Dynamic solitons in a uniaxial antiferromagnet

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Solitons in a uniaxial antiferromagnet are investigated. One-dimensional and also three-dimensional, centrally symmetric solutions describing the soliton are obtained. Two-parameter solitons are considered to which a magnetization oscillation corresponds that possesses a frequency ω in a reference system, moving together with the soliton with a velocity **v**. The parameters **v** and ω turn out to be connected with the integrals of motion of the antiferromagnet field equations in a natural way. That is, they are connected with the energy E, the momentum **P** and the projection z of the magnetization N (the z axis is taken to be the axis of the antiferromagnet). The stability of the solitons is investigated by the direct Lyapunov technique. It is shown that one-dimensional solitons are stable relative to perturbations that do not violate the one-dimensional character of the problem, the values of the parameters ω and v being arbitrary. The range of values of the parameter ω for which the three-dimensional solitons are stable is found.

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In the description of essentially nonlinear systems, an important role is played by solitons—localized states of the corresponding field. The solitons in nonlinear systems play the same role as quasiparticles in linear systems and are just as important for the study of the physical characteristics of the system.

The origin of our understanding of solitons is connected with the analysis of one-dimensional systems, in particular, systems that allow complete integration by the method of the inverse problem of scattering theory.[1] However, for the description of real cases, analysis of solitons in three-dimensional systems is necessary. The analysis shows that in the three-dimensional case, the solitons are frequently unstable,[2] and the search for systems that allow the existence of stable three-dimensional solitons is important.

Among the essentially nonlinear systems, we note magnetically ordered media, the nonlinearity of which is determined by the geometric properties of the dynamic variable. At the present time, the analysis of solitons, including non-one-dimensional, in ferromagnets has been carried out in sufficient detail (see the reviews in Refs. 3 and 4). It is important that in a number of cases, stable three-dimensional solitons do exist in the ferromagnet;[3–5] therefore, magnetic systems are especially vital for the analysis of solitons.

In the present research, we have investigated the nonlinear dynamics of antiferromagnets (AFM). The analysis is carried out on the basis of the effective equations obtained in Refs. 6–8 that described the dynamics of the AFM in terms of the normalized (unit) antiferromagnetism vector **l**.

Periodic nonlinear waves and solitons in uniaxial AFM, including three-dimensional solitons, have been investigated. On the basis of the direct Lyapunov technique, we have obtained stability criteria of solitons of any dimensionality. It is shown that the one-dimensional dynamic solitons are stable to perturbations that do not violate the one-dimensionality of the problem, at all values of the parameters of the soliton.

We have investigated three-dimensional, centrally symmetric solitons and found the region of parameters in which the solitons are stable.

1. NONLINEAR DYNAMICS AND INTEGRALS OF MOTION OF UNIAXIAL AFM

The state of an uniaxial antiferromagnet in the model of two sublattices is described by specifying the sublattice magnetization densities $\mathbf{M}_1(\mathbf{r},t)$ and $\mathbf{M}_2(\mathbf{r},t)$, $\mathbf{M}_1^2 = \mathbf{M}_2^2 = \mathbf{M}_0^2$.[9] It is convenient to introduce the magnetization vector and the antiferromagnetism vector:

$$\mathbf{M}=\mathbf{M}_1+\mathbf{M}_2, \quad \mathbf{L}=\mathbf{M}_1-\mathbf{M}_2. \tag{1}$$

Under the assumption that $|\mathbf{M}|\ll|\mathbf{L}|$, which is a natural one for AFM, we can describe the dynamics by a single dynamic equation for the normalized antiferromagnetism vector $\mathbf{l}$, $\mathbf{l} = \mathbf{L}/|\mathbf{L}|$.[6–8] This equation can be written down in the form[1]

$$\alpha[\mathbf{l}\Delta\mathbf{l}]-\frac{\alpha}{c^2}\left[\mathbf{l}\times\frac{\partial^2\mathbf{l}}{\partial t^2}\right]-\frac{2g\alpha}{c^2}\frac{\partial\mathbf{l}}{\partial t}(\mathbf{lH})$$
$$-\frac{\alpha g^2}{c^2}(\mathbf{lH})[\mathbf{l}\times\mathbf{H}]-\left[\mathbf{l}\times\frac{\partial w_a}{\partial\mathbf{l}}\right]=0. \tag{2}$$

Here c is the characteristic velocity, which is identical, at $H = 0$, with the minimum phase velocity of spin waves of the linear theory,

$$c=gM_0(\alpha\delta)^{1/2}/2, \tag{3}$$

α and δ are constants of the inhomogeneous and homogeneous volumes, respectively (see Ref. 7), g is the gyromagnetic ratio, **H** is the external magnetic field, $M_0^2 w_a(\mathbf{l})$ is the energy density of the AFM anisotropy, and Δ is the Laplace operator.

A consistent description of the dynamics includes an expression for the magnetization of the AFM in the form

$$\mathbf{M}=\frac{4}{g\delta}\left[\frac{\partial\mathbf{l}}{\partial t}\times\mathbf{l}\right]+\frac{4}{\delta}[\mathbf{H}-\mathbf{l}(\mathbf{lH})]. \tag{4}$$

In what follows, we limit ourselves to the study of purely uniaxial AFM, i.e., we shall assume that w_a depends only

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on l_z^2 (the z axis is the chosen axis). Moreover, we shall assume that the field **H** is parallel to the chosen AFM axis.

