frac{ea}{(2mI)^{1/2}} = \frac{eF}{\omega(2mI)^{1/2}} = \frac{2I}{\omega}\chi \tag{2}$$

is large, $\xi \gg 1$, the relative change of the decay probability should be determined by the square of the parameter χ, i.e., $W - W_0 \sim \chi^2 W_0$, where W_0 and W are the decay probabilities in vacuum and in a field. Under the conditions considered in Refs. 1–3, the parameter $\chi \equiv \nu(m/2I)^{1/2}(\omega/2I)$ is very small, in the range $10^{-4} < \chi < 10^{-3}$, and the parameter $\xi \equiv (m/2I)^{1/2}\nu \gtrsim 1$, or more accurately $1 < \xi < 4$. In this range of the parameters the decay probability should not differ qualitatively from the probability for the thoroughly investigated case $\xi \gg 1$.

The following qualitative arguments can also be advanced in favor of the expected decay-probability change induced by the field; these arguments clarify also the meaning of the parameters χ and ξ. The decay-formation length in vacuum is of the order of the de Broglie wavelength of the electron

$$l_f \sim 1/p \sim 1/(2mI)^{1/2}. \tag{3}$$

The field alters the decay probability substantially as its work over the length of decay formation is comparable with or larger than the decay energy I:

$$eF/(2mI)^{1/2} \gtrsim I, \tag{4}$$

i.e., if the parameter $\chi \gtrsim 1$. At $\chi \ll 1$ the field has a weak influence on the decay.

The arguments above pertain to a constant field. Qualitatively, however, they remain valid also for a slowly varying field, i.e., a field whose characteristic variation time is long compared with the time[2)] $(2mI)^{1/2}/eF$, during which the field imparts to the electron the characteristic decay energy I:

$$1/\omega \gtrsim (2mI)^{1/2}/eF. \tag{5}$$

This condition coincides with $\xi \gtrsim 1$. Interaction with a slowly varying field is multiphoton, for at $\xi \gtrsim 1$ the work of the field over the formation length exceeds the field-quantum energy, $eF\hbar(2mI)^{-1/2} \gtrsim \hbar\omega$.

Tacitly implied in the condition (5) is that $\chi \sim 1$. If the field is weak, i.e., $\chi \ll 1$, calculation[6,7] shows that the change of the decay probability by a constant field is $\Delta W \sim \chi^2 W_0$ and corresponds to a change $\Delta I \sim \chi^2 I$ in the decay energy. It is clear therefore that a weak field can be regarded as slowly varying under a condition that differs from (5) in the I is replaced by $\chi^2 I$, i.e., under the condition $\xi \gtrsim \chi$ or $\omega \lesssim I$.

The effect of an alternating field on the decay should therefore be weak at $\chi \ll 1$ and $\xi > \chi$. Actually, in the present paper we obtain for the decay probability in the case $\chi \ll 1, \xi \gtrsim \chi$ the expression

$$W = W_0[1 + \chi^2 c_2 + \chi^4 c_4(\xi) + \ldots], \tag{6}$$

where W_0 is the decay probability in vacuum, the coefficient c_2 does not depend on ξ, while the coefficient $c_4(\xi) = c_4 + c_{42}\xi^{-2}$ does not take into account the dependence of the probability on the field frequency. This result agrees with our papers,[4–7] in which we obtained for the probabilities $W(\chi,\xi)$ general expressions, their limiting expressions $W(\chi,\infty)$ as $\chi \to \infty$ which are valid for a constant field, and expansions for the latter in powers of the small χ in the form (6) with constant coefficients $c_2, c_4, \ldots$.

It was shown in Ref. 6 that the sensitivity of the decay to the external field is enhanced with decreasing decay energy I, because the characteristic field F_0 is decreased in this case. We consider therefore in the present paper β decays with energy released energy small compared with the electron mass, $I \ll m$, similar to the tritium decay $T \to He^3 + e^- + \bar{\nu}$, and obtain an expansion of the decay probability $W(\chi, \xi)$ in powers of $\chi \ll 1$ at arbitrary $\xi \gtrsim \chi$. The main conclusion is that a noticeable change of the decay probability calls for fields $F \sim F_0$. For tritium decay we have $F_0 \approx 2.4 \times 10^{11}$ Heaviside units, i.e., F_0 is quite large compared with the presently attainable fields, so that the tritium β-decay probability is little enhanced by present-day laser fields. Naturally, under the conditions of Refs. 1–3 there should be no noticeable change of the probability; see also Refs. 8–11.

Our principal result is the derivation of an analytic expression for the decay probability when the field is weak, $\chi \ll 1$, and its frequency satisfies the condition $\xi \gtrsim \chi$ or $\omega \lesssim I$, and demonstration of the fact that the principal correction terms coincide with the result of perturbation theory with respect charge, and have an unexpectedly wide range of applicability, including the constant-field limit.

2. PROBABILITY OF β DECAY IN A LASER FIELD

Tritium β decay in the field of a circularly polarized wave can be represented in the form

$$W = \sum_{s>s_0}^{\infty} \int \frac{d^3q}{q_0} CL(s,x,y),$$

$$L(s,x,y) = y^2 \left[\left(1 + \frac{I}{m} - y\right) J_s^2(z) + \xi^2 \frac{I}{mxz} \frac{d}{dz} (zJ_s(z) J_s'(z)) \pm \xi^2 \frac{2I(2s_m - s)}{mxz} J_s(z) J_s'(z) \right], \tag{7}$$

$$C = \frac{g^2}{8\pi^4} \frac{1}{2J+1} \sum_M (|C_V|^2 |M_F|^2 + |C_A|^2 |M_{GT}|^2).$$

It is assumed that the masses M_i and M_f of the initial and nuclei are so large compared with the electron mass m and the β-decay energy I,

$$M_i,\ M_f \gg m,\ I; \quad I = M_i - M_f - m, \tag{8}$$

that the nuclei can be regarded as being at rest.

If q_α and l_α are the electron 4-quasi-momentum and the neutrino 4-momentum, then the variables x and y used in (7), in the coordinate system in which the nuclei are at rest, are equal to

$$x = \frac{q_-}{m} \equiv \frac{q_0 - q_3}{m}, \quad y = \frac{l_0}{m}, \tag{9}$$

i.e., they are equal to the "minus" component of the electron momentum, which is conserved in the field, and to the neutron energy in units of m. These variables are convenient because the differential probability L has a simple dependence on them and because of the simplicity of the invariant momentum volume

$$\frac{d^3q}{q_0} = \frac{dq_1 dq_2 dq_-}{q_-} = \frac{d\varphi dq_\perp^2 dq_-}{2q_-} = d\varphi dl_0 dq_-. \tag{10}$$

The relation $dl_0 = -dq_\perp^2/2q$ used in the last equality of (10) follows from the energy conservation law

$$s\omega + m + I = q_0 + l_0 \equiv \frac{m_*^2 + q_\perp^2 + q_-^2}{2q_-} + l_0, \tag{11}$$

$$m_*^2 = m^2 + e^2a^2$$

at fixed s and q_-. The argument of the Bessel functions is proportional to the transverse momentum of the electron and is expressed in terms of x, y, and s:

$$z = eaq_\perp/\omega q_- = [2A(s - s_m)]^{1/2}, \tag{12}$$

$$s_m(x,y) = \frac{1}{2} A \left(1 + \frac{b}{\xi^2}\right) \quad A = \frac{\xi^3}{\chi x},$$

$$B = \frac{m}{I}(x-1)^2 - x + \frac{m}{I} xy. \tag{13}$$

Here s_m is the minimum number of photons needed for decay with specified values q_- and l_0 in accord with the conservation law (11). Finally, $s_0 = (m_* - m - I)/\omega$ is the minimum number of photons necessary for decay in the field of the wave and corresponding to $l_0 = 0$ and $q_0 = m$. The $\pm$ signs in the function L correspond to the right- and left-hand polarizations of the wave. The differential distribution of L does not depend on the azimuthal angle φ of the momentum $\mathbf{q}$ because the wave is circularly polarized.

