Isomorphism of certain problems of percolation theory

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The exact correspondence (isomorphism) between certain problems of the theory of transport phenomena in two-component media is discussed. Exact relations are established between the effective characteristics of the isomorphic problems. Such relations allow us to express the thermoelectric properties of a two-component (two- or three-dimensional) medium in terms of the solution to the standard electrical-conductivity problem for the medium. Similarly, the galvanomagnetic properties of an arbitrary two-dimensional two-component system can be expressed in terms of the conductivity of the medium in zero magnetic field. One of the critical exponents of the effective Hall coefficient is determined in the three-dimensional case. The isomorphism of two models of the anisotropic percolation theory is established.

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1. INTRODUCTION

The theoretical investigation of the various properties of inhomogeneous media is of considerable interest. But even the investigation of the "simplest" problem of percolation theory—the problem of the conductivity of two-component disordered systems—meets with fundamental difficulties, which have not been overcome up to now. It is natural that the difficulties increase when we go on to consider more complicated problems, e.g., the problems of the galvanomagnetic, thermoelectric, etc., effects. Therefore, there is a paucity of analytic results in the theory of transport processes in disordered media. Somewhat more favorable in this respect is the situation in the two-dimensional case, where certain problems for isotropic two-component randomly-inhomogeneous systems have been solved: the electrical conductivity,[1] the galvanomagnetic properties in a transverse magnetic field[2] (see also Ref. 3), and the thermoelectric properties[4] for different concentrations of the components. These are virtually all the exact analytic results that have been obtained for disordered media.

A central position in percolation theory is occupied by the scaling law hypothesis,[5,6] within the framework of which important physical ideas have been formulated and a number of valuable results have been obtained. At the same time the applicability of this hypothesis is limited to systems that undergo the metal-insulator phase transition in the critical region. But even here the scaling hypothesis actually provides only a qualitative description of the phenomena, leading only to relations between the critical exponents.[5,6] Therefore, to carry out a quantitative study of the properties of disordered media, we must resort to numerical and simulation experiments, which are fairly laborious even for the simplest problems.

In view of all that has been said, every kind of exact relation that can be established among the effective characteristics of inhomogeneous media is of particular importance in percolation theory. As examples of such relations in the two-dimensional case, we can cite various reciprocity relations,[1–4,7] as well a Dykhne's general relation (see Refs. 2, 3, and[1)] 8) in the problem of the galvanomagnetic effects. In all these cases the relations are established among the effective characteristics of the systems within the framework of one and the same problem, which allows us to obtain a number of exact results and relations.[1–4, 7–9] Of greatest interest, however, are the cases in which it is possible to establish an exact one-to-one correspondence, called below isomorphism, between problems of different physical natures.

In the present paper we discuss the isomorphism of certain problems of the theory of transport phenomena in two-component media. The existence of a one-to-one correspondence between any two problems is due to the presence of a nonsingular symmetry transformation that converts one problem into the other. The isomorphism enables us to derive exact relations (isomorphism relations) among the effective characteristics of these problems, which allows us to obtain results that are more interesting than those obtainable from the reciprocity relations. The isomorphism relations do not eliminate the fundamental difficulties, but reduce the original problem to a simpler problem or to one that has been studied in greater detail. Furthermore, it is possible to establish among the various effective characteristics pertaining to one problem relations (of the type of Dykhne's general relation) that do not depend on the specific structure of the system.

In Sec. 2 we find a symmetry transformation that allows us to reduce the problem of the thermoelectric properties of an isotropic two-component (two- or three-dimensional) system to the problem of the electrical conductivity of a system of the same structure. The effective thermoelectric characteristics are expressed in terms of the properties of the components and a function f describing the electrical conductivity of the original system without allowance for the thermoelectric effects. The general formulas obtained allow us to fully consider the critical behavior of the thermoelectric coefficients within the framework of the scaling law hypothesis[5,6]; no new critical exponents arise in the problem.

Using the symmetry transformation proposed by Dykhne,[2] we establish in Sec. 3 an isomorphism between the problem of the galvanomagnetic properties in a transverse magnetic field $\mathbf{H}$ and the problem of the electrical conductivity in $\mathbf{H}=0$ for the case of arbitrary two-dimensional anisotropic two-component systems. (A similar correspondence is established by another method in Ref. 8 for the

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isotropic case.) The components of the effective conductivity tensor $\hat{\sigma}_e$ are expressed in terms of the properties of the individual components and the two fundamental functions of the two-dimensional anisotropic percolation theory (see, for example, Ref. 7). We obtain an anisotropic analogue of Dykhne's general relation.

The general formulas for $\hat{\sigma}_e$ allow us to solve the problem of the conductivity of an isotropic two-component film in an oblique magnetic field **H** as well. As the intensity of **H** is increased, the smearing out region[6] of such a system changes, in contrast to the case of the transverse magnetic field,[8] in a nonmonotonic fashion, which leads to quite a complicated dependence of $\hat{\sigma}_e$ on **H**. It is sufficient to note that we can distinguish in the neighborhood of the critical point in the case when the parameters are connected by a certain relation eight magneticfield regions with different **H** dependences of $\hat{\sigma}_e$. Generally speaking, in the oblique magnetic field case the conductivity behaves anomalously[2] in two finite regions of **H**.

The galvanomagnetic characteristics of a three-dimensional medium can be expressed in terms of its electrical conductivity in two cases: 1) when the medium contains perfectly conducting inclusions and 2) when the Hall component of the conductivity tensor of the medium does not depend on the coordinates (see Sec. 4). This, in particular, allows us to determine one of the critical exponents of the effective Hall constant. In Sec. 5 we establish an isomorphism between two models of the anisotropic percolation theory.[7,10] We also consider the connection between the problem of the permittivity of a system with metallic inclusions and the problem of the conductivity of a structurally identical system containing perfectly conducting inclusions.[6,11]

2. THERMOELECTRIC PROPERTIES

2.1. The symmetry transformation

The thermoelectric, thermal-diffusion, diffusional-electric, etc., effects[12,13] are described by a set of fields $\mathbf{E}_a$ and a set of current densities $\mathbf{j}_a$ $(a = 1,..., n)$ satisfying the equations

$$\operatorname{rot} \mathbf{E}_a=0, \quad \operatorname{div} \mathbf{j}_a=0. \tag{1}$$

For an isotropic medium in zero external magnetic field the linear relation between $\mathbf{j}_a$ and $\mathbf{E}_a$ is given by "Ohm's law:"

$$\mathbf{j}_a=\sum_b \sigma_{ab}\mathbf{E}_b, \tag{2}$$

where the kinetic-coefficient matrix $\hat{\sigma}$ in an inhomogeneous system depends on the coordinates. We assume the quantities $\mathbf{j}_a$ and $\mathbf{E}_a$ to be conjugate quantities,[14] so that the matrix $\hat{\sigma}$ is symmetric.[14]

Let us carry out a linear transformation of the fields and the currents:

$$\mathbf{E}_a=\sum_b M_{ab}\mathbf{E}_b', \quad \mathbf{j}_a=\sum_b N_{ab}\mathbf{j}_b'. \tag{3}$$

Let us assume that the matrices $\hat{M}$ and $\hat{N}$ do not depend on the coordinates and that they have inverses. The equations for the currents and fields in the primed system have the form (1), (2) with the kinetic-coefficient matrix:

$$\hat{\sigma}'(\mathbf{r})=\hat{N}^{-1}\hat{\sigma}(\mathbf{r})\hat{M}. \tag{4}$$

Let us require that the fields and currents in the primed system also be conjugate quantities. Then

$$\hat{M}\hat{N}^T=1, \tag{5}$$

where the superscript T denotes transposition. The relation between the effective characteristics $\hat{\sigma}_e$ and $\hat{\sigma}_e'$ in the original and primed systems, which coincides in form with (4), can be written with allowance for (5) in the form

$$\hat{\sigma}_e=\hat{N}\hat{\sigma}_e'\hat{N}^T. \tag{6}$$

The formulas (3)–(6) establish a one-to-one correspondence between the original and the primed systems, and are valid for arbitrary isotropic media (both two- and three-dimensional ones), including those for which $\hat{\sigma}$ depends continuously on the coordinates.