It is convenient to write down Eq. (2) in angular variables for the unit vector **l**:

$$l_z=\cos\theta, \quad l_x+il_y=\sin\theta\exp(i\varphi). \tag{5}$$

In the uniaxial AFM, $w_a = w_a(\Theta)$ and Eq. (2) takes the form

$$\alpha\left[\Delta\theta-\frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2}\right]-\alpha\sin\theta\cos\theta\left[(\nabla\varphi)^2-\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}-gH\right)^2\right]-\frac{\partial w_a}{\partial\theta}=0,$$

$$\alpha\nabla[\sin^2\theta(\nabla\varphi)]-\frac{\alpha}{c^2}\frac{\partial}{\partial t}\left[\sin^2\theta\left(\frac{\partial\varphi}{\partial t}-gH\right)\right]=0. \tag{6}$$

We now investigate the nonlinear periodic waves of magnetization of the AFM. It is easy to establish the fact that the following solution of (6) corresponds to them:

$$\varphi=kx-\omega t, \quad \theta=\theta_0=\text{const}. \tag{7}$$

The dependence of the frequency of the nonlinear wave on its wave vector k and its amplitude θ_0 is determined by virtue of (6) by the equation

$$(\omega+gH)^2=c^2k^2+\frac{\partial w_a}{\partial\theta}\frac{c^2}{\alpha\sin\theta_0\cos\theta_0}. \tag{8}$$

If we choose the anisotropy energy in the simplest form

$$\beta(l_x^2+l_y^2)/2=(\beta/2)\sin^2\theta,$$

as is usually done in uniaxial magnetics, we then obtain the result that the frequency of the nonlinear wave of arbitrary amplitude does not depend on the value of the amplitude. This is a rather unusual situation for nonlinear systems and indicates a specific degeneracy of the AFM as a nonlinear system in the case of this choice of the anisotropy (we recall that in a ferromagnet, even an isotropic one, the frequency of the nonlinear wave depends on its amplitude, $\omega\propto\cos\theta_0$). In particular, the Lighthill criterion for the stability of the nonlinear wave in such a system does not allow us to draw any conclusion on the stability of the nonlinear wave.

This degeneracy appears also in the properties of solitons[10] and vanishes upon consideration of the more general form of the anisotropy energy. Therefore, we shall not, for the time being, specify the form of $w_a(\theta)$, but in specific calculations below we shall assume

$$w_a=\frac{1}{2}\beta\sin^2\theta-\frac{1}{4}b\sin^4\theta, \quad \beta>0. \tag{9}$$

In such a case, we obtain for the frequency of the nonlinear wave

$$(\omega+gH)=\pm[\omega_1^2+c^2k^2-2(\omega_1^2-\omega_t^2)\sin^2\theta_0]^{1/2}. \tag{10}$$

Here we have used the notation

$$\omega_1=gH_1, \quad \omega_t=gH_t,$$

$$H_1=M_0(\beta\delta)^{1/2}/2, \quad H_t=M_0((\beta-b/2)\delta)^{1/2}/2. \tag{11}$$

The quantity ω_1 is identical with the frequency of the homogeneous linear antiferromagnetic resonance at $H=0$ in the collinear phase of the AFM, which corresponds to $\theta=0$ or $\theta=\pi$. The characteristic value of the field H_1 determines the field of the lability of the collinear AFM phase; this phase is stable at $H<H_1$. H_1. If $b>0$, then $H_t<H_1$. The quantity H_t determines the field of the first-order transition from collinear to spin-flip phase of the AFM, in which $\theta=\pi/2$ (for more detail, see Ref. 9). It is easy to see that if $H_t<H_1$, then $\partial\omega/\partial\theta_0<0$, which, in correspondence with the Lighthill criterion, indicates the instability of the periodic nonlinear waves relative to self-modulation, i.e., the formation of solitons. In what follows, we shall assume $b>0$ and $H<H_1$; the cases $b<0$ and $H>H_1$ were considered in Ref. 10.

Equations (6) can be obtained as the Euler-Lagrange equations from the following form of the Lagrangian density function:

$$\mathscr{L}=M_0^2\left\{\frac{\alpha}{2c^2}\left[\left(\frac{\partial\theta}{\partial t}\right)^2+\sin^2\theta\left(\frac{\partial\varphi}{\partial t}-gH\right)^2\right]-\frac{\alpha}{2}[(\nabla\theta)^2+\sin^2\theta(\nabla\varphi)^2]-w_a(\theta)\right\}. \tag{12}$$

The energy and field momentum of the magnetization field of the AFM can be written in the form

$$E=M_0^2\int dx\left\{\frac{\alpha}{2c^2}\left[\left(\frac{\partial\theta}{\partial t}\right)^2+\sin^2\theta\left(\frac{\partial\varphi}{\partial t}\right)^2-g^2H^2\sin^2\theta\right]+\frac{\alpha}{2}[(\nabla\theta)^2+\sin^2\theta(\nabla\varphi)^2]+w_a(\theta)\right\}, \tag{13}$$

$$\mathbf{P}=-\frac{\alpha M_0^2}{c^2}\int dx\left\{\frac{\partial\theta}{\partial t}\nabla\theta+\sin^2\theta\left(\frac{\partial\varphi}{\partial t}-gH\right)\nabla\varphi\right\}. \tag{14}$$

Since w_a does not depend on φ, there exists one integral of the motion—the z component of the total magnetization **M**. We shall represent this integral in the form

$$N=\frac{1}{2\mu_0}\int M_z\,dx=\frac{4s}{\delta M_0a^3}\int\left(H-\frac{1}{g}\frac{\partial\varphi}{\partial t}\right)\sin^2\theta\,dx, \tag{15}$$

where s is the spin of the atom, a^3 the volume of the elementary cell, i.e., $2\mu_0 s/a^3=M_0$. The fact that N is an integer is used below in the quasiclassical quantization of the solitons.

We note an important property of Eqs. (6) and the Lagrangian function (12). If we introduce the new variable $\tilde{\varphi}$:

$$\tilde{\varphi}=\varphi-gHt, \tag{16}$$

then, in terms of $\tilde{\varphi}$, Eqs. (6) and (12) are invariant to Lorentz transformations in which the role of the characteristic velocity is played by the quantity c. Therefore, in the search for solutions of (6), it suffices to limit ourselves to the analysis of stationary soliton solutions and to obtain the moving solutions from the stationary with the help of the Lorentz transformations.