At fixed s and x the limits of integration with respect to y are determined from (11). Obviously, the lower limit is $y_1 = 0$ and the upper limit y_2 corresponds to $q_\perp = 0$, i.e.,

$$y_1(s,x) = 0, \quad y_2(s,x) = \frac{s\omega}{m} + 1 + \frac{I}{m} - \frac{x}{2} - \frac{m_*^2}{2m^2x}. \tag{14}$$

The limits $x_{1,2}(s)$ of the subsequent integration with respect to x are determined by those value of x for which $y_2(s, x) = y_1(s, x)$, i.e., by the zeros of the function $y_2(s, x)$:

$$x_{1,2}(s) = \frac{s\omega}{m} + 1 + \frac{I}{m} \pm \left[\left(\frac{s\omega}{m} + 1 + \frac{I}{m} \right)^2 - \frac{m_*^2}{m^2} \right]^{1/2}. \tag{15}$$

3. ROUGH ESTIMATE OF THE CHANGE OF THE PROBABILITY BY DECAY BY THE FIELD

We show now with the aid of rough semiquantitative calculations that in the multiphoton case, when

$$\xi^3/\chi \gg 1, \quad \xi^2/\chi \gg 1, \tag{16}$$

the field depends little on the decay, i.e., W differs little from W_0. It is convenient to interchange the order of summation and integration in (7). Then

$$W = 2\pi m^2 \sum_{s>s_0} \int_{x_1(s)}^{x_2(s)} dx \int_0^{y_2(s,x)} dy CL = 2\pi m^2 \int_0^\infty dx \int_0^\infty dy \sum_{s>s_m(x,y)} CL, \tag{17}$$

i.e., the limits of integration with respect to x and y become much simpler. We separate from the integration region $(x \geqslant 0, y \geqslant 0)$ the region of free decay, i.e., of the decay without a field. Its boundary on the x,y plane is determined by curves (14) and (15) at $s = o$ and $m_* = m$. It is easy to verify that

$B<0$ inside the free-decay region, and $B>0$ outside it, i.e., in the induced-decay region. If x and y lie inside the free-decay region the main contribution to the sum over s, as will be shown, is made by the terms from the interval

$$s_1<s<s_2, \quad s_{1,2}=A\,(1\mp(-B)^{1/2}/\xi), \tag{18}$$

whose boundaries are roots of the equation $z^2(s)-s^2=0$. Inside this interval $z(s)>s$, and for most terms

$$z^2(s)-s^2\gg 1. \tag{19}$$

Since $A\gg 1$ by virtue of the condition (16). When the condition (19) is satisfied, the square of the Bessel function can be replaced by the expression

$$J_s^2(z)\approx\frac{1+\cos 2\psi}{\pi(z^2-s^2)^{1/2}}, \quad \psi=(z^2-s^2)^{1/2}-s\arccos\frac{s}{z}-\frac{\pi}{4}, \tag{20}$$

see, e.g., Eq. 9.2.15 of Ref. 12. Then, neglecting the oscillating term in (20) and the violation of condition (19) near the boundaries of the interval (18), we obtain

$$\sum_{s=s_1}^{s_2} J_s^2(z)\approx\int_{s_1}^{s_2}\frac{ds}{\pi(z^2-s^2)^{1/2}}=1. \tag{21}$$

Outside the interval (18) and in the vicinity of its boundaries the Bessel function can be approximated by the Airy function $\Phi(u)$

$$J_s^2(z)\approx\frac{1}{\pi^2}\left(\frac{2}{s}\right)^{2/3}\Phi^2(u), \quad u=\left(\frac{s}{2}\right)^{2/3}w^2, \quad w^2=1-\frac{z^2}{s^2}\ll 1. \tag{22}$$

Outside the interval (18) we have $u(s)>0$ and the Airy function decreases exponentially. It is then easy to verify that under the condition (16) the contribution made to the sum over s by the section outside the interval (18) is small.

We thus have for x and y outside the free-decay region

$$\sum_{s>s_m(x,y)}^{\infty} J_s^2(z)\approx 1 \tag{23}$$

and therefore the sum over s of the first term of the function CL differs little from the differential probability of the free decay. This simple result is connected with the successful choice of x and y as variables.

It is clear that the sums of the second and third terms of the function CL over s are small compared with the sum of the first. In fact, the second term, which contains $z^{-1}(zJ_sJ_s')'=\frac{1}{2}(J_{s+1}^2+J_{s-1}^2)-J_s^2$, is small because at large s a change of the index by ± 1 changes $J_s(z)$ little. The third polarization term is approximated in the region (19) by the expression

$$J_s(z)J_s'(z)\approx-\sin 2\psi/\pi z, \tag{24}$$

which oscillates rapidly with change of s and therefore the sum of this term over s is small.

Thus, the differential probability summed over s is altered little by the field if x and y are in the free-decay region. A significant change of the total decay probability can be caused only by an appreciable probability of appearance of x and y in the induced-decay field, i.e., in a region inaccessible without a field, and this, as we shall show, calls for a strong field.

We consider now that outer region of x and y for which $B>0$. We can use here again for the Bessel function the approximation (22), in which now the argument $u(s)$ of the Airy function is positive for any s from the interval $s_m(x,y)<s<\infty$ and reaches a minimum at

$$s=s_c=\frac{A}{2}(3K-1)=A\left(1+\frac{2B}{3\xi^2}+\dots\right),\ K=\left(1+\frac{8B}{9\xi^2}\right)^{1/2}. \tag{25}$$

The second root of the equation $u'(s)=0$, which differs from (25) in the sign of K, is smaller than s_m, i.e., it lies in the unphysical region. We expand $u(s)$ near the minimum:

$$u(s)=u_c+\frac{1}{2}u_c''(s-s_c)^2+\frac{1}{6}u_c'''(s-s_c)^3+\dots. \tag{26}$$

Here $u_c=u(s_c)$ etc., with

$$u_c=\frac{9}{2}\left(\frac{A}{4}\right)^{2/3}(K-1)(3K-1)^{-1/3}=\frac{B}{(2\chi x)^{2/3}}\left(1-\frac{4B}{3\xi^2}+\dots\right), \tag{27}$$

$$u_c''=\left(\frac{4}{A}\right)^{4/3}K(3K-1)^{-7/3}=2^{1/3}A^{-4/3}\left(1-\frac{10B}{9\xi^2}+\dots\right), \tag{28}$$

$$u_c'''\approx\left(\frac{2}{A}\right)^{7/3}. \tag{29}$$

If $B\xi^{-2}\ll 1$ and $A\gg 1$, it can be seen from (26)–(29) that the significant values do satisfy the condition

$$|s-s_c|\lesssim A^{2/3}\ll s_c. \tag{30}$$

In this case

$$u_c'''(s-s_c)^3_{\rm eff}\sim A^{-1/3}\ll 1$$

and the function $u(s)$ is sufficiently well approximated by two terms of the expansion (26). Since the effective interval of values of s lies far from the lower limit of integration with respect to s:

$$s_c-s_m\sim A\gg|s-s_c|_{\rm eff}\sim A^{2/3},$$

we have

$$\int_{s_m}^{\infty}ds J_s^2(z)\approx 2\int_{s_c}^{\infty}ds\frac{1}{\pi^2}\left(\frac{2}{s_c}\right)^{2/3}\Phi^2\left(u_c+\frac{1}{2}u_c''(s-s_c)^2\right)$$

$$=\frac{(2/s_c)^{2/3}}{\pi(2u_c'')^{1/2}}\Phi_1(2^{2/3}u_c)\approx\frac{1}{\pi}\Phi_1(2^{2/3}u_c). \tag{31}$$

The resultant function

$$\Phi_1(x)=\int_x^{\infty}dt\Phi(t) \tag{32}$$

is equal to $\pi/3$ at $x=0$ and decreases exponentially at $x\gg 1$, see Ref. 12. The contribution of the outer region will then be noticeable only if $u_c\lesssim 1$ in a sufficiently large range of values of x and y.

