Theory of the origin of large-scale structures in hydrodynamic turbulence

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We show in this paper that helical hydrodynamic turbulence can magnify perturbation seed eddies and that this can lead to the appearance of large-scale structures. The characteristic size of the structures which appear is much larger than the size of the turbulence containing the energy and it is determined by the ratio of the topological invariant to the turbulence energy. The evolution of the instability found in this paper is accompanied by a transfer of energy from small-scale to large-scale fluid motions.

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INTRODUCTION

The problem of the origin of structures in non-equilibrium systems has recently attracted a great deal of attention. (See in this connection the monographs 1 to 3 and the literature cited there.) Hydrodynamic turbulence is clearly one of the most widespread distributed non-equilibrium systems in nature. It is well known that turbulence is characterized by two scales: λ_0, the external scale which contains the main energy and the dissipation scale $\lambda_v \ll \lambda_0$ where the energy is dissipated into heat. A turbulent system is thus an open non-equilibrium system.

The traditional presentation of turbulence is connected with that kind of disordered fluid motion in which any large-scale perturbation transfers its energy through random chaotic motions to ever smaller scales. It would seem that such ideas about turbulence are incompatible with the existence of coherent structures in it. However, recent papers (see, e.g., Refs. 4 to 7) indicate that there may appear in turbulence large-scale vortex formations, the nature of which in final analysis is not explained.

In a homogeneous isotropic turbulence a large-scale perturbation (fluctuation) transfers its energy, apparently, to small-scale pulsations, owing to damping by turbulent viscosity. If, however, the turbulence is helical, i.e., homogeneous, isotropic, but without being invariant under reflection, the situation is greatly changed, as we show in the present paper. In that case the amplification of large-scale turbulent fluctuation nuclei is possible and this leads to the appearance of large-scale coherent structure. The characteristic size L of the vortex structure is determined by the internal properties of the turbulence: $L^{-1} \sim I_\omega / E$, where $I_\omega = \int \langle \mathbf{V} \operatorname{curl} \mathbf{V} \rangle d\mathbf{r}$, $E = \int V^2 d\mathbf{r}$ are, respectively, the topological and the normal invariant. In other words, the size of the structure is in some sense determined by the "thermodynamic" parameters of the turbulence. It is thus possible in helical turbulence to transfer energy from small-scale to large-scale motions. We note that up to the evolution of the instability $\langle \mathbf{V} \rangle = 0$ and the structures found are therefore not "relics" of the average flow (such as the von Karman vortices) but are produced by the helical turbulence itself. Their characteristic size L is thus naturally connected only with the integral characteristics of the turbulence. The result obtained is interesting already for the reason that it is possible to obtain it systematically from the Navier-Stokes equations without having recourse to phenomenological models.

Generally speaking, the transfer of energy to large vortices is characteristic also of two-dimensional turbulence, but in that case there does not appear a well-defined scale of the long-wavelength fluctuations and there occurs a continuous transfer along the spectrum (see, e.g., Ref. 8). We note that