A diagram technique for hydrodynamic fluctuations

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Effects produced by long-wave fluctuations in systems of hydrodynamic type are considered. A generating functional is derived, and can be used to formulate a diagram technique with which to take into account higher orders in the interaction between fluctuations. It is shown that to the second order of perturbation theory the diagram technique is equivalent to the kinetic equation for the distribution function of the fluctuations. Corrections to the speed of sound in nematic liquid crystals are calculated as an illustration of the method.

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INTRODUCTION

During recent years the problem of interaction between long-wave (hydrodynamic) fluctuations has been the subject of intensive investigations. There exists an extensive literature devoted to this subject. In Refs. 1–3 corrections to the equations of hydrodynamics were considered for different systems, making use of the method of kinetic equations for the determination of the distribution of fluctuation in nonequilibrium situations produced by hydrodynamic motions. Corrections arise from taking into account terms of third order in the expansion of the entropy in terms of the fluctuating variables; these terms describe the interaction between the fluctuations. It was found that the corrections have a nonlocal character, and their magnitude at low frequencies exceeds that of the Barnett terms.

However, in a number of cases the method of kinetic equations turns out to be essentially useless. This happens in those cases when the interaction between the fluctuations produces essential modifications of the macroscopic equations of the system, such as, for instance, near the points of second-order phase transitions,[4,5] or in two-dimensional systems.[6–8] Taking this interaction into account requires that an adequate method be constructed for the calculation of higher-order effects. The authors have developed such a method, based on a representation of the correlators in the form of path integrals, and allowing one to formulate a standard Feynman diagram technique. A preliminary communication about this method was published as Ref. 9.

In the present paper we describe this method and show that for three-dimensional systems the kinetic equations method and the second-order corrections of perturbation theory coincide. As an illustration of the method we have derived the corrections to the speed of sound in nematic liquid crystals. As will be seen, the main contribution to the correction come from modes which exhibit minimal damping; in a nematic such modes are the oscillations of the director.

1. THE DIAGRAM TECHNIQUE

Let $\varphi_a(t,\mathbf{r})$ be a set of quantities which characterize the long-wave degrees of freedom of the system. An essential characteristic of the system is the entropy functional S depending on these quantities. The equations of motion for φ have the form

$$\frac{\partial\varphi_a}{\partial t}=-F_a+\gamma_{ab}\frac{\delta S}{\delta\varphi_b}-f_a. \tag{1}$$

The first two terms in the right-hand side of Eq. (1) are the standard hydrodynamic reactive (F_a) and dissipative terms; the third term in the right-hand side of Eq. (1) are random forces. It is tacitly assumed in Eq. (1) and in the sequel that if a quantity is labeled by an index it also depends on the corresponding spatial variable r; summation over repeated indices and integration over the corresponding spatial variable are implied. Accordingly, the symbol δ_{ab} denotes a product of the Kronecker delta in the subscripts a, b and a Dirac delta in the difference of the appropriate spatial coordinates.

The reactive terms F_a in the right-hand side of Eq. (1) are entropy-conserving, thus leading to the identity

$$(\delta S/\delta\varphi_a)F_a=0. \tag{2}$$

By differentiating this equation twice and thrice with respect to φ we obtain the following identities (where the superscript denotes functional differentiation with respect to φ_a):

$$Y_{ad}F_b{}^d+Y_{bd}F_a{}^d=0, \tag{3}$$

$$Y_{ad}F_b{}^{dc}+Y_{bd}F_a{}^{dc}+Y_{ad}{}^cF_b{}^d+Y_{bd}{}^cF_a{}^d-Y_{ab}{}^dF_d{}^c+S^{cd}Y_{af}Y_{bh}F_d{}^{fh}=0. \tag{4}$$

Here

$$Y_{ad}S^{db}=\delta_a{}^b. \tag{5}$$

In the expressions (3), (4) we have omitted terms which are linear in F_a and S^a, since at equilibrium these terms vanish.

The dissipative terms in the right-hand side of Eq. (1) must lead to the law of increase of entropy. Thus γ_{ab} is a positive definite integral kernel, which must be symmetric on account of the Onsager reciprocity theorem. The properties permit the following representation:

$$\gamma_{ab}=\overset{1/2}{\gamma_{ad}}\ \overset{1/2}{\gamma_{bd}}. \tag{6}$$

The random forces in the right-hand side of Eq. (1) can be represented in the following form:

$$f_a=-\overset{1/2}{\gamma_{ab}}\xi_b. \tag{7}$$

The correlator (correlation function) for the quantities ξ has the following form:

$$\langle\xi_a(t)\xi_b(t')\rangle=2\delta(t-t')\delta_{ab}. \tag{8}$$

By a standard reasoning one can show (see, e.g., Ref. 10) that

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such a form of the correlator (8) leads to the following Fokker-Planck equation for the distribution function W:

$$\frac{\partial W}{\partial t}=\frac{\delta}{\delta\varphi_a}(F_aW-\gamma_{ab}S^bW+\gamma_{ab}W^b). \tag{9}$$