For a weak field, when $\chi\ll 1$, the condition $u_c\lesssim 1$ is equivalent to $B\lesssim\chi^{2/3}\ll 1$ and can be satisfied only in a narrow range of x and y near the free-decay region; consequently the change of the total decay probability is small.

Finally, for $\chi \gg 1$ it follows from the condition $u_c \ll 1$ that the probability is substantial in the region where

$$x^2 \sim y^2 \sim eF/m^2 \quad \text{or} \quad q_-^2 \sim l_0^2 \sim eF, \tag{33}$$

i.e., the characteristic values of q_- and l_0 are determined by the field and do not depend on I and M. The condition $B\xi^{-2} \gtrsim 1$ that the probability be dependent on the frequency is then equivalent to the condition

$$\omega \gtrsim ea \quad \text{or} \quad \omega^2 \gtrsim eF \tag{34}$$

and likewise does not contain the parameters I and m. The independence of the probabilities of the processes in a strong field on the mass dimensionality parameters was noted repeatedly[13] and was named gauge invariance. It is connected with the fact that in a very strong field the process evolves not over a length determined by the mass parameters, see (3), but over a shorter length $(\hbar c/eF)^{1/2}$ determined by the field itself.

4. PROBABILITY OF β DECAY IN A WEAK FIELD AT $\xi > 1$

We turn now to a more detailed consideration of the probability in a weak field. The function W depends in a nontrivial manner on the parameters x and ξ and on the ratio I/m. For the reasons given in the Introduction, we consider the case of nonrelativistic decay, when $I/m \ll 1$. This means that in the region of decay formation the electron moves nonrelativistically, although on leaving this region it can become relativistic if $\xi \gtrsim (m/I)^{1/2} \gg 1$. In fact during the time of decay formation

$$t_f \sim \frac{l_f}{v} \sim \frac{m}{(2mI)^{1/2}} \frac{1}{(2mI)^{1/2}} \sim \frac{1}{I} \tag{35}$$

the electron acquires an energy $\sim I$ via the decay mechanism and via the field an energy $\sim eF(2mI)^{-1/2} = 2\chi I$ that is small compared with I if $\chi \ll 1$. Subsequently, during the time of one period the electron acquires from the field an energy $ea = \xi (2mI)^{1/2}$ that can be larger than m if $\xi \gtrsim (m/I)^{1/2}$. This energy of vibrational motion in the wave is contained in the effective mass $m_* = (m^2 + e^2a^2)^{1/2}$. Since the ratio of formation time to the field period is

$$t_f/T \sim \omega/2I = \chi/\xi \tag{36}$$

and is very small under the conditions considered by us ($\chi \ll 1, \xi \gtrsim 1$) the field does not manage to impart to the electron during the decay-formation time the energy that it acquires subsequently.

It is clear from the foregoing that if $I/m \ll 1$ the kinetic energy of the electron and the neutrino energy will be close to I, so that the effective values of the integration variables in (7) will be

$$(x-1)_{\text{eff}} \sim (I/m)^{1/2} \ll 1, \quad y_{\text{eff}} \sim I/m \ll 1. \tag{37}$$

We shall see also that a large number of photons will be absorbed in an interval of width Δs centered about $s = s_c$

$$s_c = \xi^3/\chi, \quad \Delta s \sim \xi^2/\chi. \tag{38}$$

Using this qualitative information, we can simplify the function L significantly

$$L^{\text{nonrel}} = y^2 J_s^2(z), \tag{39}$$

and can assume in the equations for z and s_m that

$$A = \frac{\xi^3}{\chi}, \quad B = \frac{m}{2I}(x-1)^2 - 1 + \frac{m}{I} y. \tag{40}$$

Finally, in the considered approximation

$$s_0 = \frac{I}{\omega}(\xi^2 - 1). \tag{41}$$

Thus, introducing the constant $f = 2\pi C m^2$, we obtain

$$W = f \sum_{s > s_0} \int dy y^2 \int dx J_s^2(z). \tag{42}$$

We replace initially the Bessel function with large index by the Langer approximation

$$J_s(z) \approx \frac{1}{\pi} \left(\frac{4u}{s^2 - z^2} \right)^{1/4} \Phi(u) \qquad \Phi(u) = \int_0^\infty dx \cos\left(ux + \frac{x^3}{3} \right), \tag{43}$$

which is the first term of an asymptotic expansion that is uniform in $z > 0$, see Eq. 9.3.35 in Ref. 12. Here $\Phi(u)$ is an Airy function of argument

$$u = \left(\frac{s}{2} \right)^{2/3} \frac{w^2}{k(w^2)}, \quad w^2 = 1 - \frac{z^2}{s^2}, \quad k(w^2) = \left(\frac{w^3/3}{\text{Arth } w - w} \right)^{2/3}. \tag{44}$$

It can be seen that the sign of u coincides with the sign of w^2, since $k(w^2) > 0$ for real w_2 in the interval $-\infty < w_2 < 1$. Since the integrand varies slowly with changing s, the sum is determined by a large number of terms and can be replaced by an integral. If due account is taken of the boundary contribution, see the Euler-Maclaurin formula on p. 26 of Ref. 12, the error due to such a substitution is exponentially small, in our case of order $e^{-\Delta s}$, where Δs is the effective number of terms that determine the sum. Thus,

$$W = \frac{f}{\pi^2} \int_{s_0}^\infty ds \int dy\, y^2 \int dx \left(\frac{4u}{s^2 - z^2} \right)^{1/2} \Phi^2(u). \tag{45}$$

We use now the relation from the theory of Airy functions[14]

$$\Phi^2(u) = 2^{-2/3} \int_{t_m}^\infty \frac{dt\Phi(t)}{(t - t_m)^{1/2}}, \quad t_m = 2^{2/3} u. \tag{46}$$

Changing the order of the integration, we can then represent the probability in the form

$$W = \frac{1}{\pi} \int_{t_0}^\infty dt \Phi(t) h(t), \tag{47}$$

$$h(t) = \frac{f}{\pi} \int_{s_1(t)}^{s_2(t)} ds \int_0^{y_2(t,s)} dy\, y^2 \int_{x_1(t,s,y)}^{x_2(t,s,y)} dx [t s^{4/3} k(w^2) - s^2 + z^2]^{-1/2}. \tag{48}$$

The limits of integration with respect to x, y, and s are obtained in the following manner. The equation $t_m(s,x,y) = t$ is first solved with respect to x; this yields the limits $d_{1,2}(t,s,y)$. We obtain next the minimum $t_m(s,y)$ of the function $t_m(s,x,y)$

with respect to x and solve the equation $t_m(s,y)=t$ for y, obtaining the upper limit $y_2(t,s)$. Since the function $t_m(s,y)$ is monotonic in y, the lower limit is $y_1=0$. We obtain next the minimum $t_m(s)$ of the function $t_m(s,y)$ with respect to y and solve the equation $t_m(s)=t$ for s, obtaining the limits $s_{1,2}(t)$. Finally, we obtain the minimum of the function $t_m(s)$ with respect to s and designate it by t_0.