We note that the introduction of the Lorentz invariance is valid at $H=0$ for arbitrary AFM, but at $H\neq0$, it is essential to assume that the AFM is uniaxial and that w_a does not depend on φ. Moreover, at $H\neq0$, the quantities E, **P**, **M** or M_Z do not possess Lorentz invariance, since the transition from φ to $\tilde{\varphi}$ contains the time explicitly. We obtain the law of transformation of E, **P** and N in the transition from the stationary to the moving soliton.

We consider the soliton solutions in the collinear AFM phase. The stationary soliton solution can be sought in the form

$$\theta=\theta(\mathbf{x}),\quad \tilde{\varphi}=-\omega t. \tag{17}$$

Substituting (17) in (6), it is easy to establish the fact that the second of Eqs. (6) is satisfied identically, while the first determines the form of the function $\theta(\mathbf{x})$:

$$\alpha\Delta\theta+\frac{\alpha\omega^2}{c^2}\sin\theta\cos\theta-\frac{\partial w_a}{\partial\theta}=0. \tag{18}$$

As boundary conditions for the function $\theta(\mathbf{x})$, we choose the conditions

$$\theta\to 0,\ \pi,\quad \nabla\theta\to 0;\qquad |\mathbf{x}|\to\infty, \tag{19}$$

which correspond to localization of the magnetization in the soliton.

Below, we shall obtain the specific form of the solution of Eq. (18) for the most interesting cases—the one-dimensional and three-dimensional spherically symmetric solitons—in a model with anisotropy of the form (9). For now, we discuss those properties of the solitons which can be investigated without knowledge of the specific form of $w_a(\theta)$ and $\theta(\mathbf{x})$.

First of all, we note that a nonzero value of the integral of motion N (15) corresponds to solitons of the form (17)–(19). As we can convince ourselves, the stability of such solitons, which it is natural to name precessional, is determined by the conservation of N, i.e., the precessional solitons are dynamic.

We now discuss the quantum-mechanical meaning of the quantity N. It is easy to see that corresponding to the soliton (17)–(19) is separation of variables: the canonical momentum P_θ, which is conjugate to the angle θ, vanishes. The rule for the quasiclassical quantization for solitons then follows. We write out the adiabatic invariant (action) in standard form (for example, see Ref. 11):

$$I=\frac{1}{2\pi}\oint P_\varphi\,d\varphi=\frac{\alpha M_0^2}{2\pi c^2}\int_0^{2\pi/\omega}dt\int d\mathbf{x}\frac{\partial\varphi}{\partial t}\left(gH-\frac{\partial\varphi}{\partial t}\right)\sin^2\theta.$$

It is easy to establish the fact that the quantity $I/\hbar$ is identical with the value of N. Thus the requirement that N be an integer corresponds to the standard quasiclassical quantization of the precessional soliton.

This conclusion remains valid also for a moving soliton, to which corresponds

$$\theta=\theta_0\left(\frac{x-vt}{(1-v^2/c^2)^{1/2}},y,z\right),\quad \tilde{\varphi}=-\frac{\omega(t-xv/c^2)}{(1-v^2/c^2)^{1/2}}, \tag{20}$$

where $\theta_0(\mathbf{x})$ is the solution of (18) and the direction of the motion is chosen as the x axis.

The solution (20) describes a two-parameter soliton; the two parameters that determine its structure are the velocity of the soliton and the frequency of precession of the magnetization in the proper frame of reference of ω. It is easy to see that the quantity N or the action I are the same both for the stationary and for the moving solitons, with the same value of the parameter ω:

$$N=\omega\int\sin^2\theta\,d\mathbf{x}=N(\omega). \tag{21}$$

Thus the value of N for the two-parameter soliton of the form (20) depends only on ω. The values of the other integrals of motion—the energy and the momentum—depend on the velocity of the soliton. Using (13) and (14) and Eq. (20), we can obtain

$$\mathbf{P}=\frac{\mathbf{v}E_0(\omega)}{(c^2-v^2)^{1/2}},\quad E=-2\mu_0HN+\frac{E_0(\omega)}{(1-v^2/c^2)^{1/2}}. \tag{22}$$

Inverting the relations (21) and (22), we can write out the energy of the soliton in the form of a function of its integrals of motion:

$$E(P,\ N)=-2\mu_0HN+(E_0^2(N)+c^2P^2)^{1/2}. \tag{23}$$

The quantity $E_0(\omega)$ or $E_0(n)$ is determined only by the form of the function $\theta_0(\mathbf{x})$. In the most interesting cases—one-dimensional ($1D$) and three dimensional, centrally symmetric ($3D$)—we get for E_0

$$1D:\ E_0=\alpha M_0^2\int_{-\infty}^{+\infty}\left\{\left(\frac{d\theta}{dx}\right)^2+\frac{\omega^2}{c^2}\sin^2\theta\right\}a^2\,dx,$$

$$3D:\ E_0=\alpha M_0^2\int_0^{\infty}\left\{\frac{1}{3}\left(\frac{\partial\theta}{dr}\right)^2+\frac{\omega^2}{c^2}\sin^2\theta\right\}4\pi r^2\,dr. \tag{24}$$

The factor a^2 is introduced in the $1D$-system for conservation of the dimensionality of the energy, and E_0 represents the energy of the soliton per single atom chain.

Equations (22) determine the desired dependence of $\mathbf{P}$ and E on the velocity of the soliton, i.e., the behavior of $\mathbf{P}$ and E under Lorentz transformations. It is easy to see that at $H=0$ the energy and the momentum make up the four-dimensional vector $P_\mu(E/c,\mathbf{P})$. If $H\neq 0$, then the energy is the sum of two terms, the first of which, as also N, is an invariant of the Lorentz transformation, while the second is a component of a four-dimensional vector.