As can be seen from Eq. (9) and the identity (2), the distribution function relaxes to the equilibrium value

$$W\infty e^S.$$

The factor in front of this expression will depend on the choice of the variables φ and becomes a constant when the reactive forces preserve phase volume

$$F_a{}^a=0.$$

The physical properties of the system are determined by the correlators of the quantities φ which can be determined by averaging over the "microcanonical" ensemble

$$\delta\left(\frac{\partial\varphi_a}{\partial t}+F_a-\gamma_{ac}S^c+f_a\right)\det\left[\delta_a{}^b\frac{\partial}{\partial t}+\frac{\delta}{\delta\varphi_b}(F_a-\gamma_{ac}S^c+f_a)\right].$$

Here the delta function and the determinant are to be understood in the functional sense; the argument of the delta function contains Eq. (1) and the determinant guarantees the normalization of the distribution. We further transform this distribution by a method which is close to that of Ref. 11. We raise the delta function into an exponential making use of the auxiliary variables p^a and express the determinant in terms of a path integral over the fermion fields ψ_a and $\bar\psi^a$. As a result of this the probability density for φ can be obtained by path integration with respect to the variables p, ψ, and $\bar\psi$ of the following distribution function:

$$W=\exp\left\{i\int dt\left[p^a\left(\frac{\partial\varphi_a}{\partial t}+F_a-\gamma_{ac}S^c+f_a\right)+\bar\psi^a\left[\delta_a{}^b\frac{\partial}{\partial t}+\frac{\delta}{\delta\varphi_b}(F_a-\gamma_{ac}S^c+f_a)\right]\psi_b\right]\right\}. \tag{10}$$

In the sequel we shall make use of correlators of various quantities which occur in Eq. (10). We average Eq. (10) over the random forces by integrating with respect to ξ with the weight corresponding to Eq. (8):

$$\exp\left(-\frac{1}{4}\int dt\xi_a\xi_a\right).$$

The distribution function obtained after averaging over ξ allows one to obtain in the standard manner[12,13] the generating functional for the correlators

$$Z(l,y,\nu,\bar\nu)=\int D\varphi\, Dp\, D\bar\psi\, D\psi\exp\left[i\int dt(L+l^a\varphi_a+y_ap^a+\bar\nu^a\psi_a+\bar\psi^a\nu_a)\right]. \tag{11}$$

The Lagrangian in Eq. (11) has the following form:

$$L=p^a\left(\frac{\partial\varphi_a}{\partial t}+\bar F_a\right)-i\bar\psi^a\left(\delta_a{}^b\frac{\partial}{\partial t}+\bar F_a{}^b\right)\psi_b-i\frac{\partial\gamma_{ca}^{1/2}}{\delta\varphi_b}\frac{\delta\gamma_{cd}^{1/2}}{\delta\varphi_h}\bar\psi^a\psi_b\bar\psi^d\psi_h+p_a\bar\psi_d\psi_f\left(\gamma_{ab}^{1/2}\frac{\delta\gamma_{db}^{1/2}}{\delta\varphi_f}-\frac{\delta\gamma_{ab}^{1/2}}{\delta\varphi_f}\gamma_{db}^{1/2}\right). \tag{12}$$

Here

$$\bar F_a=F_a-\gamma_{ab}S^b+i\gamma_{ab}p^b. \tag{13}$$

Expanding the Lagrangian (12) we obtain to second and third orders, respectively

$$L^{(2)}=-ip^aB_a{}^b\varphi_b-\bar\psi^aB_a{}^b\psi_b+ip^a\gamma_{ab}p^b, \tag{14}$$

$$L^{(3)}=-{}^1/_2ip^a(V_{1a}{}^{bc}+V_{2a}{}^{bc}+V_{2a}{}^{cb})\varphi_b\varphi_c+ip^aV_{3ab}^{\ \ c}p^b\varphi_c-\bar\psi^a(V_{1a}{}^{bc}+V_{2a}{}^{bc}+V_{2a}{}^{cb})\varphi_b\psi_c+\bar\psi^aV_{3ab}^{\ \ c}p^b\psi_c. \tag{15}$$

Here

$$B_a{}^b=i(\delta_a{}^b\partial/\partial t+F_a{}^b-\gamma_{ac}S^{cb}). \tag{16}$$

The vertices in Eq. (15) have the form

$$V_{1a}{}^{bc}=i(F_a{}^{bc}-\gamma_{ad}S^{dbc}),$$

$$V_{2a}{}^{bc}=-\frac{\delta\gamma_{ad}}{\delta\varphi_b}S^{dc},\qquad V_{3ab}^{\ \ c}=2\gamma_{bd}^{1/2}\frac{\delta\gamma_{ad}^{1/2}}{\delta\varphi_c}. \tag{17}$$

Starting with Eqs. (14) and (15) one may use the standard Feynman diagram technique for the construction of perturbation theory which takes into account the ternary interaction.

We now introduce the Green's functions which are the pair correlators defined by the generating functional (11);

$$D_{ab}(t-t')=-\langle\varphi_a(t)\varphi_b(t')\rangle,$$

$$G_a{}^b(t-t')=-\langle\varphi_a(t)p^b(t')\rangle=i\langle\psi_a(t)\bar\psi^b(t')\rangle. \tag{18}$$

The last equality can be verified starting from the structure of the generating functional (11) with the Lagrangian (12), or directly, from the distribution function (10), i.e., before averaging over the random fields (the white noise ξ); this equality follows from the fact that in the expansion of the argument of the exponential in Eq. (10) the coefficients of $p^a\varphi_b\varphi_c\cdots\varphi_d$ agree, up to a factor, with the coefficients of $\bar\psi^a\psi_b\varphi_c\cdots\varphi_d$.

