Propagation of nondecaying *TA* phonons

N. M. Guseĭnov and I. B. Levinson

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

A study has been made of the propagation of superthermal phonons $(\omega > T_0)$ in crystals in which the spontaneous decay of those modes for which it is allowed by the conservation laws occurs more rapidly than elastic scattering by defects. Under such conditions the excitation energy is concentrated in the nondecaying modes. A kinetic equation has been derived to describe the propagation of nondecaying phonons. Besides elastic scattering, it involves effective decay which accompanies elastic scattering with conversion of the nondecaying mode into a decaying mode. In the long wavelength region of the spectrum this equation leads to a linear dependence of the propagation time t of the phonon disequilibrium on the distance r, the velocity r/t being lower than that of sound. Recent experiments on the propagation of phonons in GaAs are analyzed critically and it is shown that the experimental results can be explained naturally by the theory developed in the present work.

PACS numbers: 43.35.Gk

INTRODUCTION

It is now certain that the propagation of nonequilibrium phonons created on cooling and recombination of an electron-hole plasma in a semiconductor, and also on injection from "hot" metallic sources, is accompanied by the decay of phonons.[1-3] Such phonon propagation was first discussed in Refs. 4 to 6, where an isotropic model was used with a single, naturally decaying, phonon branch. In real crystals there are, also, nondecaying modes, usually belonging to the lowest *TA* (transverse-acoustic) branch of the spectrum. This is not of special significance in those crystals for which fast conversion of nondecaying to decaying modes occurs on elastic scattering by defects, and the occupation numbers of these modes are evened out. However, in very pure crystals a situation is possible when decay takes place more frequently than elastic scattering; in this case phonons will accumulate in nondecaying modes and the model[4-6] is inapplicable. It is just such a case that will be considered in the present work; it has been stimulated by experiments in which the propagation and lifetime of high-frequency nondecaying *TA* modes in GaAs and other similar crystals have been studied.[2,7-10]

It has been found[2,8] that the propagation time of the phonon disequilibrium from source to detector, t, depends linearly on the distance r between them, $t \propto r$, while the corresponding "velocity" r/t is about one third of the velocity of the slowest transverse sound. The following interpretation was given for this. On stopping the photoexcitation of the plasma the carriers emit longwave *LO* phonons which decay into shortwave *LA* and *TA* phonons. Only the nondecaying *TA* phonons which occupy the frequency region near 1.7 THz (which corresponds to a wavelength of about 8 Å) survive. The group velocity of such shortwave *TA* phonons is 0.8×10^5 cm·s^{-1}, which determines the signal peak arrival time. It is assumed that 1.7-THz phonons are not scattered over a length $r \approx 2$ mm.

However, as will be shown in Sec. 5, the 1.7-THz *TA*-phonon free path due only to scattering by isotopes is not more than 3 μm, so that an interpretation based on ballistic propagation of these phonons over a distance of the order of millimeters must be considered doubtful.

In the present work the propagation of nondecaying *TA* phonons is studied with account taken of their elastic scattering.[1] A *TA* phonon can convert into a *LA* phonon on elastic scattering, and the latter on decaying leads to the appearance of lower frequency *TA* phonons. The result is effective decay of *TA* phonons that accompanies their propagation. It will be shown that such a propagation regime enables the linear relation $r \propto t$ and other experimental results to be explained.

1. THE KINETIC EQUATION IN CLOSED FORM FOR NONDECAYING MODES

The propagation of nonequilibrium phonons is described by the kinetic equation

$$\left[\frac{\partial}{\partial t} + \mathbf{v}(q) \nabla \right] n(q) = S(q) + J(q) \tag{1}$$

for the distribution function $n(q|\mathbf{r},t)$. Here q indicates the phonon mode, i.e., the wave vector $\mathbf{q}$ and the branch number σ; $\mathbf{v}$ is the group velocity, S and J are collision terms responsible for anharmonic three-phonon processes (decay and fusion) and for elastic scattering by point defects. It is assumed below that the occupation numbers of the nonequilibrium phonons of interest are small, $n \ll 1$, and their frequency is above the temperature describing the equilibrium phonons, $\omega > T_0$. Under these conditions the scattering of nonequilibrium by equilibrium phonons can be neglected, and only spontaneous decay need be considered for nonequilibrium phonons, ignoring fusion of phonons and induced decays.[5,6] In what follows, the occupation numbers of decaying modes are denoted by $n_1(q)$ and of nondecaying modes by $n_t(q)$; the choice of the indices l and t is related to the fact that in the isotropic model the decaying modes belong to the longitudinal branch and the nondecaying to the transverse branch.

0038-5646/83/080452-08$04.00

The collision term S for $n \ll 1$ is linearized in n; for a decaying mode it takes the form

$$S(q) = -n_l(q)/\tau(q) + B(q), \tag{2}$$

where τ is the spontaneous decay time and the incoming term is of the form

$$B(q) = \int_l dq' n_l(q') P(q' \to q). \tag{3}$$

P is here the probability of a phonon q arising on spontaneous decay of a phonon q'; it differs from zero only for $\omega_{q'} > \omega_q$. The integral covers all decaying modes, i.e.

$$\int_l dq \ldots = \sum_\sigma \int_{V_{\sigma l}} \frac{d^3q}{(2\pi)^3} \ldots, \tag{4}$$

where $V_{\sigma 1}$ is that part of the Brillouin zone where phonons of the branch σ are decaying. If the mode q is nondecaying, the outgoing term $-n/\tau$ is missing in Eq. (2); the incoming term B is of the same form as Eq. (3).

For elastic scattering by defects

$$J(q) = -n(q)/\tau^*(q) + A(q), \tag{5}$$

where τ^* is the elastic scattering relaxation time and the incoming term is

$$A(q) = \int_{l+t} dq' n(q') W(q' \to q). \tag{6}$$

The integration here is over all modes, both decaying and nondecaying; the probability W contains the factor $\delta(\omega_{q'} - \omega_q)$.

We now assume that decay is the most rapid process for decaying modes, that is,

$$\tau \ll \tau^*, t, r/v, \tag{7}$$

where t and r are the characteristic scales of temporal and spacial nonuniformity. Under these conditions phonons accumulate in nondecaying modes and, as will be seen,

$$n_t \gg n_l. \tag{8}$$

We will also assume that there are no fewer nondecaying than decaying modes, i.e., that in the important frequency region

$$\rho_t(\omega) \gtrsim \rho_l(\omega), \tag{9}$$

where ρ_t and ρ_l are the densities of states of nondecaying and decaying modes:

$$\rho_t(\omega) = \int_t dq \delta(\omega_q - \omega), \quad \rho_l(\omega) = \int_l dq \delta(\omega_q - \omega). \tag{10}$$

The integral determining ρ_t has a meaning similar to Eq. (4) with the only difference that integration is over the region $V_{\sigma t}$, that part of the Brillouin zone where the modes of branch σ are nondecaying.