The function $h(t)$ is thus an integral over a closed volume in the space x,y,s bounded by the plane $y=0$ and by a dome-shaped surface over it. The radicand of (48) is zero on this surface and positive inside the volume.

The function $t_m(s,x,y)$ is given by the relation

$$t_m(s,x,y)=[3s(\operatorname{Arth} w-w)]^{2/3},$$
$$w^2=\left(1-\frac{A}{s}\right)^2+\left(\frac{A}{s}\right)^2\frac{B}{\xi^2}, \tag{49}$$

where A and B are given in the general case by Eqs. (13) and in the nonrelativistic case by Eqs. (40). The absolute minimum t_0 is reached in the nonrelativistic case at $x=1, y=0$, and $s=s_c$, the latter determined by the transcendental equation $[\tan^2 v\equiv -w^2]$:

$$s_c=\frac{\xi^3}{\chi}\frac{\sin 2v}{2v},\quad \frac{\sin 2v}{2v}\left(2-\frac{\operatorname{tg} v}{v}\right)=1-\frac{1}{\xi^2}. \tag{50}$$

This minimum is equal to

$$t_0=-\left[\frac{3\xi^3}{2\chi}\left(\frac{\operatorname{tg} v}{v}-1\right)\sin 2v\right]^{2/3}. \tag{51}$$

The function $v(\xi)$ in (50) and (51) is monotonic, equal to $v(\xi)\approx\xi^{-1}\ll 1$ at $\xi\gg 1$ and to $v(1)=1.17$ at $\xi=1$ (the root of the equation $\tan v=2v$). Thus, the lower limit of integration in (25) is negative and has a large absolute value, $t_0\ll -1$.

The integral (47) with $t_0\ll -1$ can be calculated by the asymptotic formula obtained in Ref. 6 (see also Ref. 7, p. 100). We present here only the principal terms of this equation:

$$W=\frac{1}{\pi}\int_{t_0}^{\infty}dt\Phi(t)h(t)=h(0)+\frac{1}{3}h^{(3)}(0)+\ldots \tag{52}$$

We shall show that the first term $h(0)$ coincides with the probability W_0 of decay in vacuum, and $h^{(3)}(0)$ is the third derivative at zero and determines the correction of order χ^2 to W_0.

It is inconvenient to calculate the derivatives of $h(t)$ because of the complicated dependence of the integration limits on t. What is important, however, is that it is precisely at the integration limits $x=x_{1,2}(t,s,y)$ that the function in the radicand of (48) vanishes. If Eqs. (12) and (13) are used for z, the integral with respect to x in (48), later designated X, can be transformed into

$$X=\frac{\chi}{\xi^2}\int_{x_1}^{x_2}\left[t\frac{\xi^2}{A^2}s^{4/3}k(w^2)-\xi^2\left(\frac{s}{A}-1\right)^2-B\right]^{-1/2}x\,dx. \tag{53}$$

It can be seen from (13) that the last two terms in the radicand constitute a second-degree polynomial in x, which we designate $-\alpha(x-\beta)^2+\alpha\gamma$. Introducing now in place of x the variable u and the function $\varkappa(u)$:

$$u=\frac{x-\beta}{\gamma^{1/2}},\quad \varkappa(u)=\frac{\xi^2 s^{4/3}}{\alpha\gamma A^2}k(w^2), \tag{54}$$

we transform the integral into a sum of the two following integrals:

$$X=\frac{\chi}{\xi^2\alpha^{1/2}}(\beta J_0+\gamma^{1/2}J_1), \tag{55}$$

$$J_n=\int_{u_1}^{u_2}\frac{du\,u^n}{(1+t\varkappa(u)-u^2)^{1/2}},\quad n=0,1. \tag{56}$$

The limits $u_{1,2}$ still depend on t and are the roots of the equations $u=\mp[1+t\varkappa(u)]^{1/2}$. But if we introduce in place of u the variable v:

$$v=\frac{u}{(1+t\varkappa(u))^{1/2}}, \tag{57}$$

the limits of the integrals J_0 and J no longer depend on t:

$$J_n=\int_{-1}^{1}\frac{dv}{(1-v^2)^{1/2}}[v(1+t\varkappa)^{1/2}]^n\left(1+\frac{tv}{2(1+t\varkappa)}\frac{d\varkappa}{dv}\right),\quad n=0,1. \tag{58}$$

The integrands, however, have a complicated dependence on t because of the transcendental connection between u and v, see (57). Since we need the first three derivatives of the function $h(t)$ at $t=0$, it is natural to expand the integrals J_n in powers of t. It is easy to obtain for J_0 the expansion

$$J_0=\pi+\sum_{n=1}^{\infty}\frac{t^n}{n!}\int_{-1}^{1}\frac{dv\,v^{2n}}{(1-v^2)^{1/2}}(\varkappa^n(v))^{(n)}. \tag{59}$$

The symbol (n) denotes the nth derivative with respect to v^2. We call attention to the fact that now the integrals contain the functions $\varkappa(u)$ at the point $u=v$. The difference between u and v, determined by (57) has led to the appearance, under the integral signs, of derivatives of the function $\varkappa(u)$ at the point $u=v$.

The expansion for J_1 is of similar structure. We do not present it here since we do not need it in the nonrelativistic case ($I\ll m$). In this case

$$\beta=1,\quad \gamma^{1/2}=0,\quad \alpha=m/2I, \tag{60}$$

so that the integral X reduces to

$$X=\frac{\chi}{\xi^2}\left(\frac{2I}{m}\right)^{1/2}J_0, \tag{61}$$

and the functions $\varkappa(v)$ and w^2 take the form

$$\varkappa(v)=\frac{Q}{z\sigma^{2/3}}k(w^2),\quad w^2=-z(1-v^2), \tag{62}$$

where $Q=(\chi/\xi^2)^{2/3}$,

$$z=\frac{1-(\sigma-\xi)^2-\eta}{\sigma^2},\quad \sigma=\frac{\chi s}{\xi^2},\quad \eta=\frac{m}{I}y. \tag{63}$$

Thus, we introduce in place of s and y the variables σ and η, which have a more convenient scale of variation: $\sigma\sim 1+\xi$ and $\eta\sim 1$,

Substituting the function (62) in the expansion (62) we obtain

$$J_0=\pi\sum_{n=0}^{\infty}\left(\frac{tQ}{\sigma^{2/3}}\right)^n f_n(z),$$

$$f_n(z)=\frac{1}{\pi}\int_{-1}^{1}\frac{dv\,v^{2n}}{(1-v^2)^{1/2}}\frac{1}{n!}(k^n(w^2))^{(n)},\tag{64}$$

where the symbol (n) denotes now the nth derivative with respect to w^2. We call attention to the fact that $f_0(z)\equiv1$, and the remaining $f_n(z)$ have a nontrivial dependence on z, being finite at zero:

$$f_1(0)=-\frac{1}{5},\quad f_2(0)=-\frac{3}{700},\quad f_3(0)=\frac{2}{525},\ldots.\tag{65}$$