Let us consider a two-component system for which the matrix $\hat{\sigma}$ takes on constant values $\hat{\sigma}^{(1)}$ and $\hat{\sigma}^{(2)}$ in the first and second components respectively. Then we obtain from (4) the two matrix equations

$$\hat{A}\hat{M}=\hat{N}\hat{A}', \quad \hat{B}\hat{M}=\hat{N}\hat{B}'. \tag{7}$$

In (7), in order to reduce the number of indices in the subsequent formulas, we have introduced the notation: $\hat{\sigma}^{(1)}=\hat{A}$, $\hat{\sigma}^{(2)}=\hat{B}$, $\hat{\sigma}^{(1)\prime}=\hat{A}'$, $\hat{\sigma}^{(2)\prime}=\hat{B}'$. Let us now require that the kinetic-coefficient matrix in the primed system be diagonal:

$$A_{ab}'=A_a'\delta_{ab}, \quad B_{ab}'=B_a'\delta_{ab}, \tag{8}$$

so that the relation between $\mathbf{j}_b'$ and $\mathbf{E}_b'$ has the form of the conventional Ohm law:

$$\mathbf{j}_b'=\sigma_b'(\mathbf{r})\mathbf{E}_b', \tag{9}$$

although the quantity $\sigma_b'(\mathbf{r})$ may not have the meaning of electrical conductivity. Let us define the effective characteristics of the primed system in the standard fashion:

$$\langle\mathbf{j}_b'\rangle=\sigma_{be}'\langle\mathbf{E}_b'\rangle; \quad \sigma_{be}'=A_b'f(p, \lambda_b), \quad \lambda_b=B_b'/A_b', \tag{10}$$

where p is the concentration of the first component. Let us note that the function $f(p, \lambda_b)$ in (10) is the same as in the problem of the electrical conductivity of a system with the same structure as the original system, and with components the ratio of whose conductivities is equal to λ_b. Thus, if the system of equations (7) with the conditions (8) possesses a solution, then the formulas (3)–(6) and (10) establish an exact correspondence (isomorphism) between the original problem and the problem of the electrical conductivity of the system in question.

In the practically interesting case of $n = 2$ the solution to Eqs. (5), (7), and (8) can be found in the explicit form. Let us set $A_1'=A_{11}$, $B_1'=\lambda A_{11}$, $A_2'=A_{22}$, and $B_2'=\mu A_{22}$. Then for the elements of the matrix $\hat{N}$ we obtain

$$\begin{aligned} N_{11}&=A_{11}^{-1/2}[(\mu A_{11}-B_{11})/(\mu-\lambda)]^{1/2};\\ N_{12}&=A_{22}^{-1/2}[(B_{11}-\lambda A_{11})/(\mu-\lambda)]^{1/2}\operatorname{sign}(B_{12}-\lambda A_{12});\\ N_{21}&=A_{11}^{-1/2}[(\mu A_{22}-B_{22})/(\mu-\lambda)]^{1/2}\operatorname{sign}(\mu A_{12}-B_{12});\\ N_{22}&=A_{22}^{-1/2}[(B_{22}-\lambda A_{22})/(\mu-\lambda)]^{1/2} \end{aligned} \tag{11}$$

(21) can be obtained in the $\alpha_i \to 0$ case from the equations of the effective-medium theory.[4,15] We can verify by a direct substitution that, for arbitrary α_i, the expressions (16) and (17) are a solution to the corresponding equations (see the formulas (26), (28), and (34) in Ref. 4) if the function $f(p, h)$ is also defined within the framework of the effective-medium theory (see, for example, the formula (5.7) in Ref. 16).

2.3. The critical region

If the conductivities of the components differ greatly from each other (i.e., if $\sigma_2 \ll \sigma_1$), then a metal-insulator phase transition occurs in the system at $p = p_c$ (p_c is the critical concentration).[16,17] The general formulas (16) allow us to investigate the behavior of the thermoelectric coefficient α_e in the critical region. Let us, as in Ref. 15, consider the linear—in the α_i—approximation and the case in which $\sigma_2 \ll \sigma_1$, $\alpha_2 \gg \alpha_1$, $\varkappa_2 \sim \varkappa_1$. Then from (21) we have

$$\alpha_e \approx \alpha_1 \left[1 + \frac{\alpha_2}{\alpha_1} \frac{\sigma_2}{\sigma_1} \frac{\varkappa_1}{\varkappa_2} \left(\frac{f_\varkappa}{f_\sigma} - 1 \right) \right], \qquad (22)$$

where $f_0 = f(p, \sigma_2/\sigma_1)$ and $f_\varkappa = f(p, \varkappa_2/\varkappa_1)$.

When $\varkappa_2 \sim \varkappa_1$, the function $f_\varkappa \sim 1$ for all concentrations. As to the function f_σ, it behaves in the critical region $|\tau| \ll 1$, $h \ll 1$, ($\tau = (p - p_c)/p_c$ $h = \sigma_2/\sigma_1$) in the following manner[6]:

$$\tau>0,\ \Delta \ll \tau \ll 1:\quad f_\sigma \sim \tau^t, \qquad (23a)$$

$$|\tau| \ll \Delta:\quad f_\sigma \sim h^s, \qquad (23b)$$

$$\tau<0,\ \Delta \ll |\tau| \ll 1:\quad f_\sigma \sim h|\tau|^{-q}. \qquad (23c)$$

Here $\Delta = h^{s/t}$ is the dimension of the smearing-out region[6]; the critical exponents t, s, and q are connected by the scaling-hypothesis relation[6] $q = t(1-s)/s$.

If the inequalities $(\sigma_2/\sigma_1)^s \ll \alpha_2\sigma_2/\alpha_1\sigma_1 \ll 1$ are satisfied, then the thermoelectric coefficient α_e varies appreciably in the critical region $|\tau| \ll 1$. From (22) and (23) we have

$$\tau>0,\quad \tau^t \gg \frac{\alpha_2\sigma_2}{\alpha_1\sigma_1}:\quad \alpha_e \approx \alpha_1; \qquad (24a)$$

$$\tau>0,\quad \left(\frac{\sigma_2}{\sigma_1}\right)^s \ll \tau^t \ll \frac{\alpha_2\sigma_2}{\alpha_1\sigma_1}:\quad \alpha_e \sim \frac{\alpha_2}{\tau^t} \frac{\sigma_2}{\sigma_1}; \qquad (24b)$$

$$|\tau|^t \ll (\sigma_2/\sigma_1)^s:\quad \alpha_e \sim \alpha_2 (\sigma_2/\sigma_1)^{1-s}; \qquad (24c)$$

$$\tau<0,\quad |\tau|^t \gg (\sigma_2/\sigma_1)^s:\quad \alpha_e \sim \alpha_2 |\tau|^q. \qquad (24d)$$

Thus, the quantity α_e increases sharply when we go from $\tau > 0$ to $\tau < 0$.

The result (24a) is obtained in Ref. 15 (without any indication of the exact condition for its applicability); in this concentration range the second component can be considered to be nonconducting, so that, according to (19), $\alpha_e \approx \alpha_1$. In the "dielectric" phase ($\tau < 0$), instead of (24d), the result $\alpha_e \sim \alpha_2|\tau|$ is obtained in Ref. 15 within the framework of the effective-medium theory. The value of α_e used in Ref. 15 was also determined with the aid of a numerical experiment. It can be shown that the formulas (16) and (17) and, in consequence, (21), (22), and (24) are also valid for the lattice models. The expressions (22) and (24) agree qualitatively with the numerical results reported in Ref. 15. Unfortunately, there are no data for σ_e and $\varkappa_e$ in Ref. 15, which makes the quantitative verification of the formulas (21) and (22) impossible.

3. THE GALVANOMAGNETIC PROPERTIES. THE TWO-DIMENSIONAL CASE

3.1. The symmetry transformation

Let us consider an inhomogeneous anisotropic two-dimensional system located in a transverse magnetic field **H**, and possessing a conductivity tensor $\hat{\sigma}(\mathbf{r})$ having (in the principal axes) the form

$$\hat{\sigma} = \begin{pmatrix} \sigma_x & \sigma_a \\ -\sigma_a & \sigma_y \end{pmatrix}. \qquad (25)$$

Here σ_a is the Hall component of the tensor $\hat{\sigma}(\mathbf{r})$. In the general case the quantities σ_x, δ_y, and σ_a and the directions of the principal axes are functions of the coordinates x and y. We assume that the third principal axis of the full conductivity tensor is oriented along the normal to the plane of the system; this secures the two-dimensionality of the field and current distributions.