The function D is the pair correlator of observable quantities, the function G determines the linear response of the system to the quantity y, which, as can be seen from Eqs. (12), (13), plays the role of an external force for the quantities φ. Thus, the poles of $G(\omega)$ define the eigenvalue spectrum of the linear modes of the system. As follows from the structure of the quadratic Lagrangian (14), to the zeroth approximation

$$B_a{}^bG_b^{(0)c}(t-t')=\delta_a{}^b\delta(t-t'), \tag{19}$$

$$D_{ab}^{(0)}(t-t')=2\int dt''G_a^{(0)c}(t-t'')\gamma_{cd}G_b^{(0)d}(t'-t'').$$

All other pair correlators vanish in this approximation. Taking into account Eq. (19) and the identity (3) one can verify the following relation

$$D_{ab}^{(0)}(t-t')=iG_a^{(0)c}(t-t')Y_{cb}+iY_{ac}G_b^{(0)c}(t'-t). \tag{20}$$

In diagram language this yields the equality represented in Fig. 1. Here the dashed line represents the D function, the

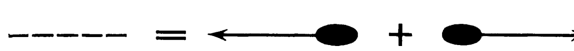

FIG. 1.

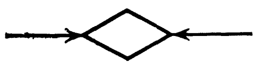

FIG. 2.

solid line the G function, and the black oval the function $-iY$.

Starting with the structure of the expression (10) one can show that the correlators which contain only powers of p vanish (this circumstance cannot change on averaging over ξ). Indeed, when calculating the correlators which do not contain $\bar{\psi}$ and ψ we can integrate the function W with respect to these variables, which yields a determinant corresponding exactly to the transition from the variables φ_a to the variables $\partial\varphi_a/\partial t + F_a + \gamma_{ab}S^b + f_a$. Integrating the distribution function so obtained over the latter variables, we obtain

$$\int D\left(\frac{\partial\varphi_a}{\partial t}+F_a-\gamma_{ab}S^b+f_a\right)\exp\left[ip^a\left(\frac{\partial\varphi_a}{\partial t}+F_a-\gamma_{ab}S^b+f_a\right)\right]$$

$$=\delta(p^a).$$

From this follows directly the assertion we made that the correlators involving only powers of p vanish. This leads to the result that the self-energy function of the φ-φ type also vanishes, since it enters into the diagram in Fig. 2 for the $\langle pp\rangle$ correlator.

Thus, the corrections to the function G are expressed by the diagram of Fig. 3, with the rectangle depicting the self-energy function which in the sequel shall be denoted by Σ.

2. THE SECOND ORDER OF PERTURBATION THEORY

We now enumerate all second-order diagrams contributing to Σ. The diagram represented in Fig. 4 yields the renormalization of the constants, and this renormalization must be included in the redefinition of the original parameters. The diagram depicted in Fig. 5 cancels out a similar diagram with a fermion line on account of Eqs. (15) and (18). Consequently, to second order in the interaction, for Σ one must only consider the diagrams of Fig. 6, where we must substitute as vertices the expressions (16) for V_1, V_2, and V_3 which we denote by black, white-black and white triangles, respectively.

The function $G(\omega)$ has poles only in the lower half-plane, since in the second diagram of Fig. 6 one must consider only the first term on the right-hand side of Eq. (20) in the expression for D.

The expression (17) shows that the diagram equality represented in Fig. 7 is valid. Thus, the diagrams of Fig. 8 cancel identically (here the small black disk denotes the sum $V_{1a}^{bc}+V_{2a}^{bc}+V_{3a}^{bc}$).

We now consider the diagram represented in Fig. 9. Owing to the symmetry of the left vertex, the vertex V_1 on the right enters in this expression in the combination

$$Y_{bh}V_{1a}{}^{ch}+Y_{ah}V_{1b}{}^{ch}$$

$$=-B_a^hY_{hb}{}^c-B_b^hY_{ha}{}^c-S^{cd}(Y_{af}Y_{bh}V_{1d}{}^{fh}+B_d^hY_{hf}Y_{ab}{}^f). \quad (21)$$

FIG. 3.

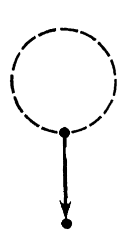

FIG. 4.