Using (61), (64), and the variables (63), we obtain for the function $h(t)$ the representation

$$h(t)=N\sum_{n=0}t^nF_n(t),\quad N=\frac{1}{8}f\left(\frac{2I}{m}\right)^{1/2},\tag{66}$$

in which

$$F_n(t)=\int_{\sigma_1(t)}^{\sigma_2(t)}d\sigma\varphi_n(t,\sigma),\quad \varphi_n(t,\sigma)=\left(\frac{Q}{\sigma^{2/3}}\right)^n\int_0^{\eta_2(t,\sigma)}d\eta\,\eta^2f_n(z).\tag{67}$$

The upper limit $\eta_2(t,\sigma)$ of the integral with respect to η is obtained from the equation

$$t_m(s,x=1,y)\equiv[3s(\operatorname{Arth}w-w)]^{2/3}_{x=1}=t,\tag{68}$$

if the latter is solved with respect to

$$y=y_2(t,s)\equiv(I/m)\eta_2(t,\sigma).$$

To determine $h(0)$ and $h^{(3)}(0)$ it suffices to know $\eta_2(t,\sigma)$ in the form of four terms of the expansion in t:

$$\eta_2(t,\sigma)=1-(\sigma-\xi)^2+tQ\sigma^{-2/3}\sigma^2-\frac{2}{5}(tQ\sigma^{-2/3})^2\sigma^2+\frac{13}{175}(tQ\sigma^{-2/3})^3\sigma^2+\ldots\tag{69}$$

Similarly, the limits $\sigma_{1,2}(t)$ of the integral with respect to σ are obtained from the equation

$$t_m(s,x=1,y=0)\equiv[3s(\operatorname{Arth}w-w)]^{2/3}_{x=1,\,y=0}=t,\tag{70}$$

if the latter is solved with respect to $s=(\xi^2/\chi)\sigma(t)$. It suffices to know the obtained two solutions at $t=0$:

$$\sigma_{1,2}(0)=\xi\mp1.\tag{71}$$

It can be seen from (66) and (67) that

$$h(0)=NF_0(0)=N\int_{\xi-1}^{\xi+1}d\sigma\int_0^{1-(\sigma-\xi)^2}d\eta\,\eta^2=\frac{32}{105}N.\tag{72}$$

This expression coincides with the probability of decay in vacuum.

For $h^{(3)}(0)$ we obtain from (66)

$$h^{(3)}(0)=N[F_0'''(0)+3F_1''(0)+6F_2'(0)+6F_3(0)].\tag{73}$$

For the first three derivatives of the functions $F_n(t)$, the following expression is valid:

$$F_n^{(k)}(t)=\int_{\sigma_1(t)}^{\sigma_2(t)}d\sigma\left(\frac{\partial}{\partial t}\right)^k\varphi_n(t,\sigma),\quad k=0,1,2,3,\tag{74}$$

i.e., we can ignore in the differentiation the dependence of the limits $\sigma_{1,2}(t)$ on t, since the integrands $\varphi_n(t,\sigma)$ and their first two derivatives with respect to t vanish at the limits of the integration with respect to σ:

$$\left(\frac{\partial}{\partial t}\right)^k\varphi_n(t,\sigma)|_{\sigma=\sigma_{1,2}(t)}=0,\quad k=0,1,2.\tag{75}$$

This follows from the representation (67) for $\varphi_n(t,\sigma)$ and from the vanishing of the function $\eta_2(t,\sigma)$ at $\sigma=\sigma_{1,2}(t)$, because the upper limit of the integration with respect to η should coincide at the limits of integration with respect to η with the lower one, and the latter is identically equal to zero:

$$\eta_2(t,\sigma_{1,2}(t))=\eta_1(t,\sigma_{1,2}(t))=0.\tag{76}$$

As a result, the first three terms of $h^{(3)}(0)$ are determined completely by the function $\eta_2(t,\sigma)$ and its first three derivatives with respect to t at $t=0$ and by the values of $f_1(0), f_2(0)$, $f_1'(0)=\frac{3}{140}$. For example,

$$F_0'''(0)=\int_{\xi-1}^{\xi+1}d\sigma(2\eta_2'^3+6\eta_2\eta_2'\eta_2''+\eta_2^2\eta_2''')_{t=0}$$

$$=Q^3\left(4\xi^4+\frac{8}{5}\xi^2-\frac{4}{875}\right).\tag{77}$$

Similarly,

$$F_1''(0)=Q^3\left(-\frac{8}{15}\xi^2+\frac{36}{875}\right),\quad F_2'(0)=-Q^3\frac{4}{875}.\tag{78}$$

Thus,

$$F_0'''(0)+3F_1''(0)+6F_2'(0)=4\chi^2\left(1+\frac{4}{175\xi^4}\right).\tag{79}$$

It follows from the foregoing that this expression is determined completely by the behavior of the function $k(w^2)$ near $w^2=0$, or more accurately by the first three terms of the expansion of $k(w^2)$ in powers of w^2. Therefore expression (79) can be called differential. The last term $6F_3(0)$ in $h^{(3)}(0)$, which can be written in the form

$$F_3(0)=\frac{16\chi^2}{15\xi^4}\int_0^{z_m}dzf_3(z)\frac{(1-z/z_m)^{5/2}}{(z+1)^3},\quad z_m=\frac{1}{\xi^2-1},\tag{80}$$

is essentially integral: it is determined by the function $f_3(z)$ and hence by $k(w^2)$ in a wide range of their variables. Thus,

$$h^{(3)}(0)=4N\chi^2\left\{1+\frac{4}{175\xi^4}+\frac{8}{5\xi^4}I(z_m)\right\},\tag{81}$$

where $I(z_m)$ denotes the integral in (80). The function $I(z_m)$ tends linearly to zero as $z_m\to0$, and to a constant of the order of 10^{-2} as $z_m\to\infty$:

$$I(z_m)=\frac{4}{3675}z_m+\ldots,\quad z_m\ll1;\quad I(\infty)=\int_0^{\infty}\frac{dzf_3(z)}{(z+1)^3}.\tag{82}$$

We note that $f_3(z)$ is finite at zero, see (65), and behaves like $z^{-1/2}$ as $z\to\infty$. Thus, we obtain for $c_2(\xi)$ at (6)

$$c_2(\xi)=\frac{35}{8}\left\{1+\frac{4}{175\xi^4}+\frac{8}{5\xi^4}I(z_m)\right\}. \qquad (83)$$

It turns out that the Langer approximation is not sufficient for the determination of $c_2(\xi)$ at $\xi\sim 1$, and terms of next higher order in s^{-2} must be taken into account in the asymptotic expansion 9.3.35 of Ref. 12. We obtain then for the square of the Bessel function

$$J_s^2(z)=\frac{1}{\pi^2}\left(\frac{4u}{s^2-z^2}\right)^{1/2}\left\{\Phi^2(u)\left(1+\frac{2a_1}{s^2}+\dots\right)\right.$$
$$\left.+\frac{2\Phi(u)\Phi'(u)}{s^{4/3}}(2^{1/3}b_0+\dots)+\frac{\Phi'^2(u)}{s^{8/3}}(2^{2/3}b_0^2+\dots)\right\}. \qquad (84)$$

a_1 and b_0 in this equation are functions of w^2:

$$a_1(w^2)=w^{-6}\left[\frac{385}{1152}-\frac{77}{192}w^2\right.$$
$$\left.+\frac{9}{128}w^4+\frac{7}{576}(5-3w^2)k^{3/2}-\frac{455}{1152}k^3\right], \qquad (85)$$

$$b_0(w^2)=\frac{k^{1/2}}{24w^4}(5-3w^2-5k^{3/2}), \qquad (86)$$

which are numerically small in a wide range of w^2 and could be neglected if the function $c_2(\xi)$ obtained above were not close to its limiting value $c_2(\infty)$. As shown by the calculation above, the values of significance are $w^2\sim -1$, i.e., $u\sim -s^{2/3}$. In this region

$$\Phi^2(u)\sim s^{-1/3},\quad \Phi(u)\Phi'(u)\sim s^{-1},\quad \Phi'^2(u)\sim s^{1/3}. \qquad (87)$$

Therefore the terms containing a_1 and b_0 are of the order of $s^{-2}\sim\chi^2\xi^{-6}$ relative to the principal term, and contribute to the function $c_2(\xi)$, or more accurately to its deviation from $c_2(\infty)$, since these terms vanish as $\xi\to\infty$. The next terms of the expansion 9.3.35 of Ref. 12, marked by dots in (84), are of order s^{-2} relative to the smallest retained ones and are therefore significant only for a correction $\sim\chi^4$.