Let us, following Dykhne,[2] transform the electric field **E** and the current density **j** into the primed system:

$$\mathbf{E} = a(\mathbf{E}' + b[\mathbf{n} \times \mathbf{j}']), \quad \mathbf{j} = a(c\mathbf{j}' + d[\mathbf{n} \times \mathbf{E}']), \qquad (26)$$

where a, b, c, and d are constant coefficients and **n** is the unit vector along the normal to the plane (x, y) of the system. The transformation (26) leaves the equations for the constant current unchanged; the conductivity tensor $\hat{\sigma}'$ in the primed system also has the form (25), with the components given by

$$\sigma_\alpha'(\mathbf{r}) = \frac{(c+bd)\sigma_\alpha(\mathbf{r})}{[c - b\sigma_a(\mathbf{r})]^2 + b^2\sigma_x(\mathbf{r})\sigma_y(\mathbf{r})} \quad (\alpha = x, y);$$

$$\sigma_a'(\mathbf{r}) = \frac{[c - b\sigma_a(\mathbf{r})][d + \sigma_a(\mathbf{r})] - b\sigma_x(\mathbf{r})\sigma_y(\mathbf{r})}{[c - b\sigma_a(\mathbf{r})]^2 + b^2\sigma_x(\mathbf{r})\sigma_y(\mathbf{r})}. \qquad (27)$$

Relations having exactly the same form are valid for the effective characteristics $\sigma_{\alpha e}'$, σ_{ae}' and $\sigma_{\alpha e}$, σ_{ae}. Below we shall need the expressions for $\sigma_{\alpha e}$ and σ_{ae} in terms of $\sigma_{\alpha e}'$ and σ_{ae}'. Solving (27) for σ_x, σ_y, and σ_a, and making the substitutions $\hat{\sigma} \to \hat{\sigma}_e$ and $\hat{\sigma}' \to \hat{\sigma}_e'$, we obtain for the components of the effective conductivity tensor $\hat{\sigma}_e$ the expressions

$$\sigma_{\alpha e} = \frac{(c+bd)\sigma_{\alpha e}'}{(1 + b\sigma_{ae}')^2 + b^2\sigma_{xe}'\sigma_{ye}'} \quad (\alpha = x, y),$$

$$\sigma_{ae} = \frac{-(1 + b\sigma_{ae}')(d - c\sigma_{ae}') + bc\sigma_{xe}'\sigma_{ye}'}{(1 + b\sigma_{ae}')^2 + b^2\sigma_{xe}'\sigma_{ye}'}. \qquad (28)$$

The expressions (26)–(28) give the exact relation between the original and primed systems, i.e., a relation which is valid for any dependence of $\hat{\sigma}$ (and the directions of the principal axes) on the coordinates.

For two-component isotropic ($\sigma_x = \sigma_y$) media, the relations (27) and (28) allow us to obtain the fundamental results of Refs. 2, 3, and 18. Thus, if as the primed system we choose the mutual system, i.e., the system obtained from the original system by interchanging the components (i.e., by making the interchange $\hat{\sigma}_1 \rightleftarrows \hat{\sigma}_2$, where $\hat{\sigma}_i$ is the conductivity tensor of the i-th component) and changing the sign of the magnetic where μ and λ ($\mu > \lambda > 0$) are the roots of the quadratic equation

$$(\xi A_{11}-B_{11})(\xi A_{22}-B_{22})-(\xi A_{12}-B_{12})^2=0. \tag{12}$$

Since the fields and the currents have been chosen to be conjugate quantities, the roots of Eq. (12) are nonnegative, and the elements of the matrix $\hat{N}$ from (11) are real.

The substitution of (11) into (6) with allowance for (10) (for $A'_1 = A_{11}, A'_2 = A_{22}$) allows us to determine the effective kinetic-coefficient matrix ($\hat{\sigma}^{(1)} = \hat{A}$, $\hat{\sigma}^{(2)} = \hat{B}$):

$$\hat{\sigma}_e=[(\mu\hat{\sigma}^{(1)}-\hat{\sigma}^{(2)})f_\lambda+(\hat{\sigma}^{(2)}-\lambda\hat{\sigma}^{(1)})f_\mu](\mu-\lambda)^{-1}. \tag{13}$$

Here we have, for brevity, introduced the notation

$$f_\lambda=f(p,\lambda),\quad f_\mu=f(p,\mu). \tag{14}$$

The functions f_λ and f_μ are determined, in accordance with (10), in the following manner. Let us write the effective electrical conductivity $\hat{\sigma}_e$ (without allowance for the thermoelectric effects) of the system under consideration in the form $\hat{\sigma}_e = \sigma_1 f(p, h)$, where $h = \sigma_2/\sigma_1$, σ_1 and σ_2 being the conductivities of the first and second components, and p is the concentration of the first component. Then the functions f_λ and f_μ are obtained from $f(p, h)$ by making the substitutions $h \to \lambda$ and $h \to \mu$.

2.2 The effective thermoelectric characteristics

For the thermoelectric effects the kinetic-coefficient matrix of the i-th component has the form (see, for example, Ref. 4)

$$\hat{\sigma}^{(i)}=\begin{pmatrix}\sigma_i & \gamma_i\\ \gamma_i & \chi_i\end{pmatrix};\quad \gamma_i=\sigma_i\alpha_i,\quad \chi_i=T^{-1}\varkappa_i+\sigma_i\alpha_i^2. \tag{15}$$

Here σ is the electrical conductivity, α is the thermoelectric coefficient, $\varkappa$ is the thermal conductivity, and T is the temperature. From (13) and (15) we find the effective thermoelectric characteristics of the system

$$\sigma_e=\frac{(\mu\sigma_1-\sigma_2)f_\lambda-(\lambda\sigma_1-\sigma_2)f_\mu}{\mu-\lambda},$$
$$\alpha_e=\frac{(\mu\gamma_1-\gamma_2)f_\lambda-(\lambda\gamma_1-\gamma_2)f_\mu}{(\mu\sigma_1-\sigma_2)f_\lambda-(\lambda\sigma_1-\sigma_2)f_\mu}, \tag{16}$$
$$\varkappa_e=\frac{\sigma_1\varkappa_1(\mu-\lambda)f_\lambda f_\mu}{(\mu\sigma_1-\sigma_2)f_\lambda-(\lambda\sigma_1-\sigma_2)f_\mu},$$

where f_λ and f_μ are the same functions figuring in (14) and μ and λ are given by the expressions

$$\begin{Bmatrix}\mu\\ \lambda\end{Bmatrix}=\frac{1}{4\sigma_1\varkappa_1}\{[(\sqrt{\sigma_1\varkappa_2}+\sqrt{\sigma_2\varkappa_1})^2+\sigma_1\sigma_2T(\alpha_1-\alpha_2)^2]^{1/2} \tag{17}$$
$$\pm[(\sqrt{\sigma_1\varkappa_2}-\sqrt{\sigma_2\varkappa_1})^2+\sigma_1\sigma_2T(\alpha_1-\alpha_2)^2]^{1/2}\}^2.$$

Notice that the formulas (16) are invariant under the interchange $\mu \rightleftarrows \lambda$, so that the quantities σ_e, α_e, and $\varkappa_e$ do not depend on the choice of the signs in (17). We shall assume that λ and μ have been chosen such that $\lambda \to \sigma_2/\sigma_1$ and $\mu \to \varkappa_2/\varkappa_1$ as $\alpha_i \to 0$.

The expressions (16) are valid for both two- and three-dimensional isotropic two-component systems of arbitrary structure. All the information about the form and distribution of the inclusions is contained in the function f. If, for any two-component system $f = f(p, h)$ is known for all p and h, then the formulas (16) give the complete solution to the problem of the thermoelectric properties of such a system. Notice that the elimination of the functions f_λ and f_μ from (16) leads to a general relation connecting the quantities σ_e, α_e, and $\varkappa_e$, and not depending on the specific structure of the medium:

$$\frac{\varkappa_e}{\sigma_e}=T[(\gamma_1\chi_2-\gamma_2\chi_1)$$
$$-(\sigma_1\chi_2-\sigma_2\chi_1)\alpha_e-(\sigma_2\gamma_1-\sigma_1\gamma_2)\alpha_e^2](\sigma_2\gamma_1-\sigma_1\gamma_2)^{-1}. \tag{18}$$

We can judge the "degree of two-componentness" of real samples by the accuracy with which the relation (18) is satisfied. The same remark applies to the formula (21) in the linear—in α_i—approximation.

Let us consider some particular cases. If one of the components is a dielectric (i.e., if $\sigma_2 = 0$), then $\lambda = 0, \mu = \varkappa_2/\varkappa_1$, and from (16) it follows (we assume that the concentration p is higher than the critical concentration p_c, so that $\sigma_e \neq 0$) that

$$\sigma_e=\sigma_1 f(p,0),\quad \alpha_e=\alpha_1,\quad \varkappa_e=\varkappa_1 f(p,\varkappa_2/\varkappa_1). \tag{19}$$

Thus, in accord with Ref. 15, when nonconducting impurities are introduced into a medium, the thermoelectric coefficient of the system does not change as the impurity concentration is raised right up to the percolation threshold. In the case when the thermoelectric coefficient does not depend on the coordinates i.e., (when $\alpha_1 = \alpha_2 = \alpha$), we have $\lambda = \sigma_2/\sigma_1$ and $\mu = \varkappa_2/\varkappa_1$, and from (16) we find, in accord with the arguments adduced in Ref. 4, that

$$\sigma_e=\sigma_1 f(p,\sigma_2/\sigma_1),\quad \alpha_e=\alpha,\quad \varkappa_e=\varkappa_1 f(p,\varkappa_2/\varkappa_1). \tag{20}$$

Thus, for $\alpha_1 = \alpha_2$, the thermoelectric effects do not give rise to corrections to σ_e and $\varkappa_e$.