We now go on to simplify the kinetic equation. On the basis of relation (7) the left-hand part of Eq. (1) and the outgoing term n/τ^* can be neglected by comparison with the outgoing term n/τ. Using Eqs. (8) and (9) the integration in A

can be limited to the nondecaying region. We then obtain the local equation

$$n_l/\tau - B\{n_l\} = A\{n_t\}, \tag{11}$$

which can be regarded as a linear integral equation in n_l. Its formal solution can be written in the form

$$n_l(q) = \int_l dq' A(q') \bar{\tau}(q' | q). \tag{12}$$

The Green function $\bar{\tau}$ of the operator in the left-hand part of Eq. (11) which enters in here has the following physical interpretation. Suppose a decaying phonon q' had been created; decaying and passing through various intermediate states, it finally enters the range of nondecaying modes. The time which the phonon spends in the process of decaying in the elementary volume of phase space of decaying modes dq near q is $\bar{\tau}(q' | q) dq$. Evidently $\bar{\tau}(q' | q)$ differs from zero only for $\omega_q < \omega_{q'}$.

We turn to the equation for a nondecaying mode. Substituting the distribution of Eq. (12) into B, we find

$$B(q) = \int_t dq' n_t(q') \hat{P}(q' \to q) \equiv \hat{B}(q), \tag{13}$$

where

$$\hat{P}(q' \to q) = \int_l dq_1 W(q' \to q_1) \eta(q_1 \to q), \tag{14}$$

$$\eta(q_1 \to q) = \int_l dq_2 \bar{\tau}(q_1 | q_2) P(q_2 \to q). \tag{15}$$

It is easy to see that $\hat{P}(q' \to q)$ differs from zero only for $\omega_q < \omega_{q'}$. The incoming term written in the form of Eq. (13) describes the appearance of nondecaying phonons of lower frequencies due to the effective decay mentioned in the Introduction. The path of this decay is seen in the structure of the kernel $\hat{P}$: on elastic scattering, a nondecaying phonon q' transforms into a decaying phonon q_1, which on decay finally leads to the appearance of a nondecaying phonon q of lower frequency. The factor η is the relative probability that upon decay of q_1 just the phonon q should appear.

Just as for a decaying mode, the integration in the term A for nondecaying mode q can also be limited to the nondecaying region and we take

$$A(q) = \int_t dq' n_t(q') W(q' \to q) \equiv \bar{A}(q). \tag{16}$$

It is convenient to divide the outgoing term in two, putting

$$1/\tau^*(q) = 1/\bar{\tau}(q) + 1/\hat{\tau}(q). \tag{17}$$

Here

$$\frac{1}{\hat{\tau}(q)} = \int_l dq' W(q \to q') \tag{18}$$

is the time for elastic scattering of a nondecaying phonon q relative to its conversion into a decaying phonon. This time can obviously be considered as the lifetime for effective scattering. Furthermore

$$\frac{1}{\bar{\tau}(q)} = \int_t dq' W(q \to q') \tag{19}$$

N. M. Guseĭnov and I. B. Levinson   453

is the elastic-scattering time of a nondecaying phonon q relative to its transformation into a nondecaying phonon.

The kinetic equation for nondecaying modes can now be written as:

$$\left[\frac{\partial}{\partial t}+\mathbf{v}(q)\,\nabla\right]n(q)=\hat{S}(q)+\tilde{J}(q), \tag{20}$$

where

$$\hat{S}(q)=-n(q)/\hat{\tau}(q)+\hat{B}(q), \tag{21}$$

$$\tilde{J}(q)=-n(q)/\tilde{\tau}(q)+\tilde{A}(q). \tag{22}$$

This equation is closed for the distribution function $n(q)$ in the nondecaying modes (the index t is omitted for simplicity); $\hat{S}$ describes an effective decay and $\tilde{J}$ the elastic scattering between nondecaying modes.

2. LONGWAVE PHONONS; SELF-SIMILAR GREEN FUNCTION

The propagation of nondecaying phonons far from the source of their excitation is determined by the Green function of Eq. (20), i.e., by the solution of this equation for a point and instantaneous source. We shall look for the Green function, assuming the source to be monochromatic with frequency $\bar{\omega}$ and the propagating phonons to be longwave. Under these conditions the Green function can be represented in self-similar form and the assumption that the source is monochromatic is unimportant.

The source, which must be written into the right hand part of Eq. (20), has the form

$$\bar{E}\delta(t)\delta(\mathbf{r})\delta(\omega_q-\bar{\omega})/\hbar\bar{\omega}\rho_t(\bar{\omega}), \tag{23}$$

where $\bar{E}$ is the energy released by the source. The solution corresponding to it is denoted by $G_q(\mathbf{r},t)$. Since Eq. (20) does not contain phonon fusion processes, it is clear that

$$G_q(\mathbf{r},\,t)=0,\quad \omega_q>\bar{\omega}. \tag{24}$$

The existence of a self-similar solution for G is related to the fact that for longwave phonons all quantities have a power-law dependence on the phonon frequency ω and the nature of this dependence is not connected with anisotropy. It is therefore convenient to indicate the frequency explicitly and to use instead of q of the set of variables $(\sigma,\,\mathbf{e},\,\omega)$, where $\mathbf{e}=\mathbf{q}/q$. We have for the quantities which describe the decay

$$\tau_\sigma(\mathbf{e},\,\lambda\omega)=\lambda^{-5}\tau_\sigma(\mathbf{e},\,\omega), \tag{25}$$

$$P_{\sigma'\to\sigma}(\mathbf{e}',\,\lambda\omega'\to\mathbf{e},\,\lambda\omega)=\lambda^{2}P_{\sigma'\to\sigma}(\mathbf{e}',\,\omega'\to\mathbf{e},\,\omega). \tag{26}$$

If the elastic scattering obeys the Rayleigh law, then

$$\tau_\sigma^*(\mathbf{e},\,\lambda\omega)=\lambda^{-4}\tau_\sigma^*(\mathbf{e},\,\omega), \tag{27}$$

$$W_{\sigma'\to\sigma}(\mathbf{e}',\,\lambda\omega'\to\mathbf{e},\,\lambda\omega)=\lambda W_{\sigma'\to\sigma}(\mathbf{e}',\,\omega'\to\mathbf{e},\,\omega). \tag{28}$$