From (46) it is easy to find that

$$2\Phi(u)\Phi'(u)=\int_{t_m}^{\infty}\frac{dt\,\Phi'(t)}{(t-t_m)^{1/2}},\quad t_m=2^{2/3}u, \qquad (88)$$

$$\Phi'^2(u)=2^{-2/3}\int_{t_m}^{\infty}\frac{dt\Phi(t)}{(t-t_m)^{1/2}}(2^{1/3}t-u). \qquad (89)$$

The probability takes therefore the form

$$W=\frac{1}{\pi}\int_{t_0}^{\infty}dt\Phi(t)\,[h(t)+h_a(t)+h_{bb}(t)]+\frac{1}{\pi}\int_{t_0}^{\infty}dt\Phi'(t)\,h_b(t)$$
$$=\frac{1}{\pi}\int_{t_0}^{\infty}dt\Phi(t)\,[h(t)+h_a(t)+h_{bb}(t)-h_b'(t)]-\frac{1}{\pi}\Phi(t_0)\,h_b(t_0), \qquad (90)$$

where the functions $h_a(t)$, $h_b(t)$, $h_{bb}(t)$ differ from $h(t)$ by additional factors $2a_1s^{-2}$, $2b_0s^{-4/3}$, $(2t-2^{2/3}u)b_0^2s^{-8/3}$ under the triple-integral sign in (48). We therefore have now in place of (52)

$$W=h(0)+\frac{1}{3}h^{(3)}(0)+h_a(0)+h_{bb}(0)-h_b'(0)+\dots, \qquad (91)$$

where the dots denote terms $\sim\chi^4$ and higher, and also oscillating terms, one of which is also the last term of (90).

To calculate $h_a(t)$, $h_b(t)$, $h_{bb}(t)$ and their derivatives, a perfectly suitable method is the one described above for $h(t)$. In the representation (66) and (67) of these functions, the entire difference reduces to a modification of $\varphi_n(t,\sigma)$, namely, to replacement of the functions $f_n(z)$ by

$$\frac{2Q^3}{\sigma^2}f_n^a(z),\quad \frac{2Q^2}{\sigma^{4/3}}f_n^b(z),\quad \frac{Q^3}{\sigma^2}\left[\frac{2tQ}{\sigma^{2/3}}f_n^{bb}(z)-\tilde f_n^{bb}(z)\right], \qquad (92)$$

respectively, where

$$f_n^a(z)=\frac{1}{\pi}\int_{-1}^{1}\frac{dv\,v^{2n}}{(1-v^2)^{1/2}}\frac{1}{n!}(k^na_1)^{(n)}, \qquad (93)$$

$$\tilde f_n^{bb}(z)=\frac{1}{\pi}\int_{-1}^{1}\frac{dv\,v^{2n}}{(1-v^2)^{1/2}}\frac{1}{n!}\left(k^nb_0^2\frac{w^2}{k}\right)^{(n)}, \qquad (94)$$

and f_n^b and f_n^{bb} differ from f_n^a in that a_1 is replaced respectively by b_0 and b_0^2. Calculating now the additional terms in (91), we get

$$\frac{1}{3}h^{(3)}(0)+h_a(0)+h_{bb}(0)-h_b'(0)$$
$$=\frac{4}{3}N\chi^2\left\{1+\frac{4}{175\xi^4}-\frac{8}{5\xi^4}f_0^b(0)\right.$$
$$\left.+\frac{8}{5\xi^4}\int_0^{z_m}\frac{dz(1-z/z_m)^{5/2}}{(z+1)^3}\left[f_3(z)+f_0^a(z)-\frac{1}{2}\tilde f_0^{bb}(z)-f_1^b(z)\right]\right\}. \qquad (95)$$

Since $f_0^b(0)=-1/70$, the additional "differential" term is cancelled by the existing term $\sim\xi^{-4}$. The previously obtained integral term is also completely cancelled by the additional integrals, since the function

$$f_3(z)+f_0^a(z)-\frac{1}{2}\tilde f_0^{bb}(z)-f_1^b(z)$$
$$=\frac{1}{\pi}\int_{-1}^{1}\frac{dv}{(1-v^2)^{1/2}}\left\{v^6\frac{1}{6}(k^3)'''+a_1-\frac{1}{2}b_0^2\frac{w^2}{k}-v^2(kb_0)'\right\}, \qquad (96)$$

defined by an integral with respect to v turns out to be identically zero. This can be proved, e.g., in the following manner. As seen from Eqs. (44), (85), and (86) the functions $(k^3)'''$, $a_1-b_0^2w^2/2k$, $(kb_0)'$ are analytic in the complex w^2 plane with a cut along the semi-axis $1\leqslant w^2<\infty$ and are represented at $|w^2|<1$ by converging power series in w^2, whose coefficients are expressed linearly in terms of the coefficients of the series in $k^{3/2}$ and k^3. Assuming $w^2=-z(1-v^2)$ and integrating these series term-by-term with respect to v, as indicated in (96), we obtain for the individual functions in (96) the series

$$f_3(z)=\sum_{m=0}^{\infty}\frac{15}{48}\gamma_m\beta_{m+3}(-z)^m,$$

$$f_0^a(z)-\frac{1}{2}f_0^{bb}(z)=\sum_{m=0}^{\infty}\frac{1}{48}\gamma_m(5\alpha_{m+3}-3\alpha_{m+2}-20\beta_{m+3})(-z)^m, \tag{97}$$

$$f_1^b(z)=\sum_{m=0}^{\infty}\frac{1}{48}\gamma_m(5\alpha_{m+3}-3\alpha_{m+2}-5\beta_{m+3})(-z)^m,$$

where α_m and β_m are the coefficients of the series for $k^{3/2}$ and k^3, $\gamma_m=(2m-1)!!/(2m)!!$, and $\gamma_0=1$. We see therefore that for the summary function (96) we obtain a series with zero coefficients. Thus, the coefficient $c_2(\xi)$ does not depend on ξ and is equal to its value $c^2=35/8$ for a constant field.

We shall show in Sec. 6 that the independence of c_2 of ξ extends also into the region $1>\xi\gtrsim\chi\ll1$, and the coefficient $c_4(\xi)$ depends on ξ, so that the probabilities in an alternating and in a constant weak field differ in the terms $\sim\chi^4$. At the same time it is appropriate to present an exact expression for the decay probability in a constant field and its asymptotic expansion at $\chi\ll1$, which will be shown to have a wider range of applicability than initially assumed.

