Dynamics of magnetic vortices in a planar ferromagnet

A. V. Nikiforov and É. B. Sonin

A. F. Ioffe Physicotechnical Institute, Academy of Sciences of the USSR, Leningrad
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The equation of motion of a vortex in a ferromagnet with the easy-plane type of anisotropy (planar ferromagnet) is derived. The velocity of such vortices depends not only on the circulation, as in hydrodynamics, but also on a second invariant: the polarization (the sign of the magnetic-moment component along the axis of the vortex core). The canonical equations of motion (Hamilton's equations) for a system of straight parallel vortices are derived.

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§1. INTRODUCTION

Much attention has been given in recent years to the investigation of the dynamics of topologically stable defects in condensed media. Among them are quantized vortices in superfluid liquids and superconductors. Similar linear defects can exist also in planar ferromagnets or antiferromagnets, which exhibit an anisotropy that tries to hold the spontaneous magnetic moment or magnetic moments of the sublattices in some crystal plane (the easy plane). They are called magnetic vortices or disclinations: on going round them the magnetic moment turns in the plane through an angle that is a multiple of 2π.

On the basis of a number of topological properties, the Bloch lines in 180-degree domain walls should also be classified with magnetic vortices. Such lines play a great role in the dynamics of domain walls, and have been the subject of a number of theoretical and experimental investigations.[1,2] Magnetic vortices are also important for the problem of the stability of helical structures in magnetically ordered systems (one of which is the A phase of superfluid He^3). The transport of spin in such structures is nondissipative, just as the transport of mass in superfluid liquids is nondissipative.[3]

The magnetic vortex has the simplest structure in the model of a planar magnetic material, in which there is no anisotropy in the easy plane (i.e., there is degeneracy with respect to the angle of turn of the magnetic moment in the easy plane) and the dipole-dipole interaction and the demagnetizing field due to it are neglected. The magnetic-vortex dynamics in such a model has been studied by Volovik and Dotsenko[4] in the Poisson-brackets formalism. The authors employ the method used in the analysis of the motion of a hydrodynamic vortex, and obtain an expression for the total momentum flux through a cylindrical surface of large radius surrounding the vortex. This flux should be equal to zero in the absence of external forces. As a result, Volovik and Dotsenko arrived at the conclusion that, in the absence of a spin flux at infinity and magnetization along the Z axis, a magnetic vortex in a planar magnetic material can move with any constant velocity. This differs significantly from hydrodynamics, where according to the Helmholtz theorem, a vortex can move only with the velocity of the medium, i.e., it is, as it were, "frozen" in the liquid.

The problem of the motion of a magnetic vortex in a planar ferromagnet has been solved by us by another method, specifically, by a method based on the derivation of the solvability condition for the inhomogeneous linear equation describing the corrections to the field of the magnetic moment, that arise as a result of the vortex motion. This method has been used before to establish the law of vortex motion in type-II superconductors[5] and in He II in the vicinity of the λ point.[6] To use it, we must have a theory that is able to describe the behavior of the medium everywhere, including the region where there is a marked departure from the ground state (the vortex core). In the case of a magnetic material such a theory is the Landau-Lifshitz phenomenological theory. The basis for its use in the present problem is the fact that, normally, all the parameters of the medium inside a vortex core change over distances significantly greater than the atomic distances. The use of the indicated method led us to a result that differs appreciably from the one obtained in Ref. 4: a vortex can move with constant velocity only when a spin flux proportional to the helical deformation of the spin system exists at infinity. In the absence of such a flux, the magnetic vortex, like the hydrodynamic vortex, is "frozen" in the medium. This is due to the fact that the ratio of the force acting on the vortex (the analogue of the Magnus force) to the vortex velocity is proportional to the difference between the magnetization along the difficult axis coinciding with the vortex axis and the magnetization at points infinitely far away from the vortex, and not just to the magnetization at infinity, as was found in Ref. 4. The vortex dynamics turns out to be dependent on the structure of the vortex core, which makes vortices behave differently in different magnetic media. In a planar antiferromagnet in zero external field the magnetization is equal to zero both on the vortex axis and at points far away from the vortex, and therefore the conclusion reached by Volovik and Dotsenko that vortex motion with arbitrary velocity is possible remains valid. But for vortices in a ferromagnet the magnetization on the vortex axis can be nonzero even in the absence of magnetization at infinity, which results in the freezing in of the vortices in the medium.