The last equality was obtained taking into account the identity (4). The first two terms in the right-hand side of Eq. (21) yield a zero contribution to Σ, since they contain the matrices B, which, when contracted with the corresponding G-functions, yield unity. Thus, the integrands produced by these terms contain no singularities on one side of the real axis, and the integral over the frequencies vanishes. On the mass shell the contribution of the last term contains an extra power of the frequency; the dissipative part of the left vertex in this diagram also has an extra power of the frequency. Therefore, all that was said leads to the result that to lowest order in the frequency of the oscillations one needs to take into account only the third term in Eq. (21), and substitute the reactive part of V_1 for the vertices. Consequently we obtain the following expression for the quantity $\Sigma(\omega, k)$:

$$\Sigma_a{}^b(\omega,k)=\frac{i}{2}\int\frac{d\nu\, d^3q}{(2\pi)^4}V_{1a}{}^{dc}G_c^{(0)m}(\omega+\nu,k+q)G_d^{(0)n}$$

$$\times(-\nu,-q)Y_{mh}Y_{nf}V_{1h}{}^{hf}S^{hb}. \quad (22)$$

The result obtained by computing this expression agrees with the results obtained by means of the kinetic equation.[1-3]

3. THE NONLINEAR HYDRODYNAMIC EQUATIONS OF NEMATICS

The nonlinear equations of the hydrodynamics of nematic liquid crystals have been derived in Ref. 14. We utilize the final form of these equations, omitting the auxiliary angular-momentum-density variable which was introduced in Ref. 14.

The quantities which characterize the hydrodynamics of a nematic are: the mass density ρ, the energy density E, the momentum density $\mathbf{g}$, and the director $\mathbf{n}$. Representing the entropy functional S in the form

$$S=\int d^3\mathbf{r}s(\mathbf{r}),$$

with $s(\mathbf{r})$ the entropy density, the differential of the entropy density can be represented in the form

$$Tds=dE-\mu d\rho-\mathbf{v}d\mathbf{g}+h_idn_i-\nabla_k(\Phi_{ki}n_i). \quad (23)$$

Here T is the temperature, μ is the chemical potential, and $\mathbf{v}$ is the velocity. On account of the identity $\mathbf{n}\cdot d\mathbf{n}=0$, the

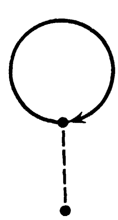

FIG. 5.

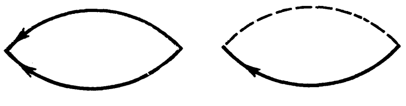

FIG. 6.

relation (23) determines only the part of $\mathbf{h}$ which is orthogonal to $\mathbf{n}$, and we will therefore assume that the longitudinal part of $\mathbf{h}$ vanishes. The pressure P is now defined as follows

$$P=\mu\rho+Ts+\mathbf{vg}-E. \tag{24}$$

The nonlinear hydrodynamics equations for nematics have the following form

$$\frac{\partial\rho}{\partial t}+\nabla\mathbf{g}=0,$$

$$\frac{\partial g_i}{\partial t}+\nabla_k\Pi_{ik}=\nabla_k\frac{\partial R}{\partial\nabla_k v_i}$$

$$\frac{\partial E}{\partial t}+\nabla\mathbf{Q}=\nabla_k\left(v_i\frac{\partial R}{\partial\nabla_k v_i}+T\frac{\partial R}{\partial\nabla_k T}+\Phi_{ki}\frac{\partial R}{\partial h_i}\right),$$

$$\frac{\partial n_i}{\partial t}+v_k\nabla_k n_i+{}^1/_2 n_k(\nabla_i v_k-\nabla_k v_i)$$

$$-{}^1/_2\lambda(\delta_{im}-n_i n_m)n_n(\nabla_n v_m+\nabla_m v_n)=\partial R/\partial h_i. \tag{25}$$

Here λ is a reactive parameter and R is the dissipative function. The reactive stress tensor and energy-flux density are, respectively

$$\Pi_{ik}=P\delta_{ik}+\rho v_i v_k+\Phi_{km}\nabla_i n_m+{}^1/_2\lambda(n_k h_i+n_i h_k)+{}^1/_2(n_k h_i-h_k n_i), \tag{26}$$

$$Q_i=\rho\mu v_i+Tsv_i+(\mathbf{vg})v_i-\Phi_{ik}\left(\frac{\partial n_k}{\partial t}-\frac{\partial R}{\partial h_k}\right)$$

$$+{}^1/_2(n_i(\mathbf{hv})-h_i(\mathbf{nv})+\lambda n_i(\mathbf{hv})+\lambda h_i(\mathbf{nv})). \tag{27}$$

The invariance of the entropy with respect to spatial rotations allows one to reduce the divergence of (26) to the divergence of a symmetric tensor

$$\Pi_{ik}^s=P\delta_{ik}+\rho v_i v_k+{}^1/_2\lambda(n_k h_i+h_k n_i)+{}^1/_2(\Phi_{km}\nabla_i n_m+\Phi_{im}\nabla_k n_m)$$
$$+{}^1/_2\nabla_m(\Phi_{ki}n_m+\Phi_{ik}n_m-\Phi_{km}n_i-\Phi_{im}n_k). \tag{28}$$

Taking account of the identity (23), the system of equations (25) yields the following equation for the entropy density:

$$\frac{\partial s}{\partial t}=-\nabla(\mathbf{v}s)+\nabla_i\frac{\partial R}{\partial\nabla_i T}$$

$$+\frac{1}{T}\left(h_i\frac{\partial R}{\partial h_i}+\nabla_i T\frac{\partial R}{\partial\nabla_i T}+\nabla_k v_i\frac{\partial R}{\partial\nabla_k v_i}-R\right). \tag{29}$$