5. PROBABILITY OF β DECAY IN A CONSTANT FIELD

The decay probability in a constant field can be regarded as the limit of the decay probability in an alternating field as $\xi\to\infty$, and it therefore satisfies expression (47) in which $t_0=-\chi^{-2/3}$ and the function $h(t)$ is equal, according to (66) and (67), to

$$h(t)=NF_0(t),\quad F_0(t)=\int_{\sigma_1(t)}^{\sigma_2(t)}d\sigma\varphi_0(t,\sigma),$$
$$\varphi_0(t,\sigma)=\frac{1}{3}\eta_2^3(t,\sigma), \tag{98}$$

since the remaining $F_n(t)$ vanish because the significant values $w^2\sim\xi^{-2}\to0$, see (49), and $k(w^2)\to1$. Since in the limit as $\xi\to\infty$ we have

$$\eta_2(t,\sigma)=1+t\chi^{2/3}-(\sigma-\xi)^2,\quad \sigma_{1,2}(t)=\xi\mp(1+t\chi^{2/3})^{1/2}, \tag{99}$$

the function $h(t)$ can be easily obtained:

$$h(t)=\frac{32}{105}N(1+t\chi^{2/3})^{7/2}. \tag{100}$$

The obtained decay probability in a constant field can be expressed in terms of a product of Airy functions:

$$W=\frac{32N}{105\pi}\int_{-\chi^{-2/3}}^{\infty}dt\Phi(t)(1+t\chi^{2/3})^{7/2}$$
$$=\frac{32N}{105\pi}\left\{\frac{\Phi'^2-z\Phi^2}{(-z)^{1/2}}+9\chi^2(-z)^{1/2}\Phi^2-\chi\Phi\Phi'+\frac{15}{4}\chi^2\frac{\Phi'^2+z\Phi^2}{(-z)^{1/2}}\right\}, \tag{101}$$

Φ and Φ' depend on $z=-(2\chi)^{-2/3}$. To this end it is necessary to change over in the integral (101) to the integration variable $\tau=t+\chi^{-2/3}$ and use the formula

$$\int_0^{\infty}d\tau\,\tau^{n-1/2}\Phi(\tau+a)=\left(\frac{d^2}{da^2}-a\right)^n\int_0^{\infty}d\tau\,\tau^{-1/2}\Phi(\tau+a) \tag{102}$$

and Eq. (46)

For $\chi\ll1$ we obtain from (101)

$$W(\chi)=W_0\left(1+\frac{35}{8}\chi^2+\frac{35}{128}\chi^4+\dots+\frac{105}{16}\chi^4\sin\left(\frac{2}{3\chi}\right)+\dots\right). \tag{103}$$

This expression is the nonrelativistic limit of the general formula (60) obtained in Ref. 6 for the pion β-decay probability; this probability coincides with the probability of relativistic β decay of heavy nuclei if the parameters in this formula are taken to be

$$\gamma=m(m+I)^{-1},\quad \delta=(m+I)M_i^{-1},\quad \chi=eFM_i^{-2}.$$

In the case $\chi\gg1$ the decay probability is described by the function

$$W=W_0\frac{5\Gamma^2(^2/_3)}{24\pi2^{2/3}}(3\chi)^{1/3}, \tag{104}$$

which does not depend on I. The reason is that the decay is now formed over a length much smaller than $(2mI)^{-1/2}$. Equation (104) is valid so long as $1\ll\chi\ll(m/2I)^{3/2}$, and at $\chi\gtrsim(m/2I)^{3/2}$ it is necessary to take into account the relativistic structure of the interaction.

Equation (103) gives a clear idea of the decay probability also in the region $\xi\gtrsim1$, where according to the arguments of Secs. 1 and 3 the probability of the decay should differ from (103) only in that the dependences of the coefficients of the powers of χ^2 and of the oscillating terms on ξ are such that the order of magnitude of neither is changed. In Sec. 4 is described the regular method of determining these dependences. The principal coefficient c_2 turns out to be independent of ξ and, as will be shown below, remains such also in the region $\chi\lesssim\xi<1$, thus suggesting that the ξ dependences of the coefficients of the powers of χ^2 are universal in a region $\xi\gtrsim\chi$ wider than initially assumed.

6. DECAY PROBABILITY IN A WEAK FIELD AT $\xi<1$

The method described in Sec. 4, based on an asymptotic expansion of large-index Bessel functions and replacement of summation by integration can no longer be used when the minimum number of photons s_0 is close to unity or is negative, i.e., at $\xi-1\lesssim(\omega/2I)=\chi/\xi$, [see (41)]. Disregarding here the difficult case of ξ very close to unity, when $|\xi-1|\lesssim(\omega/I)\ll1$, we consider the region in which $\xi<1$ but $1-\xi\sim1$.[3)] In this case the minimum number of photons s_0 is negative and large in absolute value, and a method of calculating the probability can be proposed, in which expansion in the small parameter $\alpha=(-s_0)^{-1}$ is used without replacing the sum over s by an integral.

We represent the element of momentum volume (10) in the form

$$\frac{d^3q}{q_0}=q\sin\theta\,d\theta\,d\varphi\,dl_0, \tag{105}$$

where q and θ are the quasimomentum and emission angle of the electron. In the nonrelativistic case the momentum and the argument of the Bessel function are equal to

$$q=[2m(s\omega-s_0\omega-l_0)]^{1/2},\quad z=\frac{eaq}{m\omega}\sin\theta. \qquad (106)$$

Expressing $J_s^2(z)$ in (39) by the known power series, we integrate L^{nonrel} first with respect to the angle θ and then with respect to l_0 or y in the range from $y=0$ to $y=(s-s_0)\omega/m$. Then

$$\int L^{\text{nonrel}}q\sin\theta\,d\theta\,d\varphi\,dl_0=8\pi m^2\left(\frac{\omega}{ea}\right)^7 \times\sum_{n=0}^{\infty}\frac{(-1)^n x^{2k+7}}{n!(2k-n)!(2k+1)(2k+3)(2k+5)(2k+7)}, \qquad (107)$$

when $k=n+|s|$, and

$$x=\frac{ea}{\omega}\left[\frac{2\omega}{m}(s-s_0)\right]^{1/2}=\beta[|s_0|(s-s_0)]^{1/2},\quad \beta=\frac{2\xi}{(1-\xi^2)^{1/2}}. \qquad (108)$$

The decay probability is obtained by summing (107) over $s>s_0$. It is important to change in the resultant double series from summation over n to summation over $k=n+|s|$:

$$W=4f\left(\frac{\omega}{ea}\right)^7\sum_{k=0}^{\infty}\sum_{s=\max(-k,s_0)}^{k}\frac{(-1)^{k+s}x^{2k+7}}{(k+s)!(k-s)!(2k+1)(2k+3)(2k+5)(2k+7)}.$$

In the terms for which $0\leqslant k\leqslant -s_0$, we expand x^{2k+7} in powers of s:

$$x^{2k+7}=x_0^{2k+7}\sum_{r=0}^{\infty}\binom{k+7/2}{r}(\alpha s)^r,\quad \alpha=\frac{1}{|s_0|},\quad x_0=\beta|s_0|. \qquad (110)$$

Then

$$W=4f\left(\frac{\omega x_0}{ea}\right)^7\left\{\sum_{k=0}^{-s_0}\sum_{r=0}^{\infty}\binom{k+7/2}{r}\frac{x_0^{2k}\alpha^r C_r(k)}{(2k+1)(2k+3)(2k+5)(2k+7)}+x_0^{-7}\sum_{k>-s_0}^{\infty}\frac{D_k(s_0,\beta)}{(2k+1)(2k+3)(2k+5)(2k+7)}\right\}, \qquad (111)$$

where

$$C_r(k)=\sum_{s=-k}^{k}\frac{(-1)^{k+s}s^r}{(k+s)!(k-s)!},\quad D_k(s_0,\beta)=\sum_{s=s_0}^{k}\frac{(-1)^{k+s}x^{2k+7}}{(k+s)!(k-s)!}. \qquad (112)$$

$$C_{2k}(k)=1,\quad C_{2k+2}(k)={}^1/_6k(k+1)(2k+1),$$
$$C_{2k+4}(k)={}^1/_{72}k(k+1)(k+2)(2k+1)(2k+3)(k-{}^1/_5). \qquad (113)$$

The sums $C_r(k)$ differ from zero only for even $r\geqslant 2k$. In particular

$$C_r(k)=\frac{1}{(2k)!}\left(\frac{d}{dx}\right)^r(e^{-x/2}-e^{x/2})^{2k}|_{x=0}, \qquad (114)$$

It is convenient to calculate the particular values of the sums $C_r(k)$ using the formula

and expanding the exponentials in series.