To better understand the results obtained by us, we also carry out for the vortex a momentum-balance analysis like the one performed by Volovik and Dotsenko.[4] It turns out that the expression chosen by them for the momentum (an expression which is nonunique and gauge dependent) has a singularity on the vortex axis, and we should, in construct-

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ing the momentum balance, take account of not only the momentum flux through the surface a cylinder of large radius, but also the momentum flux through the surface of a cylinder of infinitely small radius (the axes of both cylinders coincide with the vortex axis). But a momentum-balance equation correctly constructed in any gauge furnishes the same law of motion for the magnetic vortex as the law found from the solvability condition.

The law of motion of a vortex obtained in the present paper shows that a magnetic vortex, unlike the hydrodynamic vortex, is characterized not by one, but by two invariants. Besides the usual invariant—the number of rotations a magnetic moment makes on going around a vortex (the analogue of the number of circulation quanta for a vortex in quantum hydrodynamics)—we have an invariant determined by the sign of the magnetic-moment component M_z along the "difficult axis" (the axis perpendicular to the easy plane) near the vortex axis. Thus, the new invariant, which we call the polarization, can assume two values: $+1$ and -1. The magnitude and direction of the vortex velocity depend on the value of the polarization.

The dynamics of magnetic vortices is also investigated in the paper recently published by Takeno and Homma.[7] But these authors modified without justification the basic equations of motion for a spin system, as a result of which there arose in their system of equations in place of a law of conservation for the spontaneous-magnetic-moment component M_z along the difficult axis a law of conservation for M_z^2. For this reason, the equations of motion obtained by them for vortices differ from ours: their equations of motion do not contain the polarization.

The rest of the paper is organized as follows. In §2 the solvability-condition formulation method is used to derive the law of motion of a magnetic vortex in a planar magnetic material that is homogeneous at infinity. It is shown that the vortex is "frozen" in the medium, and cannot move relative to it. In §3 this result is rederived by constructing the momentum balance for the magnetic vortex. In §4 we derive the law of motion of a vortex in the presence of a helical deformation (i.e., of a spin flux) at points far from the vortex. In §5 we derive the canonical equations of motion for a system of rectilinear magnetic vortices.

§2. A MAGNETIC VORTEX IN A PLANAR MAGNETIC MATERIAL THAT IS UNIFORM AT INFINITY

The dynamics of a planar magnetic material is described by the Hamiltonian

$$\mathscr{H}=\int d\mathbf{r}\,\{\alpha(\nabla\mathbf{M})^2/2+\beta(\mathbf{Mn})^2-\mathbf{MH}\}, \tag{1}$$

where $\mathbf{M}$ is the magnetic moment per unit volume, $\mathbf{H}$ is the magnetic field, $\mathbf{n}$ is the unit vector along the direction of the anisotropy field, and α and β are the inhomogeneous-volume and anisotropy constants; here $\alpha>0$ and $\beta>0$.

We shall consider the two-dimensional problem, in which the parameters of the medium do not vary along the Z axis, whose direction coincides with the direction $\mathbf{n}$ of the anisotropy field and the direction of the magnetic field $\mathbf{H}$, i.e., the easy plane coincides with the XY plane, the energy $\mathscr{H}$ pertains to a unit length of the sample along the Z axis, and the integration in (1) is over the XY plane, $\mathbf{r}$ being a two-dimensional vector.

Since the magnitude of the magnetic moment $\mathbf{M}$ remains constant, the magnetic moment at each point is uniquely determined by two parameters: $m\approx M_z/M$ and $\Phi=\arctan(M_y/M_x)$, the angle of turn of the magnetic moment in the easy plane. In these variables the Hamiltonian has the following form:

$$\mathscr{H}=\int d\mathbf{r}\left\{\frac{\alpha M^2}{2}\frac{(\nabla m)^2}{1-m^2}+\frac{\alpha M^2}{2}(1-m^2)(\nabla\Phi)^2+\frac{\beta M^2}{2}m^2-MHm\right\}. \tag{2}$$

The Landau-Lifshitz equation is equivalent to the following Hamilton equations for the pair of cannonically conjugate variables m and Φ:

$$\frac{M}{\gamma}\frac{d\Phi}{dt}=\frac{\delta\mathscr{H}}{\delta m}-\nabla_i\frac{\delta\mathscr{H}}{\delta\nabla_i m}, \tag{3}$$

$$\frac{M}{\gamma}\frac{dm}{dt}=\nabla_i\frac{\delta\mathscr{H}}{\delta\nabla_i\Phi}, \tag{4}$$

where γ is the gyromagnetic ratio.

