Alfven waveguide

A. S. Leonovich, V. A. Mazur, and V. N. Senatorov

Siberian Institute of Terrestrial Magnetism, the Ionosphere, and Radio Wave Propagation, Siberian Division of the USSR Academy of Sciences
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Waveguide propagation of Alfven waves in a nonuniform plasma is investigated by solving the equations of two-fluid hydrodynamics under the assumption that cold dispersion is present.

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1. INTRODUCTION

The present article is devoted to determining whether natural Alfven waves can exist in a nonuniform plasma. Transverse inhomogeneity plays a special role in a magnetized plasma, since the longitudinal inhomogeneity of the plasma is, as a rule, much less than its transverse inhomogeneity as a consequence of the free flow of the plasma along the magnetic field.

Transverse dispersion of waves propagating in a transversely nonuniform plasma is important; in the case of Alfven waves, the transverse dispersion is low and is usually ignored: $\omega = k_{\parallel} A$, where A is the Alfven velocity. In such an approximation, there are no natural Alfven oscillations in a nonuniform plasma.[1] Because there are no natural oscillations, we are forced to study the evolution of the initial perturbation, i.e., the improper oscillation of the medium,[1,2] a typical example of which is a wave packet. The increase in the mean wave vector of the wave packet in accordance with the equation $d\mathbf{k}/dt = -\partial\omega/\partial\mathbf{x}$ plays the basic role in the time evolution of the packet. As a result, the packet is carried over in k-space into the region of large wave vectors, where it attenuates by viscous dissipation.[1,2]

There are two distinct effects that lead to transverse dispersion of Alfven waves. The first is due to the finite Larmor radius of the ions ρ_i and produces what is known as "hot" dispersion, which has order of magnitude $k_\perp^2\rho_i^2$. Cold dispersion results from the fact that the ratio ω/ω_i, where ω_i is the cyclotron frequency of the ions, is finite. The role of hot dispersion has been discussed in a number of studies.[1–4] In a low-pressure plasma, $\beta = 8\pi p/B^2 \ll 1$, hot dispersion prevails over cold dispersion only at very high values of the transverse wave vector: $k_\perp > \beta^{-1/4}k_\parallel$. The present article is devoted to the study of the influence of cold dispersion on the behavior of Alfven waves in a nonuniform plasma.

Dispersion of a wave causes it to propagate along the density gradient. At an appropriate density profile, the wave may become "locked" into some region in space bounded in the transverse direction. This type of natural oscillation can be referred to as an Alfven waveguide.

2. BASIC EQUATIONS

The simplest technique for studying the influence of cold dispersion is that of two-fluid hydrodynamics. Ignoring the inertia of the electrons and the electron-ion collisions, and assuming that the plasma is cold, we have

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{rot}[\mathbf{v}\times\mathbf{B}] - \frac{m_i c}{e}\,\mathrm{rot}\frac{d\mathbf{v}}{dt}, \tag{1}$$

$$m_i n\frac{d\mathbf{v}}{dt} = \frac{1}{4\pi}[\mathrm{rot}\,\mathbf{B}, \times\,\mathbf{B}], \qquad \frac{\partial n}{\partial t} + \mathrm{div}\, n\mathbf{v} = 0. \tag{2}$$

Here $\mathbf{B}$ is the magnetic field, $\mathbf{v}$ is the velocity of the plasma, and n is the density of the plasma.

In equilibrium, we may set $\mathbf{B} = (0,0,B_0)$, $B_0 = \mathrm{const}$, $\mathbf{v} = 0$, and $n = n(x)$. The perturbation in the wave will be described by means of the dimensionless vector $\mathbf{b} = \mathbf{B}'/B_0$, where $\mathbf{B}'$ is the perturbed magnetic field. Setting

$$\mathbf{b} = \mathbf{b}(x)\exp(-i\omega t + ik_z z),$$

and linearizing the initial equations, we arrive at the following system of equations for b_x and b_y:

$$\frac{d^2 b_x}{dx^2} + \left(\frac{\omega^2}{A^2} - k_z^2\right) b_x = -iuk_z^2 b_y, \tag{3}$$

$$\left(\frac{\omega^2}{A^2} - k_z^2\right) b_y = -iu\left(\frac{d^2}{dx^2} - k_z^2\right) b_x. \tag{4}$$

Here $A = A(x) \equiv B_0/(4\pi n(x) m_i)^{1/2}$ and $u = \omega/\omega_i$ is a small parameter which "entangles" Eqs. (3) and (4).

In the limit $u = 0$, Eqs. (3) and (4) separate; the first equation describes a fast magnetosonic (FMS) wave in which $b_y = 0$ and $b_x \neq 0$. Depending on the density profile, the solutions of this equation may belong to the continuous as well as the discrete spectrum. Equation (4) with $u = 0$ describes layered Alfven waves:

$$b_x = 0, \qquad b_y = C\delta(x - x_0),$$

and at each point $x = x_0$ there exists an oscillation localized at this point and having a frequency $\omega^2 = k_z^2 A^2(x_0)$.