The regions $V_{\sigma l}$ and $V_{\sigma t}$ are conical with origin at $\mathbf{q}=0$; the solid angles $\Omega_{\sigma l}$ and $\Omega_{\sigma t}$ correspond to them. The integration of Eq. (4) over decaying modes can thus be written in the new variables in the following way:

$$\int dq\ldots=\sum_l\int_{\Omega_{\sigma l}}\frac{dO}{4\pi}\int_0^{\bar{\omega}}d\omega\rho_\sigma(\mathbf{e},\,\omega)\ldots, \tag{29}$$

where dO is an element of solid angle in the direction $\mathbf{e}$ and

$$\rho_\sigma(\mathbf{e},\,\omega)=\frac{4\pi}{(2\pi)^3}\frac{\omega^2}{s_\sigma^2(\mathbf{e})v_\sigma(\mathbf{e})}, \tag{30}$$

while $s_\sigma(\mathbf{e})$ and $v_\sigma(\mathbf{e})$ are the phase and group velocities. The integral over nondecaying modes is similar, but the solid angle must be replaced by $\Omega_{\sigma t}$.

By using Eqs. (25) to (28) it can be confirmed that

$$\bar{\tau}_{\sigma'|\sigma}(\mathbf{e}',\,\lambda\omega'|\mathbf{e},\,\lambda\omega)=\lambda^{-8}\bar{\tau}_{\sigma'|\sigma}(\mathbf{e}',\,\omega'|\mathbf{e},\,\omega), \tag{31}$$

the factor λ^{-8} arises as the product λ^{-5} (from the decay rate) and λ^{-3} (from the phase volume). It can now be seen that

$$\eta_{\sigma_1\to\sigma}(\mathbf{e}_1,\,\lambda\omega_1\to\mathbf{e},\,\lambda\omega)=\lambda^{-3}\eta_{\sigma_1\to\sigma}(\mathbf{e}_1,\,\omega_1\to\mathbf{e},\,\omega). \tag{32}$$

We thus finally obtain

$$\hat{P}_{\sigma'\to\sigma}(\mathbf{e}',\,\lambda\omega'\to\mathbf{e},\,\lambda\omega)=\lambda\hat{P}_{\sigma'\to\sigma}(\mathbf{e}',\,\omega'\to\mathbf{e},\,\omega), \tag{33}$$

$$\hat{\tau}_\sigma(\mathbf{e},\,\lambda\omega)=\lambda^{-4}\hat{\tau}_\sigma(\mathbf{e},\,\omega), \tag{34}$$

i.e. the power-law dependence of the quantities determining the effective decay is the same as of the quantities describing elastic scattering. This is natural because elastic scattering is the bottleneck of the effective decay process (with conversion of a nondecaying into a decaying mode).

It is convenient, for representing the self-similar solution of Eq. (20) in natural dimensionless variables, to separate the dependence on the frequency and on the polarization factors, i.e. on σ and $\mathbf{e}$. We define the mean times:

$$\frac{1}{\hat{\tau}(\omega)}=\left\langle\frac{1}{\hat{\tau}_\sigma(\mathbf{e},\,\omega)}\right\rangle,\quad \frac{1}{\tilde{\tau}(\omega)}=\left\langle\frac{1}{\tilde{\tau}_\sigma(\mathbf{e},\,\omega)}\right\rangle, \tag{35}$$

where the brackets denote averaging of the nondecaying modes over σ and $\mathbf{e}$, i.e.

$$\langle(\ldots)\rangle=\frac{1}{\rho_t(\omega)}\int_t dq(\ldots)\delta(\omega-\omega_q). \tag{36}$$

We can now write

$$\frac{1}{\hat{\tau}_\sigma(\mathbf{e},\,\omega)}=\frac{\hat{a}_\sigma(\mathbf{e})}{\hat{\tau}(\omega)},\quad \frac{1}{\tilde{\tau}_\sigma(\mathbf{e},\,\omega)}=\frac{\tilde{a}_\sigma(\mathbf{e})}{\tilde{\tau}(\omega)} \tag{37}$$

where the dimensionless factors $\hat{a}$ and $\tilde{a}$ are independent of frequency. We also represent

$$\rho_\sigma(\mathbf{e},\,\omega)=c_\sigma(\mathbf{e})\rho(\omega),\quad \mathbf{v}_\sigma(\mathbf{e})=\mathbf{u}_\sigma(\mathbf{e})v, \tag{38}$$

where ρ is the total density of states and v is the mean velocity determining this density of states. The probability of elastic scattering can be written in the form:

$$W_{\sigma'\to\sigma}(\mathbf{e}',\,\omega'\to\mathbf{e},\,\omega)=\frac{1}{\tilde{\tau}(\omega)\rho(\omega)}\delta(\omega-\omega')b_{\sigma'\to\sigma}(\mathbf{e}'\to\mathbf{e}). \tag{39}$$

It is convenient to represent the kernel of the effective decay as:

$$\hat{P}_{\sigma'\to\sigma}(\mathbf{e}',\,\omega'\to\mathbf{e},\,\omega)=\frac{1}{\hat{\tau}(\omega')\rho(\omega')\omega'}\hat{h}_{\sigma'\to\sigma}(\mathbf{e}'\to\mathbf{e}|\omega/\omega'). \tag{40}$$

It is easy to confirm that the factors b and $\hat{h}$ in Eqs. (39) and (40) are dimensionless and the scale transformations of Eqs. (27), (28) and (33), (34) are fulfilled automatically.

We shall determine the self-similar variables

$$\boldsymbol{\xi}=\mathbf{r}/v\tilde{\tau}(\omega),\quad \eta=t/\hat{\tau}(\omega) \tag{41}$$

N. M. Guseĭnov and I. B. Levinson   454

and find a solution in the form

$$G_\sigma(e, \omega | r, t) = A\omega^\alpha f_\sigma(e | \xi, \eta). \tag{42}$$

The factor A is written so that the function f can be chosen to be dimensionless.