Let us first consider a stationary vortex in a magnetic material that is uniform at infinity. For this case the solution to Eqs. (3) and (4), which we shall denote by m_0 and Φ_0, admits of uniform precession of $\mathbf{M}$ with angular velocity Ω, with

$$\Phi_0(\mathbf{r},t)=\Omega t+\varkappa\varphi+C, \tag{5}$$

where $\varphi=\arctan(y/x)$ is the azimuthal angle in the cylindrical system of coordinates whose axis coincides with the vortex line, $\varkappa$ is the circulation, equal to the number of complete turns of $\mathbf{M}$ in the easy plane on going around the vortex, and C is a constant.

The quantity $m_0(\mathbf{r},t)=m_0(r)$ depends only on the modulus r, and is given by an equation that is obtained from (3) by substituting (5) into it:

$$\frac{1}{1-m_0^2}\left(\frac{d^2m_0}{dr^2}+\frac{1}{r}\frac{dm_0}{dr}\right)+\frac{m_0}{(1-m_0^2)^2}\left(\frac{dm_0}{dr}\right)^2-m_0(r_0^{-2}-(\nabla\Phi_0)^2)+\frac{\tilde{H}}{\beta M}r_0^{-2}=0, \tag{6}$$

where $r_0=(\alpha/\beta)^{1/2}$ is the radius of the vortex core, and the effective field

$$\tilde{H}=\Omega/\gamma+H \tag{7}$$

is equal to the true magnetic field in the system rotating with the angular velocity Ω of the precession. The asmptotic values of $m_0(r)$ are (see the Appendix):

$$r\to\infty,\quad m_0(r)\to m_\infty=\tilde{H}/\beta M, \tag{8}$$

$$r\to 0,\quad m_0(r)\to\nu. \tag{9}$$

The polarization ν, which assumes the values $+1$ and -1, indicates the sign of the component of the magnetic moment $\mathbf{M}$ along the Z axis (the difficult axis) near the vortex axis. Vortices with polarizations $\nu=1$ and $\nu=-1$ belong to different topological classes, since a continuous transition from a vortex with polarization $\nu=+1$ to a vortex with $\nu=-1$

is not possible if we require that such a transition should occur at a fixed value of m_0 at infinity and a fixed value of the magnitude of the magnetic moment[1)] **M**. The violation of any of these two conditions leads to a very large increase in the energy of the system (the anisotropy energy in the first case and the exchange energy in the second).

Let us now show that the steady-state motion of a magnetic vortex is impossible if the medium is homogenoeus, and the spin flux is zero, at infinity. For a steady-state vortex motion with velocity v we have

$$d\Phi/dt=\Omega-(\mathbf{v}\nabla\Phi),\quad dm/dt=-(\mathbf{v}\nabla m). \tag{10}$$

We shall try to find the small corrections to m_0 and Φ_0 that arise from the vortex motion, i.e., let us represent m and Φ in the form

$$m=m_0+m_1,\quad \Phi=\Phi_0+\Phi_1. \tag{11}$$

The system of equations, linearized in m_1 and Φ_1 as well as in $\mathbf{v}$, has the form

$$\frac{\Delta m_1}{1-m_0^2}+\left\{\frac{2m_0\Delta m_0}{(1-m_0^2)^2}+\frac{(\nabla m_0)^2}{(1-m_0^2)^2}+\frac{4m_0^2(\nabla m_0)^2}{(1-m_0^2)^3}\right.$$
$$\left.-[r_0^{-2}-(\nabla\Phi_0)^2]\right\}m_1+\frac{2m_0}{(1-m_0^2)^2}(\nabla m_0\nabla m_1)$$
$$=\frac{1}{\alpha\gamma M}(\mathbf{v}\nabla\Phi_0), \tag{12}$$

$$(1-m_0^2)\Delta\Phi_1-2m_0(\nabla\Phi_0\nabla m_1)-2m_0(\nabla m_0\nabla\Phi_1)$$
$$=-\frac{1}{\alpha\gamma M}(\mathbf{v}\nabla m_0). \tag{13}$$