The dissipative function R which occurs in (25) and (29) has the following expression in the quadratic approximation

$$R=\eta_{iklm}V_{ik}V_{lm}+{}^1/_2\varkappa_{ik}\nabla_i T\nabla_k T+h_i h_i/2\gamma_1, \tag{30}$$

where

$$V_{ik}={}^1/_2(\nabla_i v_k+\nabla_k v_i),\quad \varkappa_{ik}=\varkappa_0\delta_{ik}+\varkappa_1 n_i n_k,$$

$$\eta_{iknm}=\eta_1\delta_{in}\delta_{km}+\eta_2\delta_{ik}\delta_{nm}+\eta_3\delta_{ik}n_n n_m+\eta_4\delta_{in}n_k n_m+\eta_5 n_i n_k n_n n_m.$$

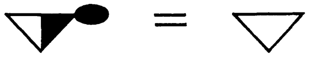

FIG. 7.

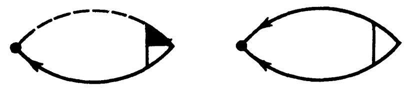

FIG. 8.

Here η are the viscosity coefficients and κ are the heat conduction coefficients. We note that by reducing the equations to the form (1) we will obtain an operator γ containing, according to Eq. (23), the quantities ηT, κT, and T/γ_1. All the viscosity coefficients η_a have the same order of magnitude and the coefficient γ_1 in Eq. (30) is of the same order and has the dimension of viscosity. We introduce the dimensionless parameters

$$b_1=\gamma_1/\eta_1,\quad b_a=\eta_a/\eta_1\quad (a=2,3,4,5). \tag{31}$$

In order to determine the quantities h_i and Φ_{ik} we use the expression of the energy density E as a function of ρ, $\mathbf{g}$, s, and $\mathbf{n}$:

$$E=g^2/2\rho+\varepsilon(\rho,s)+E_F. \tag{32}$$

Here ε is the internal energy and E_F is the Frank energy related to the inhomogeneity of the director field.

Starting from Eqs. (23) and (32) we obtain

$$h_i=(\delta_{ik}-n_i n_k)\left(\nabla_m\frac{\partial E_F}{\partial\nabla_m n_k}-\frac{\partial E_F}{\partial n_k}\right),\quad \Phi_{ik}=\frac{\partial E_F}{\partial\nabla_i n_k}. \tag{33}$$

The Frank energy is not uniquely determined. In particular, one may always add to it the divergence of some quantity. As will be shown below, we shall be interested in vertices of the type $p\nabla n\cdot\nabla n$. Therefore, neglecting terms of the type $n_k\nabla_k N\nabla_i n_i$ we can write the Frank energy in the quadratic approximation in the form

$$E_F={}^1/_2N_1\nabla_i n_i\nabla_k n_k+{}^1/_2N_2\nabla_i n_k\nabla_i n_k+{}^1/_2N_3 n_i n_k\nabla_i n_m\nabla_k n_m. \tag{34}$$

This expression corresponds to the standard form[15] with the Frank coefficients:

$$K_1=N_1+N_2,\quad K_2=N_2,\quad K_3=N_3+N_2.$$

We introduce the following notations:

$$a_1=N_1/N_2,\quad a_3=N_3/N_2. \tag{35}$$

We shall assume that at equilibrium the director is along the third axis and choose as two independent variables describing its oscillations the components n_1 and n_2 which vanish at equilibrium. The set of variables entering into the Lagrangian will be denoted by φ_E, φ_ρ, φ_i, and φ_α^n—they are the deviations from the corresponding equilibrium values (the variables p have the appropriate indices too). Here and in the sequel Latin indices always take on the values 1, 2, 3 and Greek indices run from 1 to 2.

An analysis of the linearized equations (25) shows that in the following modes are possible in a nematic: a sound mode (two eigenvalues $\omega=\pm ck$), a thermal mode, and four diffusion modes, which, if one neglects a small mixing, are respectively the diffusion of the transverse velocity and the diffusion of the director oscillations.

In the case of sound propagation its velocity must satisfy the inequality $\omega\gg\Gamma_s k^2$, where $\Gamma_s\approx\eta/\rho$ is the sound

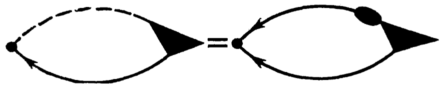

FIG. 9.

damping coefficient. Consequently we have the inequality

$$\alpha_1=\eta\omega/\rho c^2\ll 1.$$

We restrict ourselves to calculating the corrections to order zero in the small parameter α_1.