Substituting Eq. (42) into the left hand part of Eq. (20), we have

$$\left[\frac{\partial}{\partial t} + v_\sigma(e)\nabla\right] n_\sigma(e, \omega | r, t)$$

$$= \frac{A\omega^\alpha}{\hat{\tau}(\omega)}\left[\frac{\partial}{\partial \eta} + u_\sigma(e)\frac{\partial}{\partial \xi}\right] f_\sigma(e | \xi, \eta). \tag{43}$$

Substituting Eq. (42) into $\tilde{J}$, we obtain

$$\tilde{J}_\sigma(e, \omega | r, t) = \frac{A\omega^\alpha}{\hat{\tau}(\omega)}\left[-\tilde{a}_\sigma(e) f_\sigma(e | \xi, \eta) + \sum_{\sigma'}\int_{\Omega_{\sigma'}t}\frac{dO'}{4\pi}c_{\sigma'}(e')\right.$$

$$\left. \times f_{\sigma'}(e' | \xi, \eta) b_{\sigma'\to\sigma}(e'\to e)\right]. \tag{44}$$

Transformation of the incoming term in $\hat{S}$ is somewhat more complicated. Taking Eq. (24) into account, we have

$$\hat{B}_\sigma(e, \omega | r, t) = \sum_{\sigma'}\int_{\Omega_{\sigma'}t}\frac{dO'}{4\pi}\int_{\omega}^{\overline{\omega}}d\omega'\rho_{\sigma'}(e', \omega')$$

$$\times G_{\sigma'}(e', \omega' | r, t)\hat{P}_{\sigma'\to\sigma}(e', \omega'\to e, \omega). \tag{45}$$

Substituting here Eqs. (38), (40) and (42) we find

$$\hat{B}_\sigma(e, \omega | r, t) = \frac{A}{\hat{\tau}(\omega)}\sum_{\sigma'}\int_{\Omega_{\sigma'}t}\frac{dO'}{4\pi}\int_{\omega}^{\overline{\omega}}\frac{d\omega'}{\omega}c_{\sigma'}(e')$$

$$\times x^{-3}\hat{h}_{\sigma'\to\sigma}(e'\to e | x)\omega'^\alpha f_{\sigma'}(e' | \xi', \eta'), \tag{46}$$

where

$$x = \omega/\omega', \quad \xi' = r/\hat{v}\tau(\omega'), \quad \eta' = t/\hat{\tau}(\omega').$$

Noticing that

$$\xi' = \xi x^{-4}, \quad \eta' = \eta x^{-4}, \quad \omega'^\alpha = \omega^\alpha x^{-\alpha},$$

and integrating over x instead of ω', we obtain

$$\hat{S}_\sigma(e, \omega | r, t) = \frac{A\omega^\alpha}{\hat{\tau}(\omega)}[-\hat{a}_\sigma(e) f_\sigma(e | \xi, \eta)$$

$$+ \sum_{\sigma'}\int_{\Omega_{\sigma'}t}\frac{dO'}{4\pi}c_{\sigma'}(e')$$

$$\times \int_{\bar{x}}^{1}\frac{dx}{x^{\alpha+5}}f_{\sigma'}(e' | \xi x^{-4}, \eta x^{-4})\hat{h}_{\sigma'\to\sigma}(e'\to e | x)], \tag{48}$$

where $\bar{x} = \omega/\overline{\omega}$. Collecting together the separate terms of the kinetic equation (20) and noting that $\hat{\tau}$ and $\tilde{\tau}$ depend in the same way on ω, it can be seen that the frequency drops out everywhere except for the integration limit $\bar{x}$. However, if $\omega \ll \overline{\omega}$, we can consider $\bar{x} = 0$; under these conditions a self-similar solution actually exists.

The parameter α is found from the condition of conservation of the total energy of nondecaying phonons

$$E(t) = \int d^3r\varepsilon(r, t), \tag{49}$$

where the energy density of the nondecaying phonons is

$$\varepsilon(r, t) = \int_0^{\overline{\omega}}d\omega\,\omega\rho_t(\omega) A\omega^\alpha f(\xi, \eta), \tag{50}$$

while the function f (without indices) is the mean:

$$f(\xi, \eta) = \langle f_\sigma(e | \xi, \eta)\rangle. \tag{51}$$

On integrating over r and ω as was done by Kazakovtsev and Levinson,[5] it can be confirmed that $E(t)$ is independent of t only if $\alpha = 8$, and the upper limit in Eq. (50) can be replaced by infinity; this leads to the requirement $t \gg \hat{\tau}(\omega)$. The parameter α can also be found in a different way[5] by considering the total energy flux through a sphere of radius r:

$$Q(r) = \oint dS\int_t^{\infty}dt\int dq\,\omega_q v_q n_q(r, t), \tag{52}$$

and requiring this flux to be independent of r; here dS is an element of the surface of the sphere. The expression for the flux can be transformed into

$$Q(r) = \oint dS\int_0^{\infty}dt\int_0^{\infty}d\omega\rho_t(\omega)\omega A\omega^\alpha v f(\xi, \eta) \tag{53}$$

where

$$f(\xi, \eta) = \langle u_\sigma(e) f_\sigma(e | \xi, \eta)\rangle. \tag{54}$$

On integrating first over t and then over ω, it can be confirmed that $Q(r)$ is independent of r only if $\alpha = 8$ and $r \gg v\hat{\tau}(\overline{\omega})$.

The factor A is determined by the energy of the source E. If the function f_σ is normalized according to the condition

$$\frac{1}{16\pi}\int_0^{\eta}\frac{d\eta}{\eta}\cdot d^3\xi f(\xi, \eta) = 1, \tag{55}$$

we obtain

$$E = 4\pi Av^3\left[\frac{\rho_t(\omega)}{\omega^2}\right][\hat{\tau}(\omega)\omega^4]^3 \tag{56}$$

the quantities in square brackets are in fact independent of ω.

In the region

$$\omega \ll \overline{\omega}, \quad t \gg \hat{\tau}(\overline{\omega}), \quad r \gg v\hat{\tau}(\overline{\omega}) \tag{57}$$

there is thus a self-similar solution for the Green function

$$G_\sigma(e, \omega | r, t) = A\omega^8 f_\sigma(e | \xi, \eta) \tag{58}$$

with scaling variables of Eq. (41) and normalization according to Eqs. (55) and (56). This solution does not depend on the source frequency $\overline{\omega}$, but only on its total energy. If the source is spread over the spectrum in some frequency interval $\Delta\overline{\omega} \sim \overline{\omega}$, the solution of Eq. (58) and its region of applicability Eq. (57) do not change. In other words, after some effective decays the system forgets everything about the source except its total energy.

We now show that the self-similar distribution of (58) corresponds to a broadening, linear in time, of the phonon disequilibrium. In the range of Eq. (57) the upper limit in Eq. (50) can be put as $\overline{\omega} = \infty$; on changing to integration over η, we find

$$\varepsilon(r, t) \propto t^{-3}\Phi(r/vt), \tag{59}$$

where

N. M. Guseĭnov and I. B. Levinson   455

$$\Phi(x) = \int_0^\infty d\eta\, \eta^2 f(x\eta, \eta). \tag{60}$$

It can be seen from this that with the passage of time the distribution of the energy density is transformed in similar fashion, and the characteristic spatial scale increases linearly with time. The energy flux density behaves in just the same way.