Dispersion of the Alfven waves occurs if u is finite. This is quite simple to understand for a uniform plasma. In this case, the system (3),(4) yields the dispersion equation

$$\omega^2 = \left(k_z^2 + \frac{k_\perp^2 \pm (k_\perp^4 + 4u^2k^2k_z^2)^{1/2}}{2}\right) A^2,$$

where $k^2 = k_\perp^2 + k_z^2$, while the plus sign corresponds to the FMS wave and the minus sign to the Alfven wave. Small corrections due to the parameter u in the case of a FMS wave do not play any special role. On the other hand, dispersion of the Alfven wave is entirely due to the small correction

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$$\omega = k_z A\left\{1-\frac{u^2}{2}-\frac{u}{2}\left[\frac{k_\perp^2}{k_0^2}+\left(1+\frac{k_\perp^4}{k_0^4}\right)^{1/2}\right]^{-1}\right\}, \quad (5)$$

where $k_0^2 = 2uk_z^2$. Hence it is clear that the nature of the dispersion depends on the ratio $k_\perp/k_0$. Dispersion is substantial if $k_\perp \lesssim k_0$ (referred to as quasilongitudinal propagation). It is easily verified that in this case the Alfven wave has elliptical polarization (which turns into circular polarization if $k_\perp = 0$). If $k_\perp \gg k_0$, dispersion (5) becomes negligibly small.[1)] In this case, the polarization ellipse elongates and the polarization becomes nearly linear.

Let us now study the effects of finite u in a nonuniform plasma. Substituting b_y from (4) in (3), we obtain for b_x an equation in Schrödinger form:

$$d^2b_x/dx^2 - V(x)\,b_x = 0, \quad (6)$$

where the potential is

$$V(x) = -\frac{\omega^2}{A^2}+k_z^2(1+u^2)+u^2k_z^4\left[\frac{\omega^2}{A^2}-k_z^2(1-u^2)\right]^{-1}. \quad (7)$$

Depending on the type of the potential $V(x)$, which is determined by the density profile and by the value of ω, Eq. (6) can have both infinite and spatially localized solutions. In the latter case, the parameter ω in the potential (7) must be selected so that the Schrödinger equation (6) has a solution with energy $E = 0$. There may be several such values of ω. This means that the energy level $E = 0$ may correspond both to the ground state and to excited states in the well $V(x)$.

By means of Eq. (6) it is possible to express b_x'' in terms of b_x. Substituting it in (4), we obtain

$$b_y = iu\frac{\omega^2}{A^2}\left[\frac{\omega^2}{A^2}-k_z^2(1-u^2)\right]^{-1} b_x. \quad (8)$$

3. WAVEGUIDE FOR ALFVEN WAVES

We select a density profile $n(x)$ with maximum at the point $x = 0$. The expansion

$$n(x) = n_0(1-x^2/a^2) \quad (9)$$

applicable if $|x| \ll a$ may be used near the maximum point.

Let us consider a wave with frequency nearly equal to k_zA_0, where

$$A_0 = A(0) = B_0/(4\pi m_i n_0)^{1/2},$$

for which we set

$$\omega = k_zA_0(1-u^2/2-\delta/2), \quad (10)$$

and the parameter δ is small: $|\delta| \ll 1$. Then the potential (7) takes the form

$$V(x) = k_z^2\left(\frac{x^2}{a^2}+\delta+2u^2-\frac{u^2}{\delta+x^2/a^2}\right). \quad (11)$$

It has no singularities at $\delta > 0$. Its plot is shown in the figure. The narrow well in the center is due to the finite value of u. If the bottom of the curve drops below zero, for which we must have $\delta < u$, Eq. (6) can have eigensolutions in the discrete spectrum (i.e., it can have the energy level $E = 0$).

We find the solution in two limiting cases. The dimensionless parameter characterizing the potential well is the product of the depth and the square of the width. For the

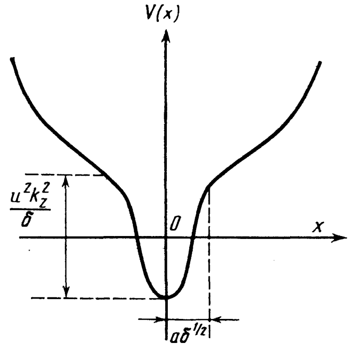

FIG. 1.

narrow well in the center, it is equal to $u^2k_z^2a^2$. If $uk_za \gg 1$, the well may be considered deep and there are many solutions in the discrete spectrum. If $uk_za \ll 1$, the well is shallow and there is only one solution.