2. STABILITY OF THE SOLITONS

We proceed to the analysis of the stability of the precessional soliton of the form (17)–(19). In this section, we can specify the form of the function $w_a(\theta)$ and of the solution $\theta(\mathbf{x})$. We require only that the function $\theta(\mathbf{x})$ approach its equilibrium value 0 and π sufficiently rapidly as $|\mathbf{x}|\to\infty$, in order that the integrals in (21) and (24) exist. As we shall see below, at certain values of ω, which we must assume to be achievable for the soliton, $\theta(\mathbf{x})$ falls off exponentially, and this condition is satisfied. Here, two types of behavior are possible for $\theta(\mathbf{x})$ in the one-dimensional case.[6] One type corresponds to a soliton of the domain-wall type:

$$\theta(+\infty)=0,\quad \theta(-\infty)=\pi,\quad d\theta/dx\leqslant 0, \tag{25a}$$

and $d\theta/dx$ vanishes only at $x=\pm\infty$, while the other type corresponds to the localized soliton:

$$\theta(\pm\infty)=0,\quad \theta(0)<\pi/2,\quad (d\theta/dx)_{x=0}=0. \tag{25b}$$

Only the localized solutions similar to (25b) (Ref. 10) correspond to the centrally symmetric soliton $[\theta=\theta(r)]$

$$\theta(\infty)=0,\quad \theta(0)<\pi/2,\quad (d\theta/dr)_{r=0}=0. \tag{26}$$

Such a separation of solitons by class is not a property only of magnetic solitons and is characteristic of many nonlinear systems.[2]

Actually, the information on the structure of solitons contained in Eqs. (25) and (26) is sufficient for the investigation of the stability of solitons. As in many other systems, the conditions for stability of the solitons must be expressed in the form of simple relations for the integrals of motion of the soliton.[2,5,12]

We make use of the direct Lyapunov technique (see, for example, Ref. 13). This method has been applied for the investigation of the stability of solitons with scalar fields, for example, in Refs. 2, 14, and 15.

For the stability of the soliton solution, it is sufficient that we can construct the Lyapunov functional such that its time derivative, calculated in accord with the equations of motion, be negative or equal to zero, while the Lyapunov functional itself be positive definite in a finite neighborhood of the soliton solution. Introducing the variables $\vartheta = \theta - \theta_0$, $\psi = \tilde{\varphi} - \tilde{\varphi}_0$, where θ_0 and $\tilde{\varphi}_0$ correspond to the soliton, the sufficient conditions for stability of the soliton can be written down in the form

$$\frac{d}{dt} V\{\vartheta, \psi\} \leqslant 0, \qquad V\{\vartheta, \psi\} \geqslant 0, \tag{27}$$

in which the equality in the second condition is achieved at $\vartheta, \psi = 0$.

We now use the Chetaev method (see Ref. 13) and seek $V\{\vartheta,\psi\}$ in the form of a combination of integrals of motion. By virtue of the Lorentz invariance of the equation for θ and $\tilde{\varphi}$, it suffices to investigate the case $v = 0$; here, the momentum does not enter into the functions $V\{\vartheta,\psi\}$, and it suffices to seek it in the form

$$V\{\vartheta, \psi\} = E\{\theta, \tilde{\varphi}\} - E_0 + A\,[N\{\theta, \tilde{\varphi}\} - N_0] + (B/2)\,[N\{\theta, \tilde{\varphi}\} - N_0]^2. \tag{28}$$

Here $E_0 = E\{\theta_0,\tilde{\varphi}_0\}$, $N_0 = N\{\theta_0,\tilde{\varphi}_0\}$. Writing $V\{\vartheta,\psi\}$ in terms of the integral of motion immediately gives the desired condition:

$$(d/dt)\, V\{\vartheta, \psi\} = 0,$$

and the coefficients A and B are so chosen as to assure positive definiteness for $V\{\vartheta,\psi\}$. To satisfy this condition, it is naturally necessary that the functional $V\{\vartheta,\psi\}$ not contain terms linear in ϑ and ψ. This can be guaranteed by choosing the coefficient A in the form

$$A = \alpha(\omega + gH)/c^2.$$

Using (13) and (15) and the value of A, we represent the Lyapunov functional (28) in an approximation that is quadratic in ϑ and ψ. In place of the variable ψ it is convenient to introduce another variable: $\mu = \psi \sin\theta_0$. As a result, we obtain

$$V\{\vartheta, \mu\} = \frac{1}{2}\int dx \left\{\left(\frac{\partial\vartheta}{\partial t}\right)^2 + \vartheta\hat{H}\vartheta + \left(\frac{\partial\mu}{\partial t}\right)^2 + \mu\hat{L}\mu\right\} + \frac{B}{2}\left[\int dx \left\{\omega \sin 2\theta_0\vartheta + \left(\frac{\partial\mu}{\partial t}\right)\sin\theta_0\right\}\right]^2, \tag{29}$$

where $\hat{H}$ and $\hat{L}$ are linear differential operators of the type of the Schrödinger type:

$$\hat{H} = -\Delta - \omega^2\cos 2\theta_0 + \frac{\partial^2 w_a}{\partial\theta_0^2},$$
$$\hat{L} = -\Delta + \frac{\Delta(\sin\theta_0)}{\sin\theta_0} = -\Delta - (\nabla\theta_0)^2 + \omega^2\cos^2\theta_0 - \frac{\partial w_a}{\partial\theta_0}\operatorname{ctg}\theta_0. \tag{30}$$

Here and below in this section, we shall use dimensionless variables, making the substitution $w_a \to \beta w_a$ and measuring the coordinates to units of $l_0 = (\alpha/\beta)^{1/2}$ and the time to units of l_0/c.