Making use of Eqs. (25) we can now write down the original Lagrangian for nematic liquid crystals, restricting ourselves to terms of the second and third order. We carry out the following linear transformation:

$$\varphi_E=\frac{T\rho^2}{c^4}(c^2-\theta Z)\tilde{\varphi}_1+\frac{Z}{c^2}\tilde{\varphi}_2,\qquad \varphi_\rho=-\frac{\rho T\theta}{c^2}\tilde{\varphi}_1+\frac{1}{c^2}\tilde{\varphi}_2,$$

$$\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}=\rho\frac{1}{k}\begin{pmatrix}k_1\\ k_2\\ k_3\end{pmatrix}\tilde{\varphi}_3+\rho\frac{1}{k_\perp k}\begin{pmatrix}-k_1k_3\\ -k_2k_3\\ k_\perp^2\end{pmatrix}\tilde{\varphi}_4$$

$$+\rho\frac{1}{k_\perp}\begin{pmatrix}-k_2\\ k_1\\ 0\end{pmatrix}\tilde{\varphi}_6, \tag{36}$$

$$\begin{pmatrix}\varphi_1{}^n\\ \varphi_2{}^n\end{pmatrix}=\frac{1}{k_\perp}\begin{pmatrix}k_1\\ k_2\end{pmatrix}\tilde{\varphi}_5+\frac{1}{k_\perp}\begin{pmatrix}-k_2\\ k_1\end{pmatrix}\tilde{\varphi}_7,$$

and the appropriate transformations for p, guaranteeing that the determinant of the transformations is unity. Here

$$Z=T\sigma+\mu,\quad \theta=(\partial\ln T/\partial\ln\rho)_\sigma,\quad s=\sigma\rho,\quad c^2=(\partial P/\partial\rho)_\sigma,$$

and c is the speed of sound. In the new variables the matrix B decomposes into two blocks. The first block B^1 for the variables 1–5 has the form

$$\begin{vmatrix}
\omega+i\dfrac{k^2\tilde{\varkappa}T}{\rho c_p} & i\dfrac{T\theta}{\rho^2c^2}\tilde{\varkappa}k^2 & 0 & 0 & 0\\
i\dfrac{\theta T^2}{c_p}\tilde{\varkappa}k^2 & \omega+i\dfrac{\theta^2T^2}{\rho c^2}\tilde{\varkappa}k^2 & -k\rho c^2 & 0 & 0\\
0 & -\dfrac{1}{\rho}k & \omega+i\Gamma_sk^2 & \dfrac{1}{\rho}i\tilde{\eta}k^2 & \lambda\dfrac{k_3k_\perp}{k\rho}N_2k^2d_1\\
0 & 0 & \dfrac{1}{\rho}i\tilde{\eta}k^2 & \omega+i\Gamma_\parallel k^2 & \tilde{\lambda}\dfrac{k}{\rho}N_2k^2d_1\\
0 & 0 & \lambda\dfrac{k_3k_\perp}{k} & \tilde{\lambda}k & \omega+i\dfrac{N_2k^2}{\gamma_1}d_1
\end{vmatrix}.$$

Here

$$\tilde{\varkappa}=\varkappa_0+\frac{k_3^2}{k^2}\varkappa_1,\quad \tilde{\eta}=(\eta_3+\eta_4)\frac{k_3k_\perp}{k^2}+2\eta_5\frac{k_3^3k_\perp}{k^4},$$

$$\Gamma_s=\frac{2}{\rho}\left(\eta_1+\eta_2+\eta_3\frac{k_3^2}{k^2}+\eta_4\frac{k_3^2}{k^2}+\eta_5\frac{k_3^4}{k^4}\right),$$

$$\Gamma_\parallel=\frac{1}{\rho}\left(\eta_1+\frac{1}{2}\eta_4+2\eta_5\frac{k_3^2k_\perp^2}{k^4}\right)$$

$$\tilde{\lambda}=-\frac{\lambda+1}{2}\frac{k_3^2}{k^2}+\frac{\lambda-1}{2}\frac{k_\perp^2}{k^2}, \tag{37}$$

$$d_1=1+a_1+(a_3-a_1)\frac{k_3^2}{k^2}.$$

The second block B^2 for the variables 6–7 has the form

$$\begin{vmatrix}
\omega+i\Gamma_\perp k^2 & \dfrac{\lambda+1}{2\rho}k_3N_2k^2d_2\\
\dfrac{\lambda+1}{2}k_3 & \omega+\dfrac{i}{\nu_4}N_2k^2d_2
\end{vmatrix},$$

where

$$\Gamma_\perp=\frac{1}{\rho}\left(\eta_1+\frac{1}{2}\eta_4\frac{k_3^2}{k^2}\right),\quad d_2=1+a_3\frac{k_3^2}{k^2}. \tag{38}$$

The matrix B^1 has five eigenvalues and five eigenvectors: a sound mode (two eigenvalues) and three diffusion modes. The matrix B^2 describes two diffusion modes. Each of the matrices B^1 and B^2 has one eigenvalue with an imaginary part much smaller than the imaginary parts of the other modes. These eigenvalues are

$$\omega_1=-\frac{i}{\eta_1}N_2k^2\tau_1,\quad \omega_2=-\frac{i}{\eta_1}N_2k^2\tau_2, \tag{39}$$

where

$$\tau_1=\frac{1}{b_1}d_1\left(1+b_1\tilde{\lambda}^2\frac{1}{1+{}^1/_2b_4+b_5x^2-b_5x^4}\right),$$

$$\tau_2=\frac{d_2}{b_1}\left[1+\left(\frac{\lambda+1}{2}\right)^2\frac{b_1x^2}{1+{}^1/_2b_4}\right],\quad x=\frac{k_3}{k}. \tag{40}$$

The eigenvectors corresponding to these diffusion modes describe oscillations of the director. In the first case the oscillations are in $\tilde{\varphi}_5$, and in the second case in $\tilde{\varphi}_7$. This is true in the zeroth approximation in the parameter $\alpha_2=N_2\rho/\eta_1^2\approx 10^{-2}\ll 1$.