For a two-dimensional randomly-inhomogeneous medium with the critical concentration $p = \frac{1}{2}$, exact expressions are found for σ_e, α_e, and $\varkappa_e$ with the aid of the reciprocity relations in Ref. 4 (see also Ref. 3). The corresponding computations are quite tedious. Let us note that the same results can easily be obtained from the general formulas (16) if we take account of the fact that in this case $f(\frac{1}{2}, h) = h^{1/2}$ (Ref. 1). The substitution of the expressions $f_\lambda = \lambda^{1/2}$ and $f_\mu = \mu^{1/2}$ into (16) leads to the formulas (17)–(19) in Ref. 4.

Usually, $T\sigma_i\alpha_i^2/\varkappa_i \ll 1$ ($i = 1, 2$), and the influence of the thermoelectric effects on the quantities σ_e and $\varkappa_e$ is slight. It is not difficult to see from (15) and (17) that, for $\alpha_i \to 0$, the corrections to σ_e and $\varkappa_e$ are quadratic in the α_i. Therefore, in the linear—in α_i—approximation we obtain from (16) the expression

$$\alpha_e=\frac{1}{\sigma_1\varkappa_2-\sigma_2\varkappa_1}\left[\alpha_1\sigma_1\varkappa_2-\alpha_2\sigma_2\varkappa_1-\sigma_1\sigma_2(\alpha_1-\alpha_2)\frac{\varkappa_e}{\sigma_e}\right], \tag{21}$$

where σ_e and $\varkappa_e$ are the same quantities figuring in (20). Notice that this result for $\alpha_i \to 0$ also follows from (18).

Let us note that a relation that coincides in form with field, then from (27) and (28) we obtain the reciprocity relations given in Refs. 2 and 3. But if as the primed system we take the initial system with the magnetic field reversed, then from (27) and (28) we obtain Dykhne's general relation[2] (see also Refs. 3, 8, and 18). The relations (27) and (28) for $\sigma_x \neq \sigma_y$ allow us to generalize the results of Refs. 2, 3, and 18 to the anisotropic case.

But we obtain the most interesting results from (27) and (28) if as the primed system we choose a system with a "natural" anisotropy (i.e., a system with zero Hall components), whose conductivity tensor in the principal axes has the form

$$\hat{\sigma}'=\begin{pmatrix}\lambda_x & 0\\ 0 & \lambda_y\end{pmatrix}. \tag{29}$$

As in Ref. 8, we shall call such a system the null system. The original and the null systems have the same structure, and the tensors $\hat{\sigma}(\mathbf{r})$ from (25) and $\hat{\sigma}'(\mathbf{r})$ from (29) have the same principal axes at every point. Thus, the null system is in fact the original system (in $\mathbf{H}=0$) with the conductivities of the components changed.

Setting $\sigma'_{ai}=0$, $\sigma'_{xi}=\lambda_{xi}$, $\sigma'_{yi}=\lambda_{yi}$ $(i=1, 2)$ in (27), we obtain six equations, from which we determine the transformation coefficients b, c, and d and the three dimensionless parameters $\lambda_{x1}/\lambda_{y1}$, $\lambda_{x2}/\lambda_{y2}$, and $\lambda_{y2}/\lambda_{y1}$ of the null system. As a result, we obtain

$$b=B/\lambda_{y1},\quad c=(\sigma_{y1}+\sigma_{a1}B)/\lambda_{y1},\quad d=-\sigma_{a1}+\sigma_{x1}B,\quad a=(c+bd)^{-1/2};$$
$$B=(\sigma_{a1}-\sigma_{a2})/(\sigma_{x1}-\lambda\sigma_{x2}); \tag{30}$$

$$\lambda_{x1}/\lambda_{y1}=\sigma_{x1}/\sigma_{y1},\quad \lambda_{x2}/\lambda_{y2}=\sigma_{x2}/\sigma_{y2},\quad \lambda_{y2}/\lambda_{y1}=\lambda, \tag{31}$$

where

$$\lambda=\frac{1}{4\sigma_{x2}\sigma_{y1}}\{[((\sigma_{x1}\sigma_{y1})^{1/2}+(\sigma_{x2}\sigma_{y2})^{1/2})^2+(\sigma_{a1}-\sigma_{a2})^2]^{1/2}$$
$$-[((\sigma_{x1}\sigma_{y1})^{1/2}-(\sigma_{x2}\sigma_{y2})^{1/2})^2+(\sigma_{a1}-\sigma_{a2})^2]^{1/2}\}^2. \tag{32}$$

In (30) we have also given the coefficient a determined from the condition $\mathbf{j}\cdot\mathbf{E}=\mathbf{j}'\cdot\mathbf{E}'$. The parameter λ is a root of the equation

$$(\lambda\sigma_{y1}-\sigma_{y2})(\sigma_{x1}-\lambda\sigma_{x2})+\lambda(\sigma_{a1}-\sigma_{a2})^2=0. \tag{33}$$

The sign in front of the second square bracket in (32) was chosen from the requirement that $\lambda\to\sigma_{y2}/\sigma_{y1}$ as $\sigma_{ai}\to 0$, and corresponds to the case $\sigma_{x1}\sigma_{y1}>\sigma_{x2}\sigma_{y2}$. Thus, the expressions (25)–(32) establish an isomorphism between the problems of the electrical conductivity and the galvanomagnetic properties of an arbitrary two-dimensional two-component system.

3.2. The effective galvanomagnetic characteristics

The isomorphism established in the preceding subsection allows us to express the galvanomagnetic characteristics of the system in question in terms of its electrical conductivity in zero magnetic field. Let us note for this purpose that the effective conductivity tensor $\hat{\sigma}'_e$ of the null (two-component) system depends on five parameters: p, the concentration of the first component, and λ_{x1}, λ_{y1}, λ_{x2}, λ_{y2}. Let us introduce for a null system that as a whole is anisotropic two functions f_x and f_y according to the equations

$$\sigma_{\alpha e}'=\lambda_{\alpha 1}f_\alpha(p;\ \lambda_{x1}/\lambda_{y1},\ \lambda_{x2}/\lambda_{y1},\ \lambda_{y2}/\lambda_{y1}),\quad \alpha=x,\ y. \tag{34}$$

The quantities f_x and f_y depend on four dimensionless arguments, and are the fundamental functions of two-dimensional anisotropic percolation theory. Notice that reciprocity relations can be established for the functions f_x and f_y (see Ref. 7).

Substituting the expressions (30)–(32) and (34) into (28), and using Eq. (33) to eliminate the quantity $(\sigma_{a1}-\sigma_{a2})^2$, we finally obtain for the components of the effective conductivity tensor $\hat{\sigma}_e$ the expressions

$$\sigma_{\alpha e}=\sigma_{\alpha 1}f_\alpha(\sigma_{x1}\sigma_{y2}-\lambda^2\sigma_{x2}\sigma_{y1})D^{-1},\quad \alpha=x,\ y,$$
$$\sigma_{ae}=\sigma_{a1}-(\sigma_{a1}-\sigma_{a2})\lambda\sigma_{x1}\sigma_{y1}(1-f_xf_y)D^{-1}, \tag{35}$$
$$D=\lambda\sigma_{x1}\sigma_{y1}(1-f_xf_y)+\sigma_{x1}\sigma_{y2}f_xf_y-\lambda^2\sigma_{x2}\sigma_{y1}.$$

We obtain, in accordance with the equations (31), the functions f_x and f_y entering into the formulas (35) from the functions, defined in accordance with (34), for the null system by making the following change of arguments:

$$\lambda_{x1}/\lambda_{y1}\to\sigma_{x1}/\sigma_{y1},\quad \lambda_{x2}/\lambda_{y1}\to\lambda\sigma_{x2}/\sigma_{y2},\quad \lambda_{y2}/\lambda_{y1}\to\lambda \tag{36}$$

with the parameter λ from (32). The formulas (35), which are valid for arbitrary two-dimensional two-component systems, solve the formulated problem. All the information about the specific structure of a particular system is contained in the functions f_x and f_y.

For isotropic media ($\sigma_{xi}=\sigma_{yi}$, $f=f_x=f_y$), the formulas (35) coincide with the corresponding expressions (obtained by another method) in Ref. 8. In this case the main properties of the function f, which depends on the two parameters p and λ, are known, and this allows us to carry out a relatively complete analysis of the galvanomagnetic properties of isotropic two-component systems.[8] But in the case of anisotropic media, to use the general formulas (35), we must know the many-parameter functions f_x and f_y, whose properties are not studied in the usual percolation theory.[5,6,16,17] Certain problems of two-dimensional anisotropic percolation theory are considered in Ref. 7, and this allows us to establish the form of these functions in a number of cases. At the same time, the problem of determining f_x and f_y in the entire domain of variation of their arguments, even if an important one, is quite complicated (but significantly simpler than the problem of the galvanomagnetic properties).