We expand the functions ϑ, $\partial\vartheta/\partial t$ and μ, $\partial\mu/\partial t$ in the corresponding complete orthonormal set of functions of the operators $\hat{H}$ and $\hat{L}$:

$$\hat{H}\vartheta_\alpha = \varepsilon_\alpha\vartheta_\alpha, \quad \hat{L}\mu_\alpha = e_\alpha\mu_\alpha, \tag{31}$$

where α is a complete set of quantum numbers. For the spherically symmetric soliton, the potentials in $\hat{H}$ and $\hat{L}$ are spherically symmetric and α represents the usual set of quantum numbers $(n, l\, m)$.[16] It is easy to establish the fact that the eigenvalues of the operator $\hat{L}$ are non-negative. Actually, from the definition of (30), it follows that $\hat{L}\sin\theta_0 = 0$, i.e., $\sin\theta_0$ is an eigenfunction of the operator $\hat{L}$ with zero eigenvalue. For the considered one-dimensional solitons and a three-dimensional soliton without nodes, $\sin\theta_0$ vanishes nowhere. Consequently, by virtue of the oscillation theorem, the zero eigenvalue for the operator $\hat{L}$ is the lowest. Since a perturbation of the form $\mu \propto \sin\theta_0$ corresponds simply to a change in the origin of φ, in which we are not interested, we can assume that the operator $\hat{L}$ is positive definite. The term $(\partial\vartheta/\partial t)^2$ is also positive definite. Thus, we need to investigate the functional (29), omitting the terms with $\mu L \mu$ and $(\partial\vartheta/\partial t)$.[2] We write

$$\vartheta = \sum_n C_\alpha\vartheta_\alpha, \qquad \frac{\partial\mu}{\partial t} = \sum_{n,i\neq 0} m_\alpha\mu_\alpha \tag{32}$$

and substitute these expansions in (29). It is easy to see that, of all the m_α, the second term in (29) contains only m_0—the coefficient of the eigenfunction proportional to $\sin\theta_0$. The problem finally reduces to the proof of the positive definiteness of the quadratic form:

$$\frac{1}{2}\left\{m_0^2 + \sum_n \varepsilon_{n0}C_nC_n + \sum_{n,i\neq 0}\varepsilon_{n,i}C_{n,i}C_{n,i}\right\} + \frac{B}{2}\left\{\sum_n \omega C_n\langle\sin 2\theta_0\vartheta_{n0}\rangle + m_0\langle\sin^2\theta_0\rangle^{1/2}\right\}^2, \tag{33}$$

where the notation $\langle f\rangle = \int f(\mathbf{x})d\mathbf{x}$ has been introduced and the set of quantum numbers α are written down in the form (n, i), where n is the radial quantum number and i determines the angular dependence of ϑ_α. By the symbol $i = 0$ we denote the spherically symmetric eigenstates of $\hat{H}$.

The part of the quadratic form containing $C_{n,i}$ at $i \neq 0$ is diagonal. For positive definiteness of this part, it is necessary that all $\varepsilon_{n,i} > 0$. The fact can easily be established that the operator $\hat{H}$ has one zero eigenvalue with $l = 1$, to which corresponds the eigenfunction

$$\frac{\partial\theta_0}{\partial\xi} = \frac{d\theta_0}{dr}\frac{\xi}{r}, \qquad \hat{H}\frac{\partial\theta_0}{\partial\xi} = 0, \tag{34}$$

where ξ is a Cartesian coordinate along any direction. In order to obtain (34), it is sufficient to differentiate the equation for $\theta_0(\mathbf{x})$ (18) with respect to ξ and compare it with $\hat{H}$.

If $\theta_0(r)$ in the soliton falls off monotonically, then the state (34) corresponds to $n = 0$. It is clear that all the remaining states of such a type with $n > 0$ and $l \geqslant 1$ have higher energies.

A perturbation of the form (34) corresponds simply to a shift in the soliton as a whole along the ξ axis and does not disturb its stability. Consequently, the soliton is stable relative to perturbations with $l \neq 0$ and it is only necessary to check the stability relative to radially symmetric perturbations.

We now investigate the quadratic form (33), setting $C_{n,i} = 0$ in it at $i \neq 0$. We diagonalize (33), denoting by λ_i its eigenvalues. The eigenvectors (33) are determined by the equations

$$(\varepsilon_n-\lambda)C_n+\omega B\langle\sin 2\theta_0\vartheta_n\rangle\Delta=0,$$
$$(1-\lambda)m_0+B\langle\sin^2\theta_0\rangle^{1/2}\Delta=0, \tag{35}$$
$$\Delta=\sum_n \omega C_n\langle\sin 2\theta_0\vartheta_n\rangle+m_0\langle\sin^2\theta_0\rangle^{1/2}.$$

Multiplying the equation for C_n by $\omega\langle\sin 2\theta_0\,\vartheta_n\rangle/(\varepsilon_n-\lambda)$, and the equation for m_0-by $\langle\sin^2\theta_0\rangle^{1/2}/(1-\lambda)$, adding them, and eliminating Δ, we obtain the sought dispersion equation that determines eigenvalues of the quadratic form (33):

$$F(\lambda)=1+\frac{B\langle\sin^2\theta_0\rangle}{1-\lambda}+B\omega^2\sum_n\frac{\langle\sin 2\theta_0\vartheta_n\rangle^2}{\varepsilon_n-\lambda}=0. \tag{36}$$

For analysis of the question of the existence of negative values of λ it is necessary to know the energy spectrum ε_n. Since the state (34) with $n=0$, $l=1$ corresponds to $\varepsilon=0$, one of the eigenvalues ε_n ($n=0$, $l=1$) is negative. We denote it by ε_0. Following the method proposed in Ref. 12, we can show that among all the remaining ε_n, not one is negative or equal to zero, i.e., $\varepsilon_1>0$, $\varepsilon_2>0$ and so on.

With account of this, we have plotted in Fig. 1 the function $F(\lambda)$ at $B>0$. We recall that we have still not fixed the constant B, and its value is chosen from the condition that we can construct a positive-definite Lyapunov functional. It is easy to establish the fact that at $B<0$, the equation $F(\lambda)=0$ has the solution $\lambda_0<\varepsilon_0<0$ (i.e., negative values of B can be disregarded.