4. CALCULATION OF THE CORRECTIONS

In order to determine Σ we must go over from the B matrix to the G matrix. The latter can be represented in the form

$$G_a{}^b=\sum_{\omega_c}\frac{1}{\omega-\omega_c}\mathbf{b}^b(\omega_c)\mathbf{a}_a(\omega_c)\quad (a,b,c=1,2,\dots,7), \tag{41}$$

where ω_c are the eigenvalues for the right eigenvector $\mathbf{a}(\omega_c)$

and the left eigenvector $\mathbf{b}(\omega_c)$ of the matrix B. The contribution of each mode to the self-energy function and thus to the dispersion of sound is the larger, the smaller the damping of the mode is (this contribution is inversely proportional to $\Gamma^{3/2}$, where Γ is the damping of the mode). Therefore in a nematic one must take into account only the modes ω_1 and ω_2 (39), which are damped by a factor of about 10^2 less than the other modes. Consequently, taking into account only these two weakly damped modes, we can represent the matrix G in the form

$$G_a^{\ b}=\frac{1}{\omega-\omega_1}\mathbf{b}_1^{\ b}\mathbf{a}_{1a}+\frac{1}{\omega-\omega_2}\mathbf{b}_2^{\ b}\mathbf{a}_{2a}. \tag{42}$$

As already noted, in zeroth order in the parameter α_2 the eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ describe the oscillations of the director. Therefore, starting with an analysis of the vertices, we also see that in zeroth order in the parameters α_1 and α_2 the contribution to the corrections to the spectrum of sound oscillations come only from the vertex types $\tilde{p}_3\tilde{\varphi}_5\tilde{\varphi}_7$, $\tilde{p}_3\tilde{\varphi}_5\tilde{\varphi}_5$, and $\tilde{p}_3\tilde{\varphi}_7\tilde{\varphi}_7$; this also means that it is necessary to calculate only the matrix element Σ_{33}. In the terminology of Refs. 1–3 this means that there are fluctuation corrections only to the stress tensor.

It is convenient to express the vertices which make a substantial contribution in terms of the variables $\tilde{p}_3$ and φ^n_α, or, denoting $\varphi^n_1=n_1$ and $\varphi^n_2=n_2$, we obtain

$$V_1=\rho^{-1}U\tilde{p}_3,$$

$$\begin{aligned}U=&(N_2+N_3)\nabla_3 n_\sigma\nabla_3 n_\sigma\frac{k_3^2}{k}+(N_1\nabla_\sigma n_\sigma\nabla_3 n_\alpha+2N_2\nabla_3 n_\sigma\nabla_\alpha n_\sigma\\&+N_3\nabla_3 n_\sigma\nabla_\alpha n_\sigma)\frac{k_3k_\alpha}{k}+(N_1\nabla_\sigma n_\sigma\nabla_\alpha n_\beta+N_2\nabla_\alpha n_\sigma\nabla_\beta n_\sigma)\frac{k_\alpha k_\beta}{k}\\&+\frac{1}{2}(\theta_1N_1\nabla_\sigma n_\sigma\nabla_\lambda n_\lambda+\theta_2N_2\nabla_m n_\sigma\nabla_m n_\sigma+\theta_3N_3\nabla_3 n_\sigma\nabla_3 n_\sigma)\frac{k^2}{k}\\&-\lambda(N_1n_\alpha\nabla_\alpha\nabla_\sigma n_\sigma+N_2n_\sigma\nabla_m^2n_\sigma+N_3n_\sigma\nabla_3^2n_\sigma)\frac{k_3^2}{k}\\&-\lambda(N_3\nabla_\alpha n_\sigma\nabla_3 n_\sigma+N_1n_\alpha\nabla_\sigma\nabla_3 n_\sigma)\frac{k_\alpha k_3}{k}\\&+\lambda(N_1n_\beta\nabla_\alpha\nabla_\sigma n_\sigma+N_2n_\beta\nabla_m^2n_\alpha+N_3n_\beta\nabla_3^2n_\alpha)\frac{k_\alpha k_\beta}{k},\end{aligned} \tag{43}$$

where

$$\theta_a=(\partial\ln N_a/\partial\ln\rho)_\sigma-1-\theta,\quad a=1,\ 2,\ 3.$$

We can now go over to a calculation of the self-energy function according to (22). Leaving out the lengthy calculations, we find

$$\Sigma(\omega,k)=\frac{1}{16\pi}\frac{i+1}{\sqrt{2}}\frac{T}{\rho}\left(\frac{\eta_1}{N_2}\right)^{3/2}\omega^{1/2}k^2\int_{-1}^{1}dx f(x), \tag{44}$$