The Eqs. (12) and (13) constitute a system of inhomogeneous linear equations for m_1 and Φ_1. In order for this system to possess a solution, the inhomogeneous terms (the right-hand sides of (12) and (13)) should be orthogonal to any solution of the corresponding system of homogeneous equations, i.e., the required solvability condition for the system of equations (12) and (13) has the following form:

$$\int_V \tilde{m}C_m dV+\int_V \tilde{\Phi}C_\Phi dV=\int_S\left\{\frac{\tilde{m}\nabla m_1-m_1\nabla\tilde{m}}{1-m_0^2}\right.$$
$$\left.+2m_0\nabla\Phi_0(\Phi_1\tilde{m}-\tilde{\Phi}m_1)+(1-m_0^2)(\tilde{\Phi}\nabla\Phi_1-\Phi_1\nabla\tilde{\Phi})\right\}dS, \tag{14}$$

$$C_m=\frac{1}{\alpha\gamma M}(\mathbf{v}\nabla\Phi_0),\quad C_\Phi=-\frac{1}{\alpha\gamma M}(\mathbf{v}\nabla m_0), \tag{15}$$

where $\tilde{m}(\mathbf{r})$ and $\tilde{\Phi}(\mathbf{r})$ are a solution to the homogeneous system, and the integration in (14) is over a surface S that, in general, consists of two parts: a surface S_∞ that is infinitely far from the vortex line and the surface S_0 of an infinitely thin cylinder surrounding the vortex axis, the contribution from which can be substantial only when the magnetization is singular on the vortex axis. The required solution to the system of homogeneous equations can be obtained by using the translational invariance. On account of this symmetry property of our problem, if $m_0(\mathbf{r})$ and $\Phi_0(\mathbf{r})$ are a solution for the stationary vortex, then $m_0(\mathbf{r}+\mathbf{d})$ and $\Phi_0(\mathbf{r}+\mathbf{d})$ are also a solution to the original system of nonlinear equations (here **d** is an arbitrary translation vector). From this it follows that the solution to the system of linear equations has the form

$$\tilde{m}(\mathbf{r})=(\mathbf{d}\nabla)m_0(r),\quad \tilde{\Phi}(\mathbf{r})=(\mathbf{d}\nabla)\Phi_0(r). \tag{16}$$

After substituting (16) into (14) we can verify that the surface integrals in (14) vanish. This follows from the asymptotic behavior of the solutions $\{\tilde{m},\Phi\}$ and $\{m_1,\Phi_1\}$ at $r\to0$ and $r\to\infty$ (which solutions are investigated in the Appendix). Finally, from the solvability condition (14) we obtain

$$2\pi\varkappa(\nu-m_\infty)([\mathbf{d}\mathbf{v}]\mathbf{e}_z)=0, \tag{17}$$

where $\mathbf{e}_z$ is the unit vector along the Z axis. Because of the arbitrariness of the translation vector **d**, it follows from (17) that $\mathbf{v}=0$, and a vortex motion in a system that is homogeneous at infinity is impossible.

§3. THE MOMENTUM BALANCE FOR THE MAGNETIC VORTEX

Using the Noether theorem for classical fields (see, for example, Ref. 9), we can easily verify that, for the Lagrangian

$$L=\frac{M}{\gamma}\int_V m\dot{\Phi}dV-\mathscr{H}, \tag{18}$$

chosen by us, the translational invariance of the Landau-Lifshitz equations leads to the conservation of momentum, whose density is[4]

$$\mathbf{p}=\frac{\delta L}{\delta\dot{\Phi}}\nabla\Phi=\frac{M_z}{\gamma}\nabla\Phi=\frac{M}{\gamma}m\nabla\Phi. \tag{19}$$