According to (36) and (32), the arguments of f_x and f_y from (35) vary with varying $\mathbf{H}$. Therefore, the general formulas (35) in principle enable us to solve the inverse problem, i.e., to determine from galvanomagnetic measurements the fundamental functions f_x and f_y of two-dimensional anisotropic percolation theory, as well. In this case the two functions f_x and f_y are expressed in terms of the three quantities σ_{xe}, σ_{ye}, and σ_{ae}, so that the components of the effective conductivity tensor $\hat{\sigma}_e$ are not independent of each other. Indeed the elimination of f_x and f_y from (35) leads to an anisotropic analogue of Dykhne's general relations,[2,3,8] an analogue which does not depend on the specific structure of the systems under consideration:

$$\rho_{ae}^{-1}\mathscr{A}-\sigma_{ae}^{-1}\mathscr{B}=1,$$
$$\mathscr{A}=\rho_{a1}\rho_{a2}(\sigma_{a1}-\sigma_{a2})/(\sigma_{a1}\rho_{a2}-\rho_{a1}\sigma_{a2}), \tag{37}$$
$$\mathscr{B}=\sigma_{a1}\sigma_{a2}(\rho_{a1}-\rho_{a2})/(\sigma_{a1}\rho_{a2}-\rho_{a1}\sigma_{a2}).$$

Here ρ_a is the off-diagonal component of the resistivity tensor (see (41)). The relation (37) (like (38) in the approximation linear in **H**) can be used to check the results of measurements, as well as to determine the "degree of two-componentness" of real inhomogeneous films. For $H\to 0$ we obtain for the effective Hall constant R_e $(R=\rho_a/H)$ from (35) or (37) the expression

$$R_e=\frac{1}{\sigma_{x1}\sigma_{y1}-\sigma_{x2}\sigma_{y2}}\left[\frac{\sigma_{x1}\sigma_{y1}\sigma_{x2}\sigma_{y2}}{\sigma_{xe}\sigma_{ye}}(R_2-R_1)+R_1\sigma_{x1}\sigma_{y1}-R_2\sigma_{x2}\sigma_{y2}\right., \tag{38}$$

where σ_{xe} and σ_{ye} are the principal values of the tensor $\hat{\sigma}_e$ for $\mathbf{H}=0$. The formula (38) generalizes the corresponding isotropic relation[9,18] to the case of an arbitrary anisotropic system.

For $\hat{\sigma}_2\to 0$ we find from (32) that

$$\lambda\approx\sigma_{x1}\sigma_{y2}/(\sigma_{x1}\sigma_{y1}+\sigma_{a1}^2). \tag{39}$$

Substituting (39) into (35), we obtain for a system with dielectric inclusions ($\hat{\sigma}_2=0$ and $p>p_c$, where p_c is the critical concentration) the expressions

$$\sigma_{\alpha e}^{(d)}=\sigma_{\alpha 1}f_{\alpha d}\frac{\sigma_{x1}\sigma_{y1}+\sigma_{a1}^2}{\sigma_{x1}\sigma_{y1}+\sigma_{a1}^2 f_{xd}f_{yd}}\quad(\alpha=x,y),$$
$$\sigma_{ae}^{(d)}=\frac{\sigma_{a1}(\sigma_{x1}\sigma_{y1}+\sigma_{a1}^2)f_{xd}f_{yd}}{\sigma_{x1}\sigma_{y1}+\sigma_{a1}^2 f_{xd}f_{yd}}. \tag{40}$$

Here $f_{\alpha d}=f_\alpha(p;\ \sigma_{x1}/\sigma_{y1},\ 0,\ 0)$. The index d denotes dielectric inclusions (s denotes perfectly conducting inclusions). For the resistivity tensor elements

$$\rho_x=\sigma_y/(\sigma_x\sigma_y+\sigma_a^2),\quad \rho_y=\sigma_x/(\sigma_x\sigma_y+\sigma_a^2),\quad \rho_a=\sigma_a/(\sigma_x\sigma_y+\sigma_a^2) \tag{41}$$

we obtain from (40) the following simple expressions:

$$\rho_{xe}^{(d)}=\rho_{x1}f_{xd}^{-1},\quad \rho_{ye}^{(d)}=\rho_{y1}f_{yd}^{-1},\quad \rho_{ae}^{(d)}=\rho_{a1}. \tag{42}$$

From (42) we conclude that the effective Hall constant $R_e=\rho_{ae}/H$ for an anisotropic film with dielectric inclusions does not, as in the isotropic case,[9,18,3] depend on the concentration of the nonconducting component right up to the metal-insulator transition point.

If the inclusions are a perfectly conducting material (i.e., if $\sigma_1\to\infty$), then, using (36) and (39), we find from (35) that

$$\sigma_{xe}^{(s)}=\sigma_{x2}f_{xs},\quad \sigma_{ye}^{(s)}=\sigma_{y2}f_{ys},\quad \sigma_{ae}^{(s)}=\sigma_{a2}. \tag{43}$$

Here the functions $f_{\alpha s}$ have been introduced with the aid of the relation $f_\alpha\to(\lambda_{\alpha 2}/\lambda_{\alpha 1})\times f_{\alpha s}(p,\lambda_{x2}/\lambda_{y2})$ for $\hat{\lambda}_1\to\infty$.

In form, the expressions (40), (42), and (43) seem to be an obvious generalization of the corresponding results of Refs. 3 and 8 to the anisotropic case. It should, however, be noted that there is a significant difference between these two cases. In Refs. 3 and 8 the quantity f_d (like f_s) is a function of only the concentration p, so that the entire dependence of, for example, $\rho_{xe}^{(d)}$ on **H** is contained in ρ_{x1}. But in the anisotropic case the quantities f_{xd} and f_{yd} depend not only on p, but also on the ratio σ_{x1}/σ_{y1}, which, generally speaking, is a function of **H**.

3.3. The conductivity in an oblique magnetic field

The general expressions (35) allow us to consider the problem of the galvanomagnetic properties of an isotropic two-component film in an oblique magnetic field **H** as well. In this case the conductivity exhibits both the features peculiar to isotropic systems in strong transverse magnetic fields[2,8] and the anomalies characteristic of extremely anisotropic media.[7]

Let us orient the y axis along the component of **H** in the plane (x, y) of the system, and denote the angle between **H** and the z axis by θ. In this case the conductivity tensor has the form (25), with the elements

$$\sigma_{xi}=\frac{\sigma_i}{1+\beta_i^2},\quad \sigma_{yi}=\frac{\sigma_i(1+\beta_i^2\sin^2\theta)}{1+\beta_i^2},\quad \sigma_{ai}=\frac{\sigma_i\beta_i}{1+\beta_i^2}\cos\theta, \tag{44}$$

where the subscript i ($i=1, 2$) numbers the components. Here we have used the same model formulas that we used in Ref. 8 (cf. (44) for $\theta=0$ with the formulas (12) in Ref. 8). In (44), $\sigma_i\propto n_i\mu_i$ is the conductivity of the i-th component for $\mathbf{H}=0$, n_i and μ_i being the carrier concentration and mobility respectively; the dimensionless quantity $\beta_i\propto\mu_i h$. The problem under consideration is a many-parameter one, so that there are a substantial number of different limiting cases. Below, for simplicity, we limit ourselves to the case of equal carrier mobilities, i.e., to the case in which $\mu_1=\mu_2=\mu$, or $\beta_1=\beta_2=\beta$.

Let the system under consideration be isotropic in $\mathbf{H}=0$. In order to use the general formulas (35), we must know the functions f_x and f_y for a two-dimensional medium with a geometrically isotropic distribution of the components and a coordinate-independent orientation of the principal axes of the tensor $\hat{\sigma}$. The conductivity of such systems is investigated in Ref. 7. As shown in Ref. 7, the properties of extremely anisotropic inhomogeneous media are distinguished by their unusualness. Thus, the effective conductivity varies appreciably even in the region of low inclusion concentrations.[7] Below we shall consider the critical region, where the properties of highly anisotropic systems are in many respects close to those of isotropic media.[7]

Let us consider a null system whose components have markedly different conductivities (i.e., for which $\lambda_2\ll\lambda_1$) in the critical region $|\tau|\ll 1$, where τ is the same quantity as in (23). If the anisotropy is weak (i.e., if $\lambda_{xi}\approx\lambda_{yi}\approx\lambda_i$), then $f_x\approx f_y\approx f$, where f is given by the expressions (23) with h replaced by λ_2/λ_1 and with $q=t, s=\frac{1}{2}$, and the sign of equality in (23b). In the case when the components are highly anisotropic (i.e., when $(\lambda_{y1}\gg\{\lambda_{x1},\ \lambda_{y2}\}\gg\lambda_{x2})$), the system as a whole is, according to Ref. 7, isotropic when $|\tau|\ll 1$ and $(\lambda_{y2}/\lambda_{x1})^{1/2}\ll 1$, and for the functions f_x and f_y defined in accordance with (34) we have