It is easy to see that the sign of the smallest eigenvalue λ is determined by the value of $F(0)$. It is easy to see from Fig. 1 that if $F(0)<0$, then all $\lambda_n>0$; consequently, the functional $V\{\vartheta,\psi\}$ satisfies the conditions (27) and the soliton is stable. Thus, the condition of stability of the soliton can be written

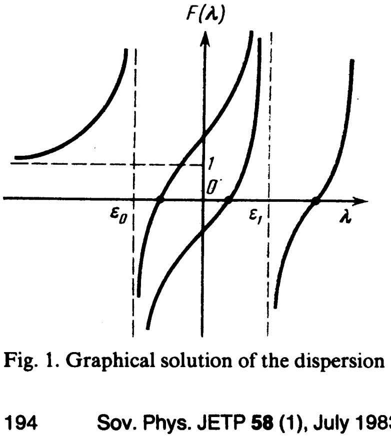

Fig. 1. Graphical solution of the dispersion equation (36).

in the form

$$\omega^2\sum_n\frac{\langle\sin 2\theta_0\vartheta_n\rangle^2}{\varepsilon_n}<-\frac{1}{B}-\langle\sin^2\theta_0\rangle<-\langle\sin^2\theta_0\rangle. \tag{37}$$

The choice of the quantity B in this formula is obvious and corresponds to the maximum value of the right-hand side of the inequality.

In order to calculate the sum in this inequality, we differentiate the equation for $\theta_0(\mathbf{x})$ with respect to ω and obtain

$$\hat{H}\,d\theta_0/d\omega=\omega\sin 2\theta_0.$$

Expanding $d\theta_0/d\omega$ in terms of ϑ_n, we easily obtain

$$\frac{d\theta_0}{d\omega}=\sum_n\frac{\omega}{\varepsilon_n}\langle\sin 2\theta_0\vartheta_n\rangle\vartheta_n,$$

whence

$$\omega\sum_n\frac{\langle\sin 2\theta_0\vartheta_n\rangle^2}{\varepsilon_n}=\int dx\sin 2\theta_0\left(\frac{d\theta_0}{d\omega}\right)=\frac{d}{d\omega}\left(\frac{N}{\omega}\right).$$

Substituting this value in the relation (37), we obtain the desired condition for stability of the soliton:

$$dN/d\omega<0. \tag{38}$$

A similar stability condition has been obtained for a number of field-theory models[2,12,17] and for the ferromagnet.[3-5]

We emphasize that the condition (38) relates only to a centrally-symmetric soliton with monotonically decreasing $\theta(r)$ (a soliton without nodes). A soliton with nodes is unstable regardless of the sign of $(dN/d\omega)$. We can show that these same conditions determine the stability of two-dimensional, axially symmetric solitons relative to two-dimensional perturbations.

Sometimes it is convenient to replace the frequencies ω with another quantity, in particular, the precession frequency Ω in the moving system of coordinates[6]

$$\varphi=-\Omega t+k(x-vt),\quad k=v\Omega/(c^2-v^2). \tag{39}$$

Comparing this equation with (17), it is easy to obtain a connection between ω and Ω:

$$\omega=\Omega(1-v^2/c^2)^{-1/2}.$$

Using this relation and the equations for P and N, (14) and (15), we can easily write the stability condition (38) in the form

$$\left(\frac{\partial N}{\partial\Omega}\frac{\partial P}{\partial v}-\frac{\partial P}{\partial\Omega}\frac{\partial N}{\partial v}\right)<0, \tag{40}$$

where $P=P(v,\Omega)$ and $N=N(v,\Psi)$ are the values of the integrals of motion of the soliton, expressed in terms of the solution in the set of coordinates moving with the speed of the soliton.

We now consider one-dimensional solitons. For a localized soliton of the type (25b), we can carry out all the same discussion as in the three-dimensional case. Since $d\theta_0/dx$ has one node, the Hamiltonian $\hat{H}$ in the case of a localized soliton has a single negative eigenvalue ε_0 and the dispersion equation has the same form as in Fig. 1. As a result, we come to the conclusion that the localized one-dimensional soliton is stable only upon satisfaction of the condition (38).

For solitons of the domain-wall type (25a), the situation is different. As before, $d\theta_0/dx$ corresponds to the zeroth eigenvalue of $\hat{H}$; however, in this case, this function does not have nodes and a minimum eigenvalue corresponds to it. Consequently, excluding the trivial displacement of the soliton as a whole, the Hamiltonian can be regarded as positive definite, and the soliton of the grain-boundary type is stable, independent of the sign of $dN/d\omega$.

In the next section, we shall obtain the explicit $N(\omega)$ dependence for three-dimensional solitons of both types and establish the fact that in specific problems, all the situations that we have considered can be realized.

3. CONCRETE STRUCTURE OF SOLITONS IN AFM

We now investigate the concrete form of solitons in a uniaxial AFM, choosing the anisotropy energy in the form (9). The equation for the angle θ in terms of the parameters ω and ω_t, introduced above in (11), is written in the form

$$c^2\Delta\theta-(\omega_1^2-\omega^2)\sin\theta\cos\theta+2(\omega_1^2-\omega_{\text{п}}^2)\sin^3\theta\cos\theta=0. \qquad (41)$$

In the one-dimensional case, the soliton solution of this equation can be written down in explicit form in terms of elementary functions. Analysis of the three-dimensional, centrally symmetric soliton of this equation and, in particular, of the $N(\omega)$ dependence can be carried out rather completely with the help of the same methods as used in Ref. 5 for the analysis of a soliton in a ferromagnet.

One-dimensional solitons. We first consider the one-dimensional solitons of the solution of Eq. (41), setting $\theta=\theta(x)$. Analysis shows[6,10] that Eq. (41) describes solitons of two different types, depending on the value of ω_t. If $\omega<\omega_t$ then the value of the angles θ tend to separate by π as $x\to+\infty$ and $x\to-\infty$.

Assuming that $\theta(+\infty)=0$, we obtain

$$\operatorname{tg}\theta=\left(\frac{\omega_1^2-\omega^2}{\omega_t^2-\omega^2}\right)^{1/2}\frac{1}{\operatorname{sh}[(x/c)(\omega_1^2-\omega^2)^{1/2}]},\qquad \omega^2<\omega_t^2. \qquad (42)$$

This solution describes a 180° domain wall of the AFM $[\theta(-\infty)=\pi,\ \theta(+\infty)=0]$. At the center of the soliton, $\theta=\pi/2$.