The momentum flux density tensor in the reference fame moving with the vortex velocity **v** is given by

$$T_{ik}=-\nabla_i m\frac{\alpha M^2}{1-m^2}\nabla_k m-\alpha M^2(1-m^2)\nabla_i\Phi\nabla_k\Phi$$
$$+\delta_{ik}\frac{\alpha M^2}{2}\frac{(\nabla m)^2}{1-m^2}+\delta_{ik}\frac{\alpha M^2}{2}(1-m^2)(\nabla\Phi)^2$$
$$+\delta_{ik}\frac{\beta M^2}{2}m^2-\delta_{ik}MHm+\delta_{ik}\frac{M}{\gamma}m(\mathbf{v}\nabla\Phi)-\frac{M}{\gamma}m\nabla_i\Phi\mathbf{v}_k. \tag{20}$$

In the absence of external forces acting on the system, the total momentum in the region surrounding the vortex axis should remain constant during the steady-state vortex motion with velocity **v**; whence it follows that

$$0=\dot{P}_i=\int_V\dot{p}_i dV=-\int_S T_{ik}n_k dS, \tag{21}$$

where **n** is the normal to the surface S surrounding the region in question, and consisting of two cylindrical surfaces S_∞ and S_0 of respectively large ($r\to\infty$) and small ($r\to0$) radii. Using the asymptotic expressions obtained in the Appendix, we can easily compute the surface integrals in (21). The integral over S_∞ gives a nonzero contribution only when $m_\infty\neq0$, while the integral over S_0, the sign of which is determined by the polarization ν, always gives a finite contribution, since, like the momentum density **p**, the last two terms in the expression for the momentum flux density diverge as $r\to0$ like $1/r$. Finally, we obtain the following condition for momentum balance:

$$0=\dot{\mathbf{P}}=-2\pi\alpha M^2\varkappa\frac{(\nu-m_\infty)r_0}{\alpha\gamma M}[\mathbf{e}_z\times\mathbf{v}]. \tag{22}$$

The right-hand side of (22) can be called the force exerted on the moving vortex by the homogeneous magnetic substance.

It is clear that, if other forces do not act on the vortex, then this force should vanish, i.e., the vortex should be stationary, in complete agreement with the result obtained in §2 by another method.

A comment should be made about the form of the Lagrangian of the system: there can be some arbitrariness involved in its choice, specifically, it can, in the general case, be written in the following form (see Ref. 10):

$$L=\frac{M}{\gamma}\int_V (C_0+m)\,\dot{\Phi}\,dV-\mathscr{H}. \tag{23}$$

The arbitrary constant C_0 does not enter into the equation of motion of the vortex. We can, through the appropriate choice of this constant, remove the singularities in the expressions for the momentum density and the momentum flux density on the axis of a vortex having only one of the two possible polarization values. Such a nonsingular momentum density is given in Ref. 11. The simultaneous removal of the singularities for oppositely polarized vortices requires a more complicated gauge transformation whose form depends on the configuration of the system of vortices.

§4. MOTION OF A MAGNETIC VORTEX UNDER THE ACTION OF A HELICAL DEFORMATION (OF A SPIN FLUX)

Let us now turn to the case in which the vortex is in a magnetic material subjected to helical deformation, as a result of the action of which the magnetic moment rotates fairly smoothly as we move along the axis of the sample. In this case Φ and m can be represented in the form

$$\Phi=\Phi_0+\Phi_s+\Phi',\quad m=m_0+m', \tag{24}$$

where $\Phi_s=\mathbf{k}\cdot\mathbf{r}$ is the angle of turn in the magnetic material with helical deformation in the absence of a magnetic vortex. Further, substituting (24) into Eqs. (3) and (4), and linearizing them in Φ', Φ_s and m', we obtain equations that differ from (12) and (13) by the replacement of Φ_1 by Φ' and m_1 by m' and the appearance of terms linear in $\mathbf{k}$, which we assume to be small ($k\ll r_0^{-1}$).

Corrections linear in $\mathbf{k}$ will appear in both the solvability condition (14) and the momentum-balance equation (22). The latter, in particular, will assume the following form:

$$0=\dot{\mathbf{P}}=2\pi\varkappa\alpha M^2\left[\mathbf{e}_z\times\left((1-m_\infty^2)r_0\mathbf{k}-\frac{(\nu-m_\infty)r_0}{\alpha\gamma M}\mathbf{v}\right)\right]. \tag{25}$$

Thus, the helical deformation leads to the appearance of forces acting on the vortex, as a result of which the vortex can move with constant velocity

$$\mathbf{v}=\alpha\gamma M(\nu+m_\infty)\mathbf{k}. \tag{26}$$

A similar expression is obtained from the solvability condition for the equations for the corrections to the magnetic structure, corrections which arise as a result of the vortex motion and the helical deformation.