$$\tau>0,\quad \Delta_\lambda\ll\tau\ll 1:\ f_x\approx 2f_d(\tau),\quad f_y\approx 2(\lambda_{x1}/\lambda_{y1})f_d(\tau);\quad f_d(\tau)\sim\tau^t;$$
$$|\tau|\ll\Delta_\lambda\ll 1:\ f_x\approx(\lambda_{y2}/\lambda_{x1})^{1/2},\quad f_y\approx(\lambda_{x1}\lambda_{y2})^{1/2}/\lambda_{y1}; \tag{45a}$$
$$\tau<0,\quad \Delta_\lambda\ll|\tau|\ll 1:\ f_x\approx{}^1/_2(\lambda_{y2}/\lambda_{x1})f_s(\tau),$$
$$f_y\approx{}^1/_2(\lambda_{y2}/\lambda_{y1})f_s(\tau); \tag{45b}$$

$$f_s(\tau)\sim|\tau|^{-t}. \quad (45c)$$

Here

$$\Delta_\lambda=(\lambda_{y2}/\lambda_{x1})^m,\quad m=1/2t. \quad (46)$$

In the formulas (45a) and (45c) $f_d(\tau)$ and $f_s(\tau)$ are functions describing the conductivity of the isotropic system in the cases when it contains dielectric and perfectly conducting inclusions respectively, one being the reciprocal of the other, i.e., $f_d f_s = 1$ (see, for example, Ref. 3). In the case when $(\lambda_{y2}/\lambda_{x1})^{1/2} \gg 1$ the system as a whole is highly anisotropic, and the functions f_x and f_y are given by the expressions (66) with h replaced by $\lambda_{y2}/\lambda_{y1}$

$$f_x\approx{}^1/_2,\quad f_y\approx 2\lambda_{y2}/\lambda_{y1};\quad |\tau|\ll 1,\quad \Delta_\lambda\gg 1. \quad (47)$$

For $\Delta_\lambda \sim 1$ the formulas (47) are, in order of magnitude, equal to (45b)

We can use the "dielectric" formulas (39) and (40) for λ, σ_{xe}, and σ_{ye} when investigating the galvanomagnetic properties of the system in question. For the quantity σ_{ae} the corresponding formula from (40) should be corrected by adding to the numerator the term $\sigma_{ae}\sigma_{x1}\sigma_{y1}$, which is not small in the $\tau\to 0$ region. In the functions f_x and f_y we should take account of the fact that $\sigma_2 \neq 0$, i.e., we should use formulas of the type (23) or the expressions (45)–(47) with the change of arguments (36) made in them.

Let us, to begin with, determine one of the most important characteristics of the system under consideration: the dimension Δ_H of the transition (or smearing-out) region. According to (36), the expression for Δ_H can be obtained from (46) by making the substitution $\lambda_{y2}/\lambda_{x1}\to\lambda\sigma_{y1}/\sigma_{x1}$. Using the formulas (44), we find from (46) and (39) that $(\beta_1=\beta_2=\beta)$

$$\Delta_H=\Delta_0\left[\frac{(1+\beta^2\sin^2\theta)^2}{1+\beta^2}\right]^m;\quad \Delta_0=\left(\frac{\sigma_2}{\sigma_1}\right)^m,\quad m=\frac{1}{2t}. \quad (48)$$

For $\theta=0$ we obtain from (48) the expression (46) in Ref. 8. In a transverse magnetic field ($\theta=0$) the quantity Δ_H decreases without restriction as $\beta\to\infty$ ($H\to\infty$). A different situation obtains in an oblique magnetic field. It is not difficult to see that, when $\theta>\pi/4$, Δ_H increases monotonically with increasing **H**. In the case $\theta<\pi/4$, as **H** is increased, the quantity Δ_H passes through a minimum and then increases without restriction as $H\to\infty$. This nonmonotonic dependence is especially strongly pronounced in the case when $\theta\ll 1$. As in an isotropic system,[8] the smearing-out region Δ_H in the interval $1\ll\beta\ll\theta^{-1}$ decreases with increasing **H**; at $\beta=\theta^{-1}$ the quantity Δ_H attains its minimum value $(\Delta_H)_{\min}=\Delta_0(2\theta)^{2m}\ll\Delta_0$. At $\beta=\theta^{-2}$ we again have $\Delta_H=\Delta_0$, and in the region $\beta\theta^2\gg 1$ we have $\Delta_H\gg\Delta_0$.

In accordance with the foregoing, we can distinguish three τ-value regions for the systems under consideration when $\theta\ll 1$ and $|\tau|\ll 1$.

1) $\Delta_0\ll|\tau|\ll 1$. In this case the system in **H** = 0 is outside the smearing-out region. But in sufficiently high **H** there arises the situation in which $\Delta_H\gg|\tau|$, so that the system will be in the smearing-out region. According to Ref. 8, anomalous conductivity is possible under these conditions.[2] Thus, in contrast to the isotropic (i.e., $\theta=0$) case,[8] in the systems under consideration, even when $|\tau|\gg\Delta_0$ (but $|\tau|\ll 1$), there is a magnetic-field-intensity range where anomalous conductivity can exist.

2) $(\Delta_H)_{\min}\ll|\tau|\ll\Delta_0\ll 1$. Under these conditions the system will, as H is increased, first leave the smearing-out region, and then find itself in it again. From this it follows that there are in this case two magnetic-field-intensity regions in which anomalous conductivity can exist.

3) If $|\tau|\ll(\Delta_H)_{\min}$, then for all **H** the system is in the smearing-out region, and there is quite a broad magnetic-field-intensity range in which anomalous conductivity is possible. It should, however, be noted that, in contrast to the isotropic case,[2,8] here, even in the $\tau=0$ case, the **H**-region of existence of anomalous conductivity is bounded from above as a result of the condition $\Delta_H\ll 1$. Notice also that anomalous conductivity is possible for any angle θ of inclination, in particular, for $\theta=\pi/2$, i.e., in a parallel magnetic field, if the conditions $|\tau|\ll\Delta_H\ll 1$ are fulfilled.

The galvanomagnetic properties of the systems under consideration in the case when $\theta\ll 1$ and $|\tau|\ll 1$ are quite complicated, and, according to the foregoing, depend essentially on the relationship among the parameters. Let us limit ourselves to the consideration of the apparently most interesting case $\tau>0$, $(\Delta_H)_{\min}\ll\tau\ll\Delta_0\ll 1$, and $\theta\ll\tau^t$, in which we can distinguish eight magnetic-field-intensity regions in the **H** dependence of $\hat\sigma_e$ (we assume $\beta_1=\beta_2=\beta$):

$$\beta\ll 1:\quad \sigma_{xe}\approx\sigma_{ye}\approx(\sigma_1\sigma_2)^{1/2},\quad \sigma_{ae}\approx 2\sigma_2\beta; \quad (49a)$$

$$1\ll\beta\ll\tau^{-t}(\sigma_2/\sigma_1)^{1/2}:\quad \sigma_{xe}\approx\sigma_{ye}\approx(\sigma_1\sigma_2)^{1/2}\beta^{-1},\quad \sigma_{ae}\approx 2\sigma_2\beta^{-1}; \quad (49b)$$

$$\tau^{-t}(\sigma_2/\sigma_1)^{1/2}\ll\beta\ll\tau^{-t}:\quad \sigma_{xe}\approx\sigma_{ye}\sim\sigma_1\tau^t,\quad \sigma_{ae}\sim\sigma_1\tau^{2t}\beta; \quad (49c)$$

$$\tau^{-t}\ll\beta\ll\theta^{-1}:\quad \sigma_{xe}\approx\sigma_{ye}\sim\sigma_1\tau^{-t}\beta^{-2},\quad \sigma_{ae}\approx\sigma_1\beta^{-1}; \quad (49d)$$

$$\theta^{-1}\ll\beta\ll\tau^t\theta^{-2}:\quad \sigma_{xe}\approx\sigma_{ye}\sim\sigma_1\theta^2\tau^{-t},\quad \sigma_{ae}\approx\sigma_1\beta^{-1}; \quad (49e)$$

$$\tau^t\theta^{-2}\ll\beta\ll\tau^t\theta^{-2}(\sigma_1/\sigma_2)^{1/2}:\quad \sigma_{xe}\approx\sigma_{ye}\sim\sigma_1\tau^t\theta^{-2}\beta^{-2},$$

$$\sigma_{ae}\sim\sigma_1\tau^{2t}\theta^{-4}\beta^{-3}; \quad (49f)$$

$$\tau^t\theta^{-2}(\sigma_1/\sigma_2)^{1/2}\ll\beta\ll\theta^{-2}(\sigma_1/\sigma_2)^{1/2}:\quad \sigma_{xe}\approx\sigma_{ye}\approx(\sigma_1\sigma_2)^{1/2}\beta^{-1},$$

$$\sigma_{ae}\approx 2\sigma_2\beta^{-1}; \quad (49g)$$

$$\beta\gg\theta^{-2}(\sigma_1/\sigma_2)^{1/2}:\quad \sigma_{xe}\approx{}^1/_2\sigma_1\theta^{-2}\beta^{-2},\quad \sigma_{ye}\approx 2\sigma_2\theta^2,\quad \sigma_{ae}\approx 2\sigma_2\beta^{-1}. \quad (49h)$$

The expressions (49a)–(49d), which are valid when $\beta\theta\ll 1$, correspond to the case of an isotropic system ($\sigma_{xi}\approx\sigma_{yi}$), and coincide with the formulas (47) in Ref. 8. (In deriving (49a)–(49d), we should set $\sigma_{xi}=\sigma_{yi}$, and use expressions of the type (23) for the functions $f=f_x=f_y$ (see also Ref.8).) Notice that (49a), (49b) and (49g), (49h) correspond to the case $\Delta_H\gg\tau$, while (49c)–(49f) correspond to the case $\Delta_H\ll\tau$. In accordance with the arguments adduced above, there are two region where anomalous conductivity can occur: see (49b) and (49g). The case (49h) corresponds to $\Delta_H\gg 1$ (the functions f_x and f_y are given by the expressions (47) with $\lambda_{y2}/\lambda_{y1}$ replaced by λ), in which case the system as a whole is highly anisotropic. At the same time, when $\theta^{-1}\ll\beta\ll\theta^{-2}(\sigma_1/\sigma_2)^{1/2}$ (the functions f_x and f_y from (45) with the appropriate substitutions made), despite the fact that the properties of the components are highly anisotropic, the medium as a whole is virtually isotropic.