If $\omega_t^2<\omega^2<\omega_1^2$, then the soliton corresponds to the same value of the angle $\theta=0$ at $x\to+\infty$ and $x\to-\infty$:

$$\operatorname{tg}\theta=\left(\frac{\omega_1^2-\omega^2}{\omega^2-\omega_t^2}\right)^{1/2}\frac{1}{\operatorname{ch}[(x/c)(\omega_1^2-\omega^2)^{1/2}]},\qquad \omega_t^2<\omega^2<\omega_1^2. \qquad (43)$$

The value of the angle θ at the center of the soliton is a maximum. In this soliton, it is less than $\pi/2$. At $\omega^2>\omega_1^2$ Eq. (41) does not have soliton solutions.

The point $\omega=\omega_t$, just as ω_1, is singular. At this point the solution of Eq. (41) describes a 90° domain wall

$$\operatorname{tg}\theta=\exp[(x/c)(\omega_1^2-\omega_t^2)^{1/2}],$$

for which the energy and the number N are infinite. Such a behavior of the soliton distinguishes the case of the antiferromagnet from the case of the ferromagnet, considered earlier.[3–4] For the ferromagnet, besides the frequency of the homogeneous ferromagnetic resonance ω_0, the value $\omega=0$ is singular. The localized solitons in the ferromagnet exist for all $\omega<\omega_0$ except $\omega=0$; at $\omega=0$, the soliton goes over to a 180° domain wall.

We calculate the value of the number of spin deflections in a homogeneous soliton. For both types of solitons, the number N is determined by the single equation (see Fig. 2)

$$N=\frac{2s}{a}\left(\frac{2\beta\alpha}{b\delta}\right)^{1/2}\frac{\omega}{\omega_1}\ln\left|\frac{(\omega_1^2-\omega^2)^{1/2}+(\omega_1^2-\omega_t^2)^{1/2}}{(\omega_1^2-\omega^2)^{1/2}-(\omega_1^2-\omega_t^2)^{1/2}}\right|. \qquad (44)$$

At $\omega^2<\omega_t^2$, N increases with increase in ω but at such frequencies the soliton has the form of a domain wall and is stable regardless of the satisfaction of the condition (38). If $\omega_t^2<\omega^2<\omega_1^2$, which corresponds to localized soliton, then $dN/d\omega<0$ and the stability condition (38) is satisfied. Thus we come to the conclusion that one-dimensional precessional solitons in AFM are stable at all achievable values of ω.

Three-dimensional solitons. In the analysis of three-dimensional, centrally symmetric solitons, we must assume that $\theta=\theta(r)$, $r=|\mathbf{x}|$. The equation for the function $\theta(r)$ can be rewritten in the form

$$2r_0^2\left(\frac{d^2\theta}{dr^2}+\frac{2}{r}\frac{d\theta}{dr}\right)+\Omega\sin 2\theta-\sin 2\theta\cos 2\theta=0, \qquad (45)$$

where the notation

$$r_0^2=\frac{c^2}{\omega_1^2-\omega_t^2}=\frac{2\alpha}{b},\qquad \Omega=\frac{\omega^2-\omega_t^2}{\omega_1^2-\omega_t^2}, \qquad (46)$$

has been introduced. If we make the substitution $\theta\to\theta/2$ in this equation, then it transforms into Eq. (28) of Ref. 5. In Ref. 5, a study and numerical integration of this equation were carried out at various values of Ω.

It can be shown that the localized solutions of this equation can exist; we write the asymptotic solution as $r\to\infty$ in this case as

$$\theta\infty(1/r)\exp(-r(1-\Omega)^{1/2}/r_0).$$

If $\Omega>1$, then the falloff of $\theta(r)$ is connected only with the factor in front of the exponential in this expression, and the decrease is insufficiently rapid for the convergence of the integrals in E_0 and N.

In order to show the necessity of the condition $\Omega>0$, we multiply Eq. (45) by $(d\theta/dr)r^4$ and integrate it over r from $r=0$ to $r=\infty$. Carrying out the integration by parts, it is not difficult to obtain

$$\Omega\int_0^\infty\sin^2\theta r^3\,dr={}^1/_4\int_0^\infty\sin^2 2\theta\, r^3\,dr. \qquad (47)$$

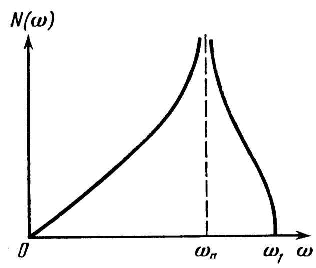

Fig. 2. Plot of $N(\omega)$ for a one-dimensional soliton (schematic).

It follows from the positiveness of the integrals in (47) that $\Omega > 0$.

Thus, the localized solitons exist at $0 < \Omega < 1$, i.e., $\omega_t^2 < \omega^2 < \omega_1^2$, in the same interval of values of the frequency as the localized one-dimensional solitons.

The form of the solution depends significantly on the parameter Ω. If $\Omega \ll 1$, then the soliton represents a spherical region of radius R, $R + 2r_0/\Omega \gg r_0$, in which the angle θ is close to $\pi/2$, and the magnetization $\mathbf{M}$ is almost parallel to the z axis, and is equal to $\widetilde{M}$, $\widetilde{M} = 4\omega/g\delta$. Change in the magnetization from $\theta \approx \pi/2$ to $\theta = 0$ takes place in a narrow range of values of r (of the order of $r_0 \ll R$).