As a result, the representation (111) turns into an expansion in powers of the small parameter α^2:

$$W=4f\left(\frac{2I}{m}\right)^{7/2}(1-\xi^2)^{7/2}\left\{\sum_{m=0}^{\infty}\alpha^{2m}S_m(\beta)+R\right\} \qquad (115)$$

with coefficients that depend only on ξ:

$$S_m(\beta)=\sum_{k=0}^{\infty}\binom{k+7/2}{2k+2m}\frac{C_{2k+2m}(k)\beta^{2k}}{(2k+1)(2k+3)(2k+5)(2k+7)},\quad \beta=\alpha x_0=\frac{2\xi}{(1-\xi^2)^{1/2}}. \qquad (116)$$

In the expression for $S_m(\beta)$ we have extended the summation over k to ∞, neglecting the exponentially small terms $\sim\exp(s_0)$. The letter R in (115) denotes the residue of the series in k, i.e., the second term in the curly brackets of (111). The terms of the series R are exponentially small, $\sim\exp(s_0)$. This follows from the representation of the sum $D_k(s_0,\beta)$ by a Sommerfeld-Watson integral, in which the integration contour extends to infinity and the edges of the cut $-\infty<s<s_0$, where the integral can be easily investigated. Using (116) and (113), we obtain for the first three coefficients

$$S_0(\beta)=\frac{1}{105}\left(1+\frac{1}{4}\beta^2\right)^{7/2}=\frac{1}{105}(1-\xi^2)^{-7/2},$$
$$S_1(\beta)=\frac{\beta^2}{384}\left(1+\frac{1}{4}\beta^2\right)^{1/2}=\frac{\xi^2}{96}(1-\xi^2)^{-3/2}$$
$$S_2(\beta)=\frac{\beta^4}{46080}\left(1+\frac{1}{4}\beta^2\right)^{-5/2}\left(\frac{7}{32}-\frac{1}{\beta^2}\right)=\frac{\xi^4}{6144}(1-\xi^2)^{1/2}\left(1-\frac{8}{15\xi^2}\right). \qquad (117)$$

If a is expressed in terms of χ and ξ, then W is represented by a power series in χ^2 with coefficients that are polynomials in ξ^{-2}:

$$W=W_0\left\{1+\frac{35}{8}\chi^2+\frac{35}{128}\chi^4\left(1-\frac{8}{15\xi^2}\right)+\ldots\right\}. \quad (118)$$

This asymptotic representation, obtained for $\xi<1$ differs from the representation that is valid for $\xi>1$ [see Eq. (103) and the pertinent comments] only in the absence of a rapidly oscillating part. It can be shown that when the parameter ξ decreases near the point $\xi=1$ the exponentially small terms contained in both representations become appreciable and annihilate the oscillating part of the representation for $\xi>1$. This qualitative difference between the representations for $\xi>1$ and $\xi<1$ is possibly due to the fact that at $\xi=1$ a decay channel with emission of photons into the wave becomes open.

The fact that the coefficient of χ^2 does not depend on ξ means that in a weak field the process is sensitive to the field frequency not under the condition $\xi\lesssim 1$, but under the condition $\xi\lesssim\chi$, i.e., $\omega\gtrsim I$. This can be understood in such a way that for the process that proceeds also in the absence of a field the time of formation of small correction terms for the total probability remains of the order of I^{-1} and is small compared with the period ω^{-1} of the change of the field if $\omega\ll I$.

We are grateful to I. S. Shapiro for discussions and remarks.

[1)]The units used are $\hbar=c=1$; m is the electron mass and a is the wave-potential amplitude.

[2)]The estimate follows from the known equation that connects the time, the path, and the acceleration, wherein the path is I/eF and the acceleration eF/m.

[3)]The point $\xi=1$ is the condensation point of the sequence of points $\xi_k, k=1,2,3,\ldots$, in the interval $\xi_2=0.794\leqslant\xi_h\leqslant\xi_1=\infty$, at which the exponentially small terms are transformed, when ξ approaches 1, into rapidly oscillating ones:

$$\chi^4\exp\left(-\frac{\operatorname{Im} f}{\chi}\right)\sin\left(\frac{\operatorname{Re} f}{\chi}+\gamma\right)\to\chi^4\sin\left(\frac{f}{\chi}+\gamma\right),$$

$$f=f(\xi),\quad \gamma=\gamma(\xi).$$

Thus, oscillating terms accumulate near $\xi=1$, whereas in the interval $\xi_3=1.139<\xi<\xi_1=\infty$ there is only one oscillating term, while at $\xi<\xi_2$ there are none at all.

[1]W. Becker, W. H. Louisell, J. D. McCullen, and M. O. Scully, Phys. Rev. Lett. **47**, 1262 (1981).
[2]W. Becker, W. H. Louisell, J. D. McCullen, R. Schlicher, and M. O. Scully, *ibid.* **48**, 653 (1982).
[3]W. Becker, W. H. Louisell, J. D. McCullen, R. Schlicher, and M. O. Scully, Appl. Phys. **B28**, 310 (1982).
[4]A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 1768 (1964) [Sov. Phys. JETP 19, 1191 (1964)].
[5]A. I. Nikishov and V. I. Ritus, *ibid.* **50**, 255 (1966) [23, 268 (1966)].
[6]V. I. Ritus, *ibid.* **56**, 986 (1969) [29, 532 (1969)].
[7]V. I. Ritus, Quantum Electrodynamics of Phenomena in an Intense Field, Trudy FIAN, Vol. 111, Nauka, 1979.
[8]H. R. Reiss, Phys. Rev. Lett. **48**, 652 (1982).
[9]H. R. Reiss, Appl. Phys. **B28**,310 (1982).
[10]J. I. Gersten and M. H. Mittleman, Phys. Rev. Lett. **48**, 651 (1982).
[11]I. M. Ternov, V. N. Rodionov, and O. F. Dorofeev, Proc. 5th Internat. Seminar on Problems of High Energy Physics and Quantum Field Theory, Protvino, 1982, Vol. 1, p. 340. Preprints No. 08/1982 and 14/1982, Moscow State Univ. Physics. Dept.
[12]M. Abramowitz and I. A. Stegun, eds., Handb. of Math. Functions, Dover, 1964.
[13]V. O. Papanyan and V. I. Ritus, Zh. Eksp. Teor. Fiz. **61**, 2231 (1971); **65**, 1756 (1973) [Sov. Phys. JETP **34**, 1195 (1972); **38**, 879 (1974)].

Translated by J. G. Adashko