If the level $E = 0$ is near the bottom of the deep well, the characteristic dimension Δ of localization of the solution is much less than the width $a\delta^{1/2}$ of the well, i.e., $\Delta^2 \ll a^2\delta$. Then the potential is quadratic and its solutions are well known. We have

$$\delta = \delta_n = u - u^2 - \sqrt{2}(k_za)^{-1}(n+1/2),$$

which, in accordance with (10), means that

$$\omega = \omega_n = k_zA_0\left[1-\frac{u}{2}+\frac{1}{\sqrt{2}\,k_za}\left(n+\frac{1}{2}\right)\right].$$

In order of magnitude, the number of states in the discrete spectrum with $\delta > 0$ is $n \sim uk_za \gg 1$. In the ground state, the eigenfunction has the form

$$b_x = \exp(-x^2/\Delta^2), \qquad \Delta = 2^{1/4}(a/k_z)^{1/2}.$$

If $\Delta \ll a$, meaning that

$$k_za \gg 1, \quad (12)$$

the expansion (9) is applicable. We assume everywhere below that the condition (12) is satisfied. In the case of a deep well, even the stronger inequality $uk_za \gg 1$ holds from which follows the condition $\Delta^2 \ll a^2\delta$ proposed earlier.

In the case of a shallow well, $uk_za \ll 1$, the dimension of localization of the eigenfunction is much greater than the width of the well. In this case, the result can be obtained by matching the solution within the narrow center well to that outside it (see, e.g., Ref. 5, p. 196). The equation outside the well has the form

$$\frac{d^2b_x}{dx^2}-k_z^2\left(\frac{x^2}{a^2}+\delta+2u^2\right)b_x = 0.$$

Its solution, which is decreasing as $x \to +\infty$, is designated $b_x^{(+)}$. We have

$$b_x^{(+)} = D_p[(2k_z/a)^{1/2}x],$$

where

$$p = -1/2[1+k_za(\delta+2u^2)],$$

and $D_p(z)$ is a parabolic-cylinder function (see Ref. 6, Russ. p. 1081). Equating the logarithmic derivatives of this solution and of the solution within the well, we obtain

$$\left.\frac{d\ln b_x^{(+)}}{dx}\right|_{x=0}=-\frac{1}{2}\int_{-\infty}^{\infty}\frac{u^2k_z^2\,dx}{\delta+x^2/a^2}.$$

Since $k_za(\delta+2u^2)\ll 1$, using the formulas of the theory of parabolic-cylinder functions, we have for the unique solution

$$\delta=\frac{\pi^2}{4}\frac{\Gamma^2(1/4)}{\Gamma^2(3/4)}u^4(k_za)^3.$$

The eigenfunction extends far outside the narrow central well of the figure and coincides there with the solution $b_x^{(+)}$:

$$b_x^{(+)}=D_{-1/2}\left[\left(\frac{2k_z}{a}\right)^{1/2}|x|\right]. \qquad (13)$$

The solution within the well in fact reduces to an abrupt change of the derivative and can be interpreted as a kink of the solution (13) at the point $x=0$. The characteristic scale of this solution is $\Delta=(a/k_z)^{1/2}$. By virtue of the condition $k_za\gg 1$, we have $\Delta\ll a$, and, consequently, the expansion (9) is valid.

The polarization of the resulting solutions is of interest. According to (8),

$$b_y=-iu(\delta+x^2/a^2)^{-1}b_x.$$

Hence it follows that in the case of a deep well, $uk_za\gg 1$, we have $b_y(x)/b_x(x)=-i$, i.e., the polarization of a van Alfven wave is circular. This result corresponds to the fact that in a deep well the characteristic wave vector $k_x\sim 1/\Delta=(k_z/a)^{1/2}$ satisfies the quasi-longitudinal propagation condition

$$k_x/k_z\sim(k_za)^{-1/2}\ll u^{1/2}.$$

In the case of a shallow well, the polarization is elliptical, the semi-axes of the ellipse depending quite strongly on the coordinate x. If $x=0$, we have

$$|b_y/b_x|\sim(uk_za)^{-3}\gg 1,$$

though even at $x=u^{1/2}a\ll\Delta$ the ratio of the axes is inverted, $|b_y/b_x|<1$.

4. CONCLUSION

In the present article we have studied the influence of a finite value of the parameter $u=\omega/\omega_i$ on the propagation of Alfven waves in a nonuniform plasma. We have shown that natural (waveguide) solutions can exist for Alfven waves near the plasma-density maximum. This effect can play an important role in the physics of the magnetosphere and in the physics of the sun. Conditions necessary for the existence of waveguide solutions in the earth's magnetosphere may occur in the neighborhood of the plasmopause as well as in the magnetosphere ducts. As for the physics of the sun, Alfven waveguides may appear in the corona solar loops.

[1)] If $k_\perp>(m_i/m_e)^{1/2}k_z$, the inertia of the electrons begins to influence the dispersion of the Alfven wave.

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