It is important to emphasize that the self-similar solution and the ensuing linear nature of the broadening of the disequilibrium occur independently of the relation between the relaxation times for elastic scattering and for effective decay $\bar\tau$ and $\hat\tau$; it is only important that they should have the same frequency dependence. If decay and scattering depend differently on frequency, as has been suggested,[4-6] a self-similar solution exists only in those cases for which scattering can be neglected (quasiballistics) or when scattering occurs appreciably more frequently than decay (quasidiffusion).

The question of whether the system of equations for the self-similar functions f_σ contains small parameters is very important. It can be seen that all the quantities in this system are due only to the angular and polarization dependences of the spectrum and of scattering probabilities. If we assume that small parameters do not arise from such quantities, the functions f_σ, $\langle f_\sigma \rangle$ and Φ also do not contain small parameters. Under these conditions it can be confirmed that for a linear broadening of the phonon disequilibrium the corresponding velocity r/t is of the order of the velocity of sound v. Actually, however, in many cases the number of nondecaying modes in the low frequency region appreciably exceeds the number of decaying modes: $\rho_t \gg \rho_l$ (see Sec. 3). In this case $\bar\tau \ll \hat\tau$, i.e., elastic scattering of a nondecaying phonon with conversion into a decaying phonon takes place considerably more rarely than without conversion. In other words, effective decay takes place more rarely than elastic scattering. In this case, one can pass from Eq. (20) to the diffusion equation for the function

$$n(\omega\,|\,\mathbf{r},\, t) = \langle n_q(\mathbf{r},\, t) \rangle. \tag{61}$$

It is important to emphasize, however, that the transition to the diffusion equation does not destroy the linear transformation of the spatial scale, and only a reduction in the speed of the rate r/t of broadening of the phonon disequilibrium results. We shall consider this situation in the isotropic model in order not to burden the discussion with unimportant complications.

3. DIFFUSION OF NONDECAYING PHONONS IN THE ISOTROPIC MODEL

In the isotropic model all longitudinal LA modes are decaying and all transverse TA modes are nondecaying. We also assume the probability of scattering not to depend on the direction of the phonon momenta, but only on the polarizations. Under these conditions

$$\hat\tau(q) = \tau_{TA\to LA}(\omega), \tag{62}$$

$$\bar\tau(q) = \tau_{TA\to TA}(\omega), \tag{63}$$

while it follows from the principle of detailed balancing that

$$\tau_{TA\to TA}(\omega)/\dot\tau_{TA\to LA}(\omega) = \rho_{LA}(\omega)/\rho_{TA}(\omega). \tag{64}$$

For all frequencies ω for which the TA branch exists, the ratio (64) is small; in the long wavelength region it is $s_{TA}^3/2s_{LA}^3 \sim 0.1$, and it is even smaller in the short wavelength region.

Since scattering within the TA branch takes place more frequently than scattering with $TA\to LA$ conversion, i.e., more frequently than effective decay, we can study the quasi-diffusion equation[5] instead of Eq. (20) for the isotropic part of the distribution $n_{TA}(\omega) = n(\omega)$. It has the following form:

$$[\partial/\partial t - D(\omega)\nabla^2]\, n(\omega\,|\,r,\, t) = \hat S\{n(\omega)\}, \tag{65}$$

where the diffusion coefficient is

$$D(\omega) = \tfrac{1}{3} v_{TA}^2(\omega)\, \dot\tau_{TA\to TA}(\omega), \tag{66}$$

and the term responsible for decay is

$$\hat S\{n(\omega)\} = -\frac{n(\omega)}{\dot\tau_{TA\to LA}(\omega)} + \int_\omega^\infty d\omega'\, \rho_{TA}(\omega')\, n(\omega')\, \hat P(\omega' \to \omega). \tag{67}$$

Here $v_{TA}(\omega)$ is the group velocity of a TA phonon and the kernel $\hat P$ is obtained from the kernel $\hat P$ of Eq. (13) by averaging over e and e'.

In the long wavelength region Eq. (65) belongs to the type of equations studied elsewhere.[5] For a point source it has the self-similar solution

$$G(\omega\,|\,r,\, t) = A\omega^8 f(\xi,\, \eta), \tag{68}$$

where

$$\eta = t/\dot\tau_{TA\to LA}(\omega), \quad \xi = r/l(\omega),$$

$$l(\omega) = [D(\omega)\,\dot\tau_{TA\to LA}(\omega)]^{1/2} = s\tau_{TA\to LA}(\omega), \tag{69}$$

$$s \equiv s_{TA}(s_{TA}^3/6s_{LA}^3)^{1/2}.$$

The solution (68) is valid for $\omega \ll \bar\omega$, $t \gg \tau^*_{TA\to LA}(\bar\omega)$, and $r \gg l(\bar\omega)$. Asymptotic expansions are known for the function f.[5] For $\xi \gg \eta$

$$f(\xi,\, \eta) = c_1 e^{-\eta}\eta^{-1/2}D(\zeta), \quad \zeta = \xi^2/4\eta,$$

$$D(\zeta) = 2^{1/2}\zeta^{-1/4}e^{-\zeta/2}D_{-1}(\sqrt{2\zeta}), \tag{70}$$

where D_{-1} is the parabolic cylinder function. For $\xi \ll \eta$

$$f(\xi,\, \eta) = c_2\eta^{-1} \quad (\eta \ll 1),$$

$$f(\xi,\, \eta) = c_3\eta^{-1/2}e^{-\eta} \quad (\eta \gg 1). \tag{71}$$

In Eqs. (70) and (71) c_1, c_2, and c_3 are numbers of the order unity, dependent on the form of the function $h(x)$ which is obtained from the function $\hat h$ occurring in Eq. (40) by averaging over e and e'. The factor A is expressed through the source energy $\bar E$; the normalization (55) yields

$$\bar E = 4\pi A s^3 [\rho_{TA}(\omega)/\omega^2]\,[\dot\tau_{TA\to LA}(\omega)\,\omega^4]^3. \tag{72}$$

The quantities in square brackets do not, in fact, depend on frequency.