As has already been noted, the helical deformation is connected with the flux of the spin component along the Z axis.[3] In particular, Eq. (4) constitutes a conservation law for this component of the total spin. It follows from it that the flux density of the spin component along the Z axis in a system moving with the vortex velocity $\mathbf{v}$ has the form

$$\mathbf{J}=-\alpha\gamma(M^2-M_z^2)\nabla\Phi-M_z\mathbf{v}. \tag{27}$$

From it we can easily verify that the momentum conservation law can be represented in the following form:

$$\dot{\mathbf{P}}=-\frac{2\pi\varkappa}{\gamma}r_0[\mathbf{e}_z\times(\mathbf{J}|_\infty-\mathbf{J}|_0)], \tag{28}$$

where $\mathbf{J}|_\infty$ and $\mathbf{J}|_0$ are respectively the flux densities of the spin component at infinity and on the vortex axis. This form of the law allows us to see some analogy between the force exerted by the spin flux on the magnetic vortex and the Magnus force exerted by the fluid flux on a vortex in hydrodynamics, though in the second case the corresponding expression does not contain a flux $\mathbf{J}|_0$ near the vortex axis.

§5. THE CANONICAL EQUATIONS FOR A SYSTEM OF RECTILINEAR MAGNETIC VORTICES

Let us now generalize our results for a single vortex to the case of a system of any number N of rectilinear magnetic vortices in a magnetic material that is homogeneous at infinity. Each vortex is located in a deformation field produced by the remaining vortices. Everywhere, including the neighborhoods of the vortex axes (the vortex cores), this field can be represented in the form of a superposition of the deformations induced by the individual vortices, i.e., in the form

$$\nabla\Phi(\mathbf{r})=\sum_{j=1}^{N}\nabla\Phi_j(\mathbf{r})=\sum_{j=1}\{\varkappa_j[\mathbf{e}_z\times(\mathbf{r}-\mathbf{r}_j)]/(|\mathbf{r}-\mathbf{r}_j|^2)\} \tag{29}$$

($\mathbf{r}_j$ is the radius vector of the j-th vortex, and $\varkappa_j$ is its circulation). In computing the deformation determining the velocity of some i-th vortex with the aid of the formula (26), we should choose as $\mathbf{k}$ in (26) the value of $\nabla\Phi(\mathbf{r})$ for $\mathbf{r}\to\mathbf{r}_i$, discarding in (29) the term connected with the self-action, i.e., the term with $i=j$. The justification for such a procedure is similar to the one that is cited in the derivation of the equations of motion of hydrodynamic vortices.[12] As a result, we obtain

$$\mathbf{v}_i=\alpha\gamma M(\nu_i+m_\infty)\sum_{j\neq i}\{\varkappa_j[\mathbf{e}_z\times(\mathbf{r}_i-\mathbf{r}_j)]/(|\mathbf{r}_i-\mathbf{r}_j|^2)\}. \tag{30}$$

The individual terms in (30) represent the forces exerted on the vortex by the rest of the vortices; these forces determine not acceleration, but velocity. The equations of motion (30) for the system of vortices can be written in the form of Hamilton's equations.

The procedure for deriving the Hamiltonian as a function of the vortex coordinates is the same for magnetic vortices[7] as for hydrodynamic vortices.[12] In deriving it, we should proceed from the Hamiltonian (2), discarding all the gradients in it except $\nabla\Phi$, and setting $m\approx m_\infty$, $\nabla\Phi=0$. This assumption is valid everywhere except in small neighborhoods of the vortex axes (the vortex cores). Next, performing the integration, and retaining only the terms that depend on the vortex coordinates, we obtain

$$\mathscr{H}=\alpha M^2\cdot 2\pi(1-m_\infty^2)\sum_{i<k=1}^{N}\varkappa_i\varkappa_k\ln(r_0/|\mathbf{r}_k-\mathbf{r}_i|). \tag{31}$$

The equations of motion (30) can now be rewritten in the form of Hamilton's equations for the Hamiltonian (31):

$$\frac{d\mathbf{r}_i}{dt}=-\left[2\pi\frac{M}{\gamma}(\nu_i-m_\infty)\varkappa_i\right]^{-1}\left[\mathbf{e}_z\times\frac{\partial\mathscr{H}}{\partial\mathbf{r}_i}\right]. \tag{32}$$

If as the canonical variable we choose the coordinate x_i, the canonical momentum conjugate to it is

$$\tilde{p}_i=\left[2\pi l\frac{M}{\gamma}(v_i-m_\infty)\varkappa_i\right]y_i, \tag{33}$$

where l is the dimension of the system along the Z axis. (Let us recall that the Hamiltonian is the energy of the system per unit length along the Z axis.)