If now $1 - \Omega \ll 1$, then the soliton has the following dependence:

$$\theta=(1-\Omega)^{1/2} f\left(\frac{r}{r_0}(1-\Omega)^{1/2}\right), \tag{48}$$

where $f(x)$ is some function with amplitude and region of localization of the order of unity.[5]

The properties of the solution just stated allow us to obtain the asymptotic dependence $N(\Omega)$ as $\Omega \to 0$ and $\Omega \to 1$. Actually, at small values of Ω ($\omega^2 - \omega_t^2 \ll \omega_1^2 - \omega_t^2$) the principal contribution to N is made by the spherical region of radius R, the density of magnetization of which is close to $\widetilde{M}$. Taking this circumstance into account, we obtain

$$N(\omega)=\frac{4\pi}{3}R^3\frac{\widetilde{M}}{2\mu_0}=\frac{1}{3}N_3\frac{\omega}{\omega_1}\left(\frac{\omega_1^2-\omega_t^2}{\omega^2-\omega_t^2}\right)^3,$$

$$\omega^2-\omega_t^2 \ll \omega_1^2-\omega_t^2, \tag{49}$$

where N_3 is the characteristic number of spin deflections:

$$N_3=64\pi s(\beta/\delta)^{1/2}(r_0/a)^3, \quad N_3 \gg 1. \tag{50}$$

In the other extreme case ($1-\Omega \ll 1$ or $\omega_1^2 - \omega^2 \ll \omega_1^2 - \omega_t^2$) the equation for $N(\omega)$ with account of (48) can be written in the form

$$N(\omega)=AN_3\frac{\omega}{\omega_1}\left(\frac{\omega_1^2-\omega_t^2}{\omega_1^2-\omega^2}\right)^{1/2}, \quad A=\frac{1}{8}\int f^2(x)x^2\,dx. \tag{51}$$

It is easily seen from (49)–(51) that, just as in the one-dimensional case, $N(\omega) \to \infty$ as $\omega \to \omega_t$. However, as $\omega \to \omega_1$, the behavior of $N(\omega)$ is different in the three-dimensional soliton than in the one-dimensional, namely, $N(\omega) \propto (\omega_1^2 - \omega^2)^{-1/2}$. Thus, $N(\omega)$ increases without limit both as $\omega \to \omega_t$ and $\omega \to \omega_1$. This means that $dN/d\omega$ changes sign at some value of the frequency $\omega = \omega_*$, $\omega_t < \omega < \omega_*$. Consequently, the soliton is stable at $\omega_t < \omega < \omega_*$ and unstable at $\omega_* < \omega < \omega_1$.

For the analysis of the dependence $N(\omega)$ at $1 - \Omega \sim \Omega \sim 1$, it is necessary to use the results of the numerical integration of Eq. (45). In contrast with the ferromagnet,[5] the energy anisotropy of the AFM, which admits of existence of solitons, depends on two parameters, β and b or ω_1 and ω_p. It turns out that the form of the specific $N(\omega)$ dependence and the value of ω_* depend on the relation between ω_1 and ω_t. Figure 3 shows the dependence of N on the quantity

$$\Delta\omega=(\omega_1-\omega)/(\omega_1-\omega_t)$$

in two extreme cases: $\omega_t \ll \omega_1$ and $\omega_1 - \omega_t \ll \omega_1$. In both cases,

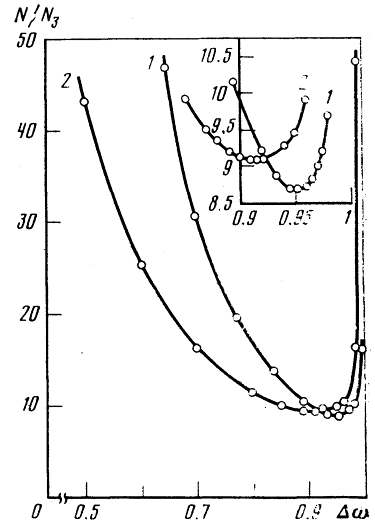

Fig. 3. Dependence of N on $\Delta\omega$ for different relations of ω_1 and ω_t. Curve 1 corresponds to $\omega_t \ll \omega_1$, curve 2 corresponds to $\omega_1 - \omega_t \ll \omega_1$. The points indicate the results of numerical calculation.

the quantity ω_* is close to ω_1:

$$\omega_*=0.96\omega_1 \quad \text{at} \quad \omega_t \ll \omega_1,$$

$$\omega_*=\omega_t+0.915(\omega_1-\omega_t) \quad \text{at} \quad \omega_1-\omega_t \ll \omega_1.$$

Thus the three-dimensional solitons in AFM are stable over a wide range of frequencies, from ω_t to ω_*. We note that the separation of the three-dimensional solitons into stable low-frequency and unstable high-frequency is characteristics for most of the systems, known at the present time, in which three-dimensional solitons can exist (see Refs. 2, 3, 5 and 12).

4. CONCLUSION

We have proved the existence of stable soliton solutions, including three-dimensional, spherically symmetric, in uniaxial AFM in a description of this magnet in terms of the effective equations for the antiferromagnetism unit vector $\mathbf{l}$.[2] However, the equations of such a type appear in a number of other branches of physics. For example, at the present time, Lorentz-invariant models of the dynamics of a unit vector are widely discussed in the nonlinear field theory (see, for example, Ref. 18). Equation (2) is the anisotropic generalization of the chiral σ-model of the $\mathbf{n}$ field,[18] and the existence of stable solitons in this model can represent interest for field theory.

The state of the A phase of superfluid helium-3 is described by unit vectors: $\mathbf{l}$ and $\mathbf{d}$ (see Ref. 19). A deep analogy between the dynamics of He_3A and the antiferromagnet was noticed by Anderson (Ref. 19, p. 212). If we assume that in the texture of He_3A the vector $\mathbf{l}$ does not depend on the coordinates, then the Leggett equations, which are described in terms of the vector $\mathbf{d}$, coincide literally with the equation for the antiferromagnetism vector [see Eq. (25) of Ref. 20].

The dipole interaction, which fixes the mutual orientation of $\mathbf{l}$ and $\mathbf{d}$, plays the role of the energy of the anisotropy for the vector $\mathbf{d}$, while the direction of $\mathbf{l}$ fixes the axis of the anisotropy. In superfluid He_3, only topological, non-one-dimensional solitons have been discussed to date,[21] while the stability of the solitons considered in the present work is determined not by the presence of a topological charge, but by the properties of the dynamical equation (2). The corresponding dynamical equation for antiferromagnetism and He_3 allow us to describe the dynamical solitons in He_3A.

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[1)]In Ref. 8, Eq. (2) was obtained from analysis of the dynamical symmetry of AFM without use of the sublattice model.

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