Notice that the measurement of the conductivity of systems that undergo the metal–insulator phase transition in oblique magnetic fields allows us to determine the functions f_x and f_y in the critical region, and thereby verify the validity of the scaling-law hypothesis for two-dimensional two-component anisotropic media.[7]

4. GALVANOMAGNETIC PROPERTIES. THE THREE-DIMENSIONAL CASE

Let us consider an isotropic three-dimensional two-component medium in a magnetic field **H**. If we orient the z axis along **H**, then the conductivity tensor $\hat{\sigma}(\mathbf{r})$ has the usual form

$$\hat{\sigma}=\begin{pmatrix} \sigma_x & \sigma_a & 0 \\ -\sigma_a & \sigma_x & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}. \quad (50)$$

The problem is to determine the effective characteristics σ_{xe}, σ_{ze}, and σ_{ae} of such a system.

The symmetry transformation (26) is a characteristic of the two-dimensional case, and does not have a three-dimensional analogue. Therefore, it is not possible to establish for three-dimensional media such general relations as (35). Nevertheless, in two cases it is possible to relate the galvanomagnetic properties of a three-dimensional system with its electrical conductivity computed without allowance for the Hall components.

The equations for the constant current do not change under the transformation

$$\mathbf{E}=\mathbf{E}', \quad \mathbf{j}=\mathbf{j}'+\hat{C}\mathbf{E}'; \quad \hat{\sigma}'(\mathbf{r})=\hat{\sigma}(\mathbf{r})-\hat{C}, \quad (51)$$

where $\hat{C}$ is an arbitrary (coordinate-independent) antisymmetric tensor. Let the Hall component for some two-component medium be coordinate independent, i.e., let $\sigma_{al}=\sigma_{a2}=\sigma_a$. Then, setting $\hat{C}=\hat{\sigma}_a$, where $\hat{\sigma}_a$ is the antisymmetric part of the conductivity tensor $\hat{\sigma}(\mathbf{r})$, in (51), we arrive at the conclusion that the original galvanomagnetic-effect problem reduces in the primed system to the problem of the electrical conductivity of an anisotropic medium with zero Hall components (i.e., with $\sigma_a'=0$). Thus,

$$\sigma_{xe}=\sigma_{xe}', \quad \sigma_{ze}=\sigma_{ze}', \quad \sigma_{ae}=\sigma_a. \quad (52)$$

In (52) we have taken account of the fact that $\sigma_{ae}'=0$, i.e., that $\sigma_{ae}=\sigma_a$. It is not difficult to see that the equalities (52) are, in accordance with Ref. 19, valid for any **r** dependence of $\hat{\sigma}$ when $\sigma_a=$ cost. The quantities σ_{xe}' and σ_{ze}' in (52) do not depend on σ_a. Therefore, introducing for two-component systems the functions f_x and f_z by analogy with (34), we obtain

$$\sigma_{\alpha e}=\sigma_{\alpha 1}f_\alpha(p;\ \sigma_{x1}/\sigma_{z1},\ \sigma_{x2}/\sigma_{z1},\ \sigma_{z2}/\sigma_{z1}) \quad (\alpha=x,\ z), \quad \sigma_{ae}=\sigma_a. \quad (53)$$

The condition $\sigma_{a1}=\sigma_{a2}$ cannot be fulfilled for all **H**. Using for the elements of the conductivity tensor $\hat{\sigma}$ the usual model formulas ($\sigma_{zi}=\sigma_i$, while σ_{xi} and σ_{ai} are given by the expressions (44) with $\theta=0$), we conclude that the equalities $n_1\mu_1^2=n_2\mu_2^2$ for $H\to 0$ and $n_1=n_2$ for $H\to\infty$ follow from the condition $\sigma_{a1}=\sigma_{a2}$.

Let us now consider a system consisting of a matrix with conductivity tensor $\hat{\sigma}_2$ of the form (50) and perfectly conducting ($\hat{\sigma}_1\to\infty$) inclusions. In this case, setting $\hat{C}=\hat{\sigma}_{a2}$, we arrive in the primed system to one of the problems of anisotropic percolation theory[7,10,20]: the problem of the conductivity of an anisotropic medium (with $\sigma_a'=0$) with perfectly conducting inclusions. Thus, $\sigma_{xe}=\sigma_{xe}'$, $\sigma_{ze}=\sigma_{ze}'$, and $\sigma_{ae}=\sigma_{a2}$. Introducing the functions f_{xs} and f_{zs} in accordance with the relation

$$\sigma_{\alpha e}^{(s)'}=\sigma_{\alpha 2}f_{\alpha s}(p,\sigma_{x2}/\sigma_{z2}) \quad (\alpha=x,z), \quad (54)$$

we finally obtain ($p<p_c$):

$$\sigma_{xe}^{(s)}=\sigma_{x2}f_{xs}(p,\sigma_{x2}/\sigma_{z2}), \quad \sigma_{ze}^{(s)}=\sigma_{z2}f_{zs}(p,\sigma_{x2}/\sigma_{z2}), \quad \sigma_{ae}^{(s)}=\sigma_{a2}. \quad (55)$$

Outside the smearing-out region the expressions (55) are also valid in the case of a large, but finite $\hat{\sigma}_1$.

Let us use the results obtained to analyze the critical behavior of the effective Hall constant R_e (for $H\to 0$) in systems that undergo the metal-insulator transition. In the "insulator" phase ($\tau<0$), outside the smearing-out region,[6] the inclusions of the first component can be considered to be a perfectly conducting material if $\sigma_{x1}\gg\sigma_{x2}$ and $\sigma_{a1}\gg\sigma_{a2}$, i.e., $n_1\mu_1\gg n_2\mu_2$ and $n_1\mu_1^2\gg n_2\mu_2^2$ (for $h\to 0$). Both inequalities are satisfied if $\mu'\sim\mu_2$ and $n_1\gg n_2$. In this case we obtain for $R_e=H^{-1}\sigma_{ae}/\sigma_{xe}^2$ [$H\to 0$] from (55) with allowance for (23c) the relation

$$R_e\sim R_2|\tau|^{2q}; \quad \tau<0, \quad \Delta\ll|\tau|\ll 1, \quad (56)$$

where Δ is the same quantity figuring in (23). From (56) we find the following relation between the critical exponents f (see Ref. 9) and q:

$$f=2q. \quad (57)$$

The relation (57) is proposed (in the form of a hypothesis) in Ref. 9 for a consistent description of the critical behavior of R_e in a broad range of parameter values. The present analysis shows that the relation (57) is exact.

The case $\sigma_2\ll\sigma_1$, $\sigma_{a1}=\sigma_{a2}$ (i.e., $n_2\mu_2\ll n_1\mu_1$ $n_2\mu_2{}^2=n_1\mu_1{}^2$) is realized when $n_2\ll n_1$ and $\mu_2\gg\mu_1$, with $R_2=R_1(\sigma_1/\sigma_2)^2\gg R_1$. For such a system, the effective Hall constant can be found in the entire critical region. Setting, for $H\to 0$, $\sigma_{xe}=\sigma_e=\sigma_1 f(p,h)$, and using for $f(p,h)$ the expressions (23), we obtain for $R_e=R_1[f(p,h)]^{-2}$ from (53) the relations

$$\tau>0, \quad \Delta\ll\tau\ll 1: \quad R_e\sim R_1\tau^{-2t}, \quad (58a)$$

$$|\tau|\ll\Delta: \quad R_e\sim R_1(\sigma_1/\sigma_2)^{2s}, \quad (58b)$$

$$\tau<0, \quad \Delta\ll|\tau|\ll 1: \quad R_e\sim R_2|\tau|^{2q}, \quad (58c)$$

where $\Delta=(\sigma_2/\sigma_1)^{s/t}$, $q=t(1-s)/s$.