§6. CONCLUSION

We have derived the equations of motion of rectilinear magnetic vortices for the simple planar-ferromagnet model, in which we neglect the dipole-dipole interactions, the anisotropy in the easy plane, and the dissipation.

The obtained equations of motion resemble in form the corresponding equations of motion of vortices in hydrodynamics; in particular, in the absence of spin fluxes (in analogues of mass fluxes in hydrodynamics) the vortices are "frozen" in the medium, and cannot move relative to it. But there is also an important divergence from the case of hydrodynamic vortices, since the dynamics of magnetic vortices is determined not be one, but by two topological invariants: to the ordinary invariant—the circulation—is added a new invariant—the polarization (the sign of the magnetic-moment component along the vortex axis in the vortex core).

It follows from our analysis that the magnetic vortex can move relative to the medium only in the case when the magnetization along the difficult axis vanishes both on the vortex axis and at points far from the vortex; this occurs only in an antiferromagnet (in the absence of even weak ferromagnetism) in zero magnetic field. For this case the results obtained by Volovik and Dotsenko[4] remain valid.

In conclusion, let us note that the results of the present paper agree with the result obtained in Ref. 13, with which were able to get acquainted only after we had submitted our paper for publication.

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APPENDIX

To investigate the asymptotic behavior of the magnetic moment of a stationary vortex in a homogeneous ferromagnet, let us analyze the equation (6) for $m_0(r)$. To find the asymptotic form for $r\to\infty$, we can discard all the gradient terms in Eq. (6), after which we obtain (for $\tilde{H}\neq 0$)

$$r\to\infty,\quad m_0(r)=m_\infty(1+\varkappa^2r_0^2/r^2). \tag{A.1}$$

For $\tilde{H}=0$ the solution to Eq. (6) for $r\to\infty$ is

$$m_0=\mathrm{const}\,(r/r_0)^{-1/2}\exp(-r/r_0). \tag{A.2}$$

For $r\to 0$, it is convenient to make the change of variable $m_0=\cos\theta_0$ in Eq. (6); θ_0 then satisfies the following linear equation:

$$\frac{\partial^2\theta_0}{\partial r^2}+\frac{1}{r}\frac{\partial\theta_0}{\partial r}+\left[(1-vm_\infty)r_0^{-2}-\frac{\varkappa^2}{r^2}\right]\theta_0=0; \tag{A.3}$$

near the vortex axis

$$\theta_0=\sqrt{a/2}(1-m_\infty)^{|\varkappa|/2}(r/r_0)^{|\varkappa|}\ \text{for}\ v=+1,$$

$$\theta_0=\pi-\sqrt{a/2}(1+m_\infty)^{|\varkappa|/2}(r/r_0)^{|\varkappa|}\ \text{for}\ v=-1, \tag{A}$$

and consequently

$$m_0=v-av(1-vm_\infty)^{|\varkappa|}(r/r_0)^{2|\varkappa|}, \tag{A.4}$$

where a is some numerical factor of the order of unity.

To find the corrections, due to the motion, to the structure of the vortex, let us substitute the asymptotic forms of m_0 and Φ_0, (A.1), (A.4) and (5), into Eqs. (12) and (13); we then obtain the following linearized system of equations:

For $r\to 0$:

$$\frac{\Delta m_1}{2a(1-vm_\infty)^{|\varkappa|}(r/r_0)^{2|\varkappa|}}-m_1\left[r_0^{-2}-\frac{2\varkappa^2}{r^2}\right]$$

$$-\frac{|\varkappa|}{r_0a(1-vm_\infty)^{|\varkappa|}(r/r_0)^{2|\varkappa|+1}}\left(\frac{\partial m_1}{\partial r}\right)+\frac{2\varkappa v}{r^2}\frac{\partial\Phi_1}{\partial\varphi} \tag{A.5}$$

$$=\frac{\varkappa}{r}\frac{1}{r_0^2\gamma\beta M}\mathbf{v}\mathbf{e}_\varphi,$$

$$2a(1-vm_\infty)^{|\varkappa|}(r/r_0)^{2|\varkappa|}\Delta\Phi_1-\frac{2\varkappa v}{r^2}\frac{\partial m_1}{\partial\varphi}$$

$$+4|\varkappa|r_0^{-1}a(1-vm_\infty)^{|\varkappa|}(r/r_0)^{2|\varkappa|-1}\frac{\partial\Phi_1}{\partial r}$$

$$=2|\varkappa|va(1-vm_\infty)^{|\varkappa|}\frac{(r/r_0)^{2|\varkappa|-1}}{r_0^3\gamma\beta M}\mathbf{v}\mathbf{e}_r, \tag{A.6}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ are the unit vectors of the cylindrical system of coordinates. The solution to this system is (neglecting the small terms)

$$m_1=-\frac{\varkappa}{|\varkappa|}\mathbf{v}\mathbf{e}_\varphi a(1-vm_\infty)^{|\varkappa|}(r/r_0)^{2|\varkappa|+1}\frac{1}{r_0\gamma\beta M},$$

$$\Phi_1=v\mathbf{v}\mathbf{e}_r r\frac{1}{r_0^2\gamma\beta M}. \tag{A.7}$$

For $r\to\infty$

$$\frac{\Delta m_1}{1-m_\infty^2}-m_1\left[r_0^{-2}-\frac{\varkappa^2}{r^2}-\frac{4(1+3m_\infty^2)\varkappa^4m_\infty^2}{r_0^2(1-m_\infty^2)^2(r/r_0)^6}\right]$$

$$-\frac{4\varkappa^2m_\infty^2}{r_0(1-m_\infty^2)^2(r/r_0)^3}\frac{\partial m_1}{\partial r}+\frac{2\varkappa m_\infty}{r^2}\frac{\partial\Phi_1}{\partial\varphi}=\frac{\varkappa}{r}\frac{1}{r_0^2\gamma\beta M}\mathbf{v}\mathbf{e}_\varphi, \tag{A.8}$$

$$(1-m_\infty^2)\Delta\Phi_1-\frac{2\varkappa m_\infty}{r^2}\frac{\partial m_1}{\partial\varphi}+\frac{4\varkappa^2m_\infty^2}{r_0(r/r_0)^3}\frac{\partial\Phi_1}{\partial r}$$

$$=\frac{2\varkappa^2m_\infty}{(r/r_0)^3}\frac{1}{r_0\gamma\beta M}\mathbf{v}\mathbf{e}_r; \tag{A.9}$$

the solution being

$$m_1=-\frac{\varkappa}{r}\frac{1}{\gamma\beta M}\mathbf{v}\mathbf{e}_\varphi,\quad \Phi_1=-\frac{2\varkappa^2m_\infty}{1-m_\infty^2}\frac{\ln(r/r_0)}{r}\frac{1}{\gamma\beta M}\mathbf{v}\mathbf{e}_r. \tag{A.10}$$

The corrections Φ' and m' arising from the helical deformation can be determined in exactly the same way:

for $r\to 0$

$$m'=0,\quad \Phi'=-\mathbf{k}\mathbf{e}_r r; \tag{A.11}$$

for $r\to\infty$

$$m'=\frac{2m_\infty\varkappa}{r}r_0^2\mathbf{k}\mathbf{e}_\varphi,\qquad \Phi'=\frac{4\varkappa^2m_\infty}{1-m_\infty^2}\frac{\ln(r/r_0)}{r}r_0^2\mathbf{k}\mathbf{e}_r. \tag{A.12}$$

[1)] The magnetic vortices being considered by us correspond to the elements of the relative homotopy group $\pi^2(S^2,S^1)$ with degree of mapping $\frac{1}{2}$ and $-\frac{1}{2}$, i.e., those elements for which the points of a plane intersecting a vortex are mapped into the upper or lower half of the sphere S^2, which is the degeneracy space for the order parameter in the vortex core[8] (this was pointed out to us by G. E. Volovik).

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