The energy density propagating from a point source is

$$\varepsilon(r, t) = \frac{1}{16\pi}\frac{\bar E}{(st)^3}\, g\left(\frac{r}{st}\right), \tag{73}$$

where

$$g(z) = \int_0^\infty d\eta \, \eta^2 f(z\eta, \eta). \qquad (74)$$

It follows from normalizing f according to Eq. (55) that

$$\frac{1}{4} \int_0^\infty dz z^2 g(z) = 1. \qquad (75)$$

The asymptotic functions g can be found by using Eqs. (70) and (71); they are:

$$g(0) = c_0, \quad g'(0) = 0, \quad g(z) = c_\infty/z^5 \quad (z \gg 1), \qquad (76)$$

where c_0 and c_∞ are numbers of the order unity.

It follows from Eq. (73) that the transition to the diffusion equation does not alter the nature of the relation between the temporal and spatial scales; it remains linear. The reason is that diffusion is accompanied by decay with the mean phonon frequency decreasing and the diffusion coefficient increasing. Since the functions f and g do not contain small parameters, the propagation velocity in the diffusion regime is of the order of s, which is appreciably less than the velocity of sound. The time evolution of ε at a fixed position in space is shown in Fig. 1; the distribution of ε at a fixed instant is shown in Fig. 2.

The results obtained allow a clear physical interpretation in terms of phonon generations[4,6] produced by effective decay. The generation with frequency ω lives for a time $\tau^*_{TA \to LA}(\omega)$ and during this time propagates over a diffusion length $l(\omega)$ by diffusion. Since $l(\omega)$ increases rapidly with decreasing ω, the total distance r traversed by TA phonons from the source by the given instant t is of the order of the length $l(\omega)$ for the last generation, i.e., the one that which exists at the moment t. This means that phonons with frequency determined by the relation

$$l(\omega) \sim r \qquad (77)$$

reach a detector at a distance r. On the other hand, since the lifetime $\tau^*_{TA \to LA}(\omega)$ also increases rapidly with decreasing ω, the frequency of the generation existing at the moment t is determined by the condition

$$\tau_{TA \to LA}(\omega) \sim t. \qquad (78)$$

Eliminating from Eqs. (77) and (78) the frequency of phonons arriving at the detector, we find the arrival time $t \sim r/s$. The actual frequency of detected phonons is much less than the source frequency:

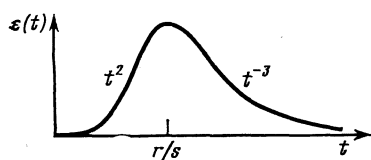$$\omega/\bar\omega \sim [r/l(\bar\omega)]^{-1/4}. \qquad (79)$$

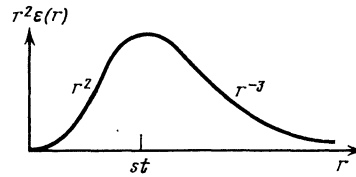FIG. 1. Detector signal for quasidiffusive propagation of a perturbation from a point source.

FIG. 2. Spatial distribution of energy density at a fixed time.

4. SCATTERING TIMES IN GaAs

Scattering by isotopes

An exact formula for the probability of scattering by isotopes in a crystal with one atom per unit cell is given in Ref. 12. It is easy to generalize this expression for the case of scattering in a crystal with several atoms per cell and to obtain the following expression for the scattering rate:

$$\left(\frac{1}{\tau_q}\right)_{\sigma \to \sigma'} = \frac{\pi}{2} a_0^3 \omega^2 \rho_{\sigma'}(\omega) \sum_\alpha \overline{\frac{\delta M_\alpha^2}{M_0^2}} \langle |e_{\alpha,\sigma q} e_{\alpha,\sigma' q'}|^2 \rangle_\omega. \qquad (80)$$

Here q and q' are the initial and final phonon momenta, σ and σ' are the initial and final branches, α is the number of atom in the cell, and the polarization vectors are averaged according to

$$\sum_\alpha \overline{M}_\alpha |e_{\alpha,\sigma q}|^2 = \overline{M}_0, \qquad (81)$$

where M_α is the mass of the atoms and M_0 the mass of a unit cell of volume a_0^3; a bar indicates averaging over the isotopes, $\omega = \omega_{\sigma q} = \omega_{\sigma' q'}$, and the brackets mean averaging for q' over an isoenergetic surface $\omega_{\sigma' q'} = \omega$. In a cubic crystal

$$\langle |e_{\alpha,\sigma q} e_{\alpha,\sigma' q'}|^2 \rangle_\omega = \frac{1}{3} |e_{\alpha,\sigma q}|^2 \langle |e_{\alpha,\sigma' q'}|^2 \rangle_\omega. \qquad (82)$$

Equation (80) with Eq. (82) inserted is equivalent to an earlier form[13] if the difference in the way of normalizing the polarization vectors is taken into account.

For the acoustic branch in the longwave region $|e'|^2 \approx 1$, so that an isotropic scattering time is obtained for a cubic crystal; it is convenient to express the time in the following way:

$$1/\tau_{\sigma \to \sigma'}(\omega) = \eta_{\sigma \to \sigma'} \omega_D (\omega/\omega_D)^4, \qquad (83)$$

where ω_D is the Debye frequency and

$$\eta_{\sigma \to \sigma'} = \frac{1}{12\pi} a_0^3 \left(\frac{\omega_D}{s_{\sigma'}}\right)^3 \delta, \quad \delta = \sum_\alpha \overline{\frac{\delta M_\alpha^2}{M_0^2}}; \qquad (84)$$

here s_σ is the mean velocity of sound for the branch σ, determined from the density of states for this branch. The factor in front of δ is of the order of unity.

Decay

In the isotropic model the TA phonons are nondecaying, while LA phonons decay via two channels:

$$LA \to LA + TA, \quad TA + TA. \qquad (85)$$

This statement is valid both in the long wavelength and short wavelength regions of the spectrum. In the isotropic model the decay of TA phonons is forbidden by the dispersion (if there is no dispersion, collinear decays are possible). Anisot-

ropy larger than the dispersion removes this prohibition. The magnitude of the anisotropy does not depend strongly on frequency, so that the prohibition is relaxed more easily in the long wavelength region where the dispersion is small. This has been illustrated by numerical calculation[14] for an fcc lattice with central forces between nearest neighbors. In this model there are only decaying modes for the lowest TA branch at $q \lesssim 0.5\ q_{max}$; they lie in cones with axes along $\langle 100 \rangle$ in an aperture angle approximately $60°$, so that the fraction of decaying TA modes in the long wavelength region is $\sim 10\%$. This is most likely also the picture for GaAs. Calculations of the two-phonon density of states[13] and the fact that the anisotropy of the velocity of sound in GaAs is close to that which follows from the model indicated above attest to this.