Shklovskiĭ[9] has proposed for R_e an interpolation formula that is consistent with the scaling-law hypothesis:

$$R_e\approx R_2\sigma_2{}^2\sigma_e{}^{-2}+R_1(\tau^2+\Delta^2)^{-g/2}, \quad g=t(2s-1)/s.$$

Here Δ is the dimension of the smearing-out region (see (23) and (58)); the expression for the critical exponent g takes account of the relation (57). For $s = 0.62$ and $t = 1.6$ (Ref. 9) we have $g \approx 0.6$, which differs from[9] $\nu \approx 0.9$. Another exponent from Ref. (9) assumes, when (57) is taken into account, the form $k = 2(1-s) \approx 0.76$.

In both of the cases considered above the first term in (59) is large compared to the second, and (59) coincides with the expressions (56) and (58). At the same time for a system with dielectric inclusions ($\sigma_2 = 0, \tau > 0$) only the second term in (59) makes a nonzero contribution. In this case, according to (59), the effective Hall component σ_{ae} varies in the following manner as $\tau \to 0$ ($H \to 0$):

$$\sigma_{ae}^{(d)} \sim \sigma_1 \beta_1 \tau^{t/s}, \quad \tau > 0 \tag{60}$$

(in this case $\sigma_{xe} = \sigma_{ze} \sim \sigma_1 \tau^t$). Let us point out that, for the quantities R_e and σ_{ae}, all the critical exponents (see Ref. 9 with allowance for the equality (57)) can be expressed in terms of t and s, (i.e., in terms of the critical exponents for the $\mathbf{H} = 0$ electrical conductivity problem. It is possible that this indicates the existence of some approximate isomorphism between these problems for $|\tau| \ll 1$ and $H \to 0$. In view of this, it is of considerable interest to obtain the result (60) by a direct method.

5. THE CONDUCTIVITY OF ANISOTROPIC MEDIA

It is natural in the investigation of the electrical conductivity of inhomogeneous anisotropic systems[7,10,20] to limit ourselves initially to the study of the simplest models. First, there is the system with geometrically isotropic distribution of the components and coordinate-independent orientation of the principal axes of the conductivity tensor $\hat{\sigma}$. An example of such a medium is given in Subsec. 3.3. Second, there is the system whose components are isotropic, and the anisotropy of the medium as a whole is determined by the shape and orientation of the inclusions (a geometrically anisotropic medium). We shall show that these models are isomorphic in one important particular case, and find an exact relation between their properties (cf. Ref. 10, where a system with inclusions of ellipsoidal shape is considered).

Let the structure of a two-dimensional system with geometrical anisotropy be obtainable from some isotropic (random or regular) network through a uniform extension along one of the axes, e.g., the x axis. Let us transform the coordinates, the electric field $\mathbf{E}$, and the current density $\mathbf{j}$ into the primed system:

$$x = \nu x', \quad y = y'; \quad E_x = E_x', \quad E_y = \nu E_y'; \quad j_x = \nu j_x', \quad j_y = j_y'. \tag{61}$$

Here $\nu = \text{const}$ is the coefficient of extension. The transformation (61) converts the anisotropic network into an isotropic one, and does not change the constant-current equations. But the conductivity of the primed system is described by a tensor of the form (29) with $\sigma_x'(\mathbf{r}) = \nu^{-1}\sigma(\mathbf{r})$ and $\sigma_y'(\mathbf{r}) = \nu\sigma(\mathbf{r})$, where $\sigma(\mathbf{r})$ is the local isotropic electrical conductivity of the original system. Thus, the primed system is a medium with a geometrically isotropic distribution of the components, which are described by conductivity tensors of the form (29) with elements

$$\sigma_{xi}' = \nu^{-1}\sigma_i, \quad \sigma_{yi}' = \nu\sigma_i \quad (i = 1, 2), \tag{62}$$

where σ_1 and σ_2 are the isotropic conductivities of the components in the original system.

The effective characteristics of the original $(\sigma_{xe}, \sigma_{ye})$ and primed $(\sigma_{xe}', \sigma_{ye}')$ systems are also connected by relations of the type (62):

$$\sigma_{xe} = \nu\sigma_{xe}', \quad \sigma_{ye} = \nu^{-1}\sigma_{ye}'. \tag{63}$$

Let us introduce the functions g_x and g_y for a system with geometrical anisotropy in the following manner:

$$\sigma_{\alpha e} = \sigma_1 g_\alpha(p, h; \nu), \quad h = \sigma_2/\sigma_1 \quad (\alpha = x, y). \tag{64}$$

For the primed system, let us define functions f_x and f_y in a manner similar to (34). Then from (63), (62), and (64) we finally obtain the relations

$$g_\alpha(p, h; \nu) = f_\alpha(p; \nu^{-2}, \nu^{-2}h, h), \quad \alpha = x, y. \tag{65}$$

Relations of the type (65) can be established for three-dimensional media as well.

The limiting case of a system with geometrical anisotropy is a layered medium ($\nu \to \infty$) in which $\sigma(\mathbf{r})$ depends only on one coordinate, in the present case y. The effective characteristics of a layered medium are easily determined; as a result, from (64) and (65) we obtain

$$f_x(p; 0, 0, h) = p + (1-p)h, \quad f_y(p; 0, 0, h) = h/(ph + 1 - p). \tag{66}$$

Hence for $p \approx \frac{1}{2}$ and $h \ll 1$ we obtain after the substitution $h \to \lambda_{y2}/\lambda_{y1}$ the expressions (47).

In conclusion, let us discuss the consequences (for anisotropic media) of the simple, but useful isomorphism between the problem of the permittivity of a system with metallic inclusions and the problem of the conductivity of a structurally identical system with perfectly conducting inclusion.[11] Let the matrix be uniaxial (i.e., let $\varepsilon_{x2} = \varepsilon_{y2} < \varepsilon_{z2}$), and let the orientation of the principal axes of the tensor $\hat{\varepsilon}(\mathbf{r})$ be coordinate independent. Then, for the principal values of the effective permittivity tensor $\hat{\varepsilon}_e$ we have, in accordance with Ref. 11, the relations ($p < p_c$)

$$\varepsilon_{xe} = \varepsilon_{x2} f_{xs}(p, \varepsilon_{x2}/\varepsilon_{z2}), \quad \varepsilon_{ze} = \varepsilon_{z2} f_{zs}(p, \varepsilon_{x2}/\varepsilon_{z2}), \tag{67}$$

where f_{xs} and f_{zs} are the same functions figuring in (54) and (55). According to arguments adduced in Refs. 20 and 7, a system with perfectly conducting inclusions becomes virtually completely isotropic as $p \to p_c$ ($\tau < 0$, $|\tau| \ll 1$). Therefore, for the $\hat{\varepsilon}_e$-tensor elements we obtain (in the case when $\varepsilon_{z2} \gg \varepsilon_{x2}$)

$$\varepsilon_{xe} \approx \varepsilon_{ze} \sim \varepsilon_{z2} |\tau|^{-q}, \tag{68}$$

where q is the "isotropic" critical exponent. Thus, in the critical region even a highly anisotropic medium with metallic inclusions possesses qualitatively the same dielectric properties as the isotropic medium.[6,11]

6. CONCLUSION

The examples considered in the present paper demonstrate that certain problems of percolation theory are secondary. Their solution with the aid of isomorphism relations is entirely determined by the solution to certain primary (basic) problems. Notice that the establishment of the isomorphism relations allows us to, in particular, relate separate analytical results of percolation theory. Thus, of the three exact results mentioned in the Introduction, the second and third are, when allowance is made for the isomorphism of the corresponding problems, a consequence of the first. In other cases the isomorphism relations enable us to obtain new results from the known ones.

The secondary problems do not require separate investigations with the aid of numerical and simulation experiments, thus allowing us to concentrate our efforts on the basic problems. The role of the latter problems is greater, but so also is the need for quality of their investigations both in the sense of accuracy and in respect of the ranges of values of the parameters. In particular, it is necessary to carry out a thorough investigation of the function $f(p, h)$ in the entire domain of variation of the arguments p and h. Notice that we can invert the isomorphism relations and use them to carry out a more thorough study of the basic problems. Finally, let us note that we can judge the applicability of the two-component medium model to real samples by the degree of accuracy with which the relations of the type of Dykhne's general relation are fulfilled.

The examples cited do not, apparently, exhaust the consequences of the symmetry transformations used in the present paper. But of even greater interest is the search for new transformations that will allow the establishment of isomorphisms between other problems. The establishment of all possible isomorphisms will, in principle, enable us to reduce the entire variety of phenomema associated with the theory of transport processes to a set of basic problems. Of top-priority is the search for the correspondence, if it exists (even if with a limited region of applicability), between the problem of the galvanomagnetic properties of three-dimensional media and the problem of the conductivity of anisotropic systems with the same structure.

[1)] Let us, seizing the opportunity, correct a typographical error in Ref. 8. Reference No. 4 in Ref. 8 should read: 4. B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. **81**, 665 (1981) [Sov. Phys. JETP **54**, 355 (1981)].

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