$$\begin{aligned}f(x)=&\frac{1}{d_1^2}\frac{1}{(2\tau_1)^{3/2}}\frac{1}{k^4}\Big[k_3^4f_1^2+\frac{1}{4}k^4f_3^2+\frac{1}{2}k_3^2k_\perp^2f_2^2\\&+{}^3/_8k_\perp^4(1+a_1-\lambda a_1)^2(1-x^2)^2+{}^3/_8k_\perp^4\lambda^2(1+a_3x^2)^2+k_3^2k^2f_1f_3\\&+k_3^2k_\perp^2f_1(1-x^2)(1+a_1-\lambda a_1)-k_3^2k_\perp^2f_1\lambda(1+a_3x^2)\\&+{}^1/_2k^2k_\perp^2f_3(1-x^2)(1+a_1-\lambda a_1)-{}^1/_2k_2^2k_\perp^2(1+a_3x^2)\lambda f_3\\&-{}^3/_4k_\perp^4(1+a_1-\lambda a_1)\lambda(1-x^2)(1+a_3x^2)\Big]\\&+\frac{1}{d_2^2}\frac{1}{(2\tau_2)^{3/2}}\frac{1}{k^4}\Big\{k^4[{}^1/_4(\theta_2+a_3\theta_3x^2)^2+{}^3/_8(1-x^2)^2\\&+{}^3/_8\lambda(1+a_3x_2)^2\\&+{}^1/_2(1-x^2)(\theta_2+a_3\theta_3x^2)-{}^1/_2(\theta_2+\theta_3a_3x^2)\lambda(1+a_3\theta_3x^2)\\&-{}^1/_2\lambda(1+a_3x^2)(1-x^2)]+k^2k_3^2[-{}^1/_4(1-x^2)^2-{}^3/_4\lambda^2(1+a_3x^2)\\&+(x^2+x^2a_3-\lambda-\lambda a_3x^2)(\theta_2+a_3\theta_3x^2)+(1-x^2)(x^2+a_3x^2-\lambda-\lambda a_3x^2\\&-{}^1/_2\theta_2-{}^1/_2a_3\theta_3x^2)-\lambda(x^2+x^2a_3-\lambda-\lambda a_3x^2)(1+a_3x^2)\\&+{}^1/_2\lambda(\theta_2+\theta_3a_3x^2)(1+a_3x^2)-\lambda(1-x^2)(1+a_3x^2)]\\&+k_3^4[(x^2+x^2a_3-\lambda-\lambda a_3x^2)^2+{}^3/_8(1-x^2)^2+{}^3/_8\lambda^2(1+a_3x^2)^2\\&+(x^2+x^2a_3-\lambda-\lambda a_3x^2)(\theta_2+a_3\theta_3x^2)-(1-x^2)(x^2+a_3x^2-\lambda-\lambda a_3x^2)\\&+\lambda(x^2+x^2a_3-\lambda-\lambda a_3x^2)(1+a_3x^2)-{}^1/_2(1+a_3x^2)(1-x^2)]\Big\}\\&+\frac{1}{d_1d_2}\frac{1}{(\tau_1+\tau_2)^{3/2}}\frac{1}{k^4}\{{}^1/_2k_3^2k_\perp^2a_1^2(1+\lambda)^2x^2(1-x^2)+{}^1/_4k_\perp^4\\&\times[a_1(1-x^2)-\lambda a_1(1-x^2)-2-2a_3x^2]^2\},\end{aligned}$$

$$f_1(x)=x^2+x^2a_3-\lambda a_1-\lambda a_1x^2+\lambda+\lambda a_3x^2,$$

$$f_2(x)=(a_1+2+a_3-\lambda a_3+\lambda a_1)(1-x^2),$$

$$f_3(x)=\theta_1a_1(1-x^2)+\theta_2+\theta_3a_3x^2,$$

where the notation for the parameters correspond to the expressions (31), (35), (37), (38), and (40). The dependence of $\Sigma(\omega,k)$ on the frequency turns out to be the same as for the systems which were considered before in Refs. 1–3.

For sound oscillations the correction to the spectrum $\omega=ck-i\Gamma_sk^2$ necessitated by Σ_{33} has the form

$$\delta\omega=-{}^1/_2\Sigma_{33}(ck,\ k). \tag{45}$$

To estimate the self-energy function we make use of the following relation

$$\Sigma\approx\frac{1+i}{64\pi}\frac{T\omega^{1/2}}{\rho}\left(\frac{\gamma_1}{N_2}\right)^{3/2}k^2. \tag{46}$$

As can be seen from the final formula (44) the quantity Σ_{33} has a complicated dependence on the angles, but this dependence is smooth, and the order of magnitude of Σ_{33} is the same in all directions.

The real part of the sound oscillation spectrum must be much larger than its imaginary part, yielding the inequality $\omega\ll\rho c^2/\eta_1$. Therefore we obtain the following inequality for the magnitude of the relative correction to the speed of sound

$$\frac{\delta c}{c}\ll\frac{T}{128\pi}\frac{1}{\rho c^2}\left(\frac{\rho c^2}{N_2}\right)^{3/2}\left(\frac{\gamma_1}{\eta_1}\right)^{3/2}.$$

Making use of the numerical values for MBBA,[5] one can determine $\delta c/c\ll10^{-2}$–10^{-3}. The corrections turn out to be negligible. However near the point of transition of the nematic into an isotropic liquid one can expect the Frank constants to decrease, on account of their strong temperature dependence (since this is a weak first-order transition). This may lead to a growth of the fluctuations and since the corrections are proportional to $N^{-3/2}$, also to a considerable growth of the dispersion of the sound velocity and of the sound damping.

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