The probability of decay of long wavelength phonons can be conveniently written in the following form:

$$1/\tau_o(\omega,\ e) = \gamma_o(e)\,\omega_D(\omega/\omega_D)^5, \qquad (86)$$

where γ is an adiabatic parameter of the order of $\omega_D/M_0 s^2$ and is independent of the frequency. Accurate values of γ can be calculated by using measured third order elastic moduli. As far as we know, such a calculation has not been made for GaAs, but it has been made for Ge, admittedly in the isotropic model.[15] Since the phonon spectra of GaAs and of Ge are very close, we used for GaAs the values of γ obtained for Ge:

$$\gamma_{LA \to LA+TA} = 2 \cdot 10^{-2}, \qquad \gamma_{LA \to TA+TA} = 4 \cdot 10^{-3}.$$

5. DISCUSSION OF EXPERIMENTS

It was estimated [2,7,8] that short wavelength TA phonons of frequency about 1.7 THz and wavelength 8 Å propagate in GaAs over a distance of the order of a few mm without being scattered by isotopes. Since the usual estimates of the mean free path according to a formula like (83) give an appreciably smaller value, it has been suggested[16] that these phonons, which lie close to the zone edge, are weakly scattered by isotopes or that the oscillation amplitude of the single isotope, Ga, is zero for modes at the zone edge. However, if such a selection rule exists, it is unlikely that it applies to all four directions of q in which "ballistic propagation of shortwave TA phonons" is observed, including the [211] direction of not very high symmetry. Furthermore, if such a selection rule exists it is valid either for the light or for the heavy atom. Yet the isotope in GaAs is the light atom, while in InP and GaP, where the "ballistic propagation effect" is also observed, the isotopes indium and gallium are the heavy atoms. The calculations of oscillation amplitude carried out by Gurevich[13] also confirm this. The relative oscillation amplitude of gallium at the zone edge in the [110] direction is not at all small:

$$|e(Ga)|^2/|e(As)|^2 \approx 0.47.$$

We shall calculate the actual scattering time of 1.7 THz phonons, using Eq. (80). The parameters for GaAs are as follows: $a_0^3 = 45$ Å, $\omega_D = 7.13$ THz $= 4.48 \times 10^{13}$ s^{-1}, $s_{LA} = 5.2 \times 10^5$ cm·s^{-1}, $s_{TA} = 3.1 \times 10^5$ cm·s^{-1}, $\delta = 4.6 \times 10^{-5}$. The dispersion is small for LA phonons in

the frequency region of 1.7 THz, so that when calculating $\tau^*{}_{TA \to LA}$, we can consider

$$\rho_{LA}(\omega) = \frac{1}{2\pi^2} \frac{\omega^2}{s_{LA}{}^3}, \qquad \langle |e_{Ga,LA}|^2 \rangle_\omega \approx 1. \qquad (87)$$

According to Gurevich,[13] we have $|e_{Ga,TA}|^2 \approx 0.6$ for $q\|[110]$. Using this value and Eq. (87), we obtain $\tau^*{}_{TA \to LA} = 0.3\ \mu s$, which for a group velocity $v_{TA} = 0.8 \times 10^5$ cm·s^{-1} corresponds to a mean free path of 0.3 mm. In calculating $\tau^*{}_{TA \to TA}$, the dispersion of TA phonons must be taken into account. On evaluating the density of states (for two branches) according to the isotropic model, we have

$$\rho_{TA}(\omega) = \pi^{-2} q_{TA}{}^2(\omega)/v_{TA}(\omega), \qquad (88)$$

where q_{TA} is the wave vector and v_{TA} is the group velocity. Assuming[13]

$$\langle |e_{Ga,\ TA}|^2 \rangle_\omega \approx |e_{Ga,\ TA}|^2 \approx 0.6,$$

we find $\tau^*{}_{TA \to TA} = 3$ ns, which gives a mean free path of 3 μm. A similar value is obtained if the weighted density of states[13] is used.

These calculations indicate that ballistic propagation of 1.7-THz TA phonons over a distance of several mm is impossible. We therefore consider that the linear relation between the distance to the detector and the arrival time of the signal is to be explained not by ballistic propagation but by a peculiarity of the scattering leading to the existence of the self-similar solution (42) and the linear propagation law for the phonon disequilibrium which follows from it, Eq. (52). The small value of r/t compared with the velocity of sound is explained by elastic scattering occurring more frequently than effective decay; self-similar propagation then corresponds to the quasidiffusive regime which, according to Eq. (69) leads to a reduction in the propagation velocity s. In GaAs $s = 0.58 \times 10^5$ cm·s^{-1} which is extremely close to the propagation velocity of the signal peak measured,[2] $v_{peak} = 0.78 \times 10^5$ cm·s^{-1}.

The primary TA phonons with $\bar{\omega} = 1.7$ THz can be considered as the source, so that the self-similar solution [Eq. (68)] is valid for $\omega \ll 1.7$ THz, $t \gg 0.3$ s, and $r \gg 0.2$ mm. On the other hand, it follows from Eq. (79) that phonons with $\omega \approx 0.9$ THz arrive at the detector for $r = 2$ mm. The conditions for applicability of the self-similar solution are not badly fulfilled for describing the signal at the detector. The dispersion of TA phonons in the 0.9 THz region is slight, justifying the use of the long wavelength description. We now find the scattering time in the important region of 0.9 THz. In this frequency region Eq. (83) can be used for isotope scattering, where

$$\eta_{LA \to LA} = \eta_{TA \to LA} = 3.5 \cdot 10^{-5}, \qquad \eta_{TA \to TA} = \eta_{LA \to TA} = 3.3 \cdot 10^{-4}.$$

This gives

$$\overset{.}{\tau}_{LA \to TA} = \overset{.}{\tau}_{TA \to TA} \approx 0.3\ \mu s, \qquad \overset{.}{\tau}_{TA \to LA} \approx 3\ \mu s.$$

According to Eq. (86) the decay time of 0.9 THz LA phonons $\tau_{LA} \approx 20$ ns. It is seen that decaying phonons decay much more rapidly than they are scattered and that assumption (7) is satisfied in the important frequency region.

N. M. Guseĭnov and I. B. Levinson   458

A weak dependence of signal arrival velocity on propagation direction is observed in the experiments; for example, the velocity in the three principal directions [100], [110] and [111] varies[8] for GaAs from 1×10^5 to 0.8×10^5 cm·s^{-1}. It seems that this scatter lies within the limits of the experimental accuracy, but in principle a dependence on direction can exist. We point to the following fact in this connection. In the quasidiffusion approximation the energy flow from a point source in a cubic crystal possesses spherical symmetry, independently of whether the isotropic model is used or not; the reason is that in a cubic crystal the diffusion tensor reduces to a scalar. Therefore, in the quasidiffusion approximation the signal velocity in GaAs should be the same in all directions. However, the accuracy of this approximation is in fact not very high; it is determined by the parameter

$$\dot{s}_{TA}\dot{\tau}_{TA\to TA}/l \sim (\dot{\tau}_{TA\to TA}/\dot{\tau}_{TA\to LA})^{1/3} \approx 0.3.$$

Violation of the quasidiffusion approximation can lead to some anisotropy.

The so-called "velocity bunching" was also studied experimentally, when phonons were excited by scanning a laser spot with velocity u over a length L', with the detector located on the continuation of the line at a distance d from its end. In this case the phonon energy density at the place where the detector is positioned, i.e., its signal, is according to Eq. (79)

$$\varepsilon(t) = \frac{\beta W}{16\pi} \int_0^{t^*} dt' \frac{1}{[s(t-t')]^3} g\left(\frac{L-ut'}{s(t-t')}\right),$$

$$(89)$$

$$t^* = \min\{t, L'/u\}, \qquad \beta = (1-R)(h\nu - E_g)/h\nu.$$

Here W is the laser power, β the fraction of the quantum energy $h\nu$ transformed into LO phonons on relaxation of electron-hole pairs, E_g is the width of the forbidden band, R the reflection coefficient, $L = L' + d$ is the distance from the initial scanning point to the detector, and $t = 0$ corresponds to the start of the scanning.

For numerical calculations of $\varepsilon(t)$ the function g was approximated by the expression

$$g(z) = A(1 + az^5)^{-1},$$

$$(90)$$

satisfying the asymptote (76) and the normalization (55). The parameter a can be found from the measured[2,8] maximum
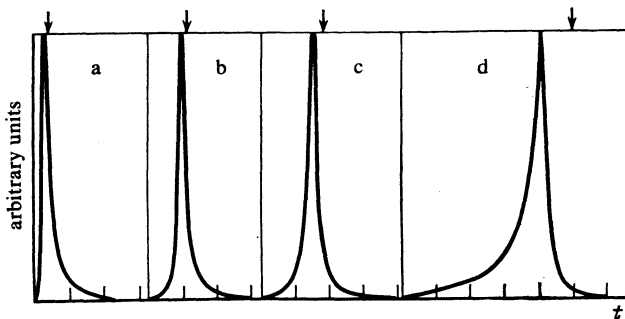
FIG. 3. Calculated detector signal ($L = 2.59$ mm, $d = 0.42$ mm) for different scanning speeds: a) $u = 3.8$; b) $u = 0.47$; c) $u = 0.29$; d) $u = 0.1$ (in units of 10^5 cm·s^{-1}). One division on the abscissa axis is 5 μs; the arrows indicate the measured[7] signal-peak arrival time. The signals are normalized to the same amplitude at the maximum.
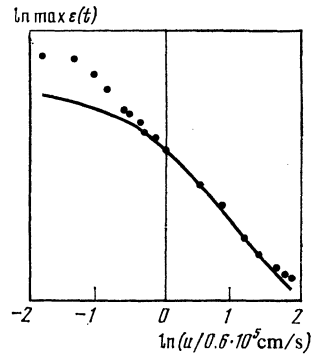
FIG. 4. Dependence of the peak signal amplitude on the scanning speed ($L = 2.59$ mm, $d = 0.42$ mm). The points are experimental,[7] the curve is calculated; for normalization the theoretical curve is made to coincide with the experimental points at $u = s$.

spatial distribution of the signal $r^2\varepsilon(r,t)$ from a point source. Using the signal in GaAs for $t = 2.05$ μs, we find $a = 0.98$. The results of a numerical calculation of $\varepsilon(t)$ are shown in Fig. 3. It can be seen that as the scanning speed u decreases, the steep leading and mildly sloping rear edges change over to mild leading and steep rear edges, and for $u \approx s$ the signal is symmetrical. It is just this behavior that was observed experimentally.[7]

The numerically calculated dependence of the peak signal amplitude, max $\varepsilon(t)$, on the scanning velocity u is shown in Fig. 4 and agrees with measured value quite well. The experimental points for $u \gtrsim s$ fall on a straight line with slope -1, corresponding to a relation max $\varepsilon(t) \propto u^{-1}$. This relation follows from Eq. (89) at $u \gg s$.

The authors are grateful to L. Shchur for the numerical calculations.

[1] D. Hiet and J. P. Maneval, Phonon Scattering in Condensed Matter, Ed. H. J. Maris, Plenum Press, New York (1980), p. 145.
[2] R. G. Ulbrich, V. Narayanamurti, and M. A. Chin, Phys. Rev. Lett. 45, 1432 (1980).
[3] R. E. Horstman and J. Wolter, Phys. Lett. A 85, 369 (1982).
[4] D. V. Kazakovtsev and I. B. Levinson, Pis'ma Zh. Eksp. Teor. 27, 194 (1978) [JETP Lett. 27, 181 (1978)].
[5] D. V. Kazakovtsev and Y. B. Levinson, Phys. Status Solidi B 96, 117 (1979).
[6] Y. B. Levinson, Mol. Cryst. Liq. Cryst. 57, 23 (1980).
[7] P. Hu, V. Narayanamurti, and M. A. Chin, Phys. Rev. Lett. 46, 192 (1981).
[8] R. G. Ulbrich, V. Narayanamurti, and M. A. Chin, J. Phys. (Paris) 42, C6-226 (1981).
[9] K. T. Tsen, D. A. Abramsohn, and R. Bray. Phys. Rev. B 26, 4770 (1982).
[10] D. B. McWhan, P. Hu, M. A. Chin and V. Narayanamurti, Phys. Rev. B 26, 4774 (1982).
[11] N. M. Guseinov and Y. B. Levinson, Solid State Commun. 45, 371 (1983).
[12] V. L. Gurevich, The Kinetics of Phonon Systems [in Russian], Nauka, Moscow (1980).
[13] M. Lax, V. Narayanamurti, P. Hu, and W. Weber, J. Phys. (Paris) 42, C6-161 (1981).
[14] H. J. Maris, Phys. Lett. 17, 228 (1965).
[15] R. S. Markiewicz, Phys. Rev. B 21, 4674 (1980).
[16] M. Lax, P. Hu, and V. Narayanamurti, Phys. Rev. B 23, 3095 (1981).

Translated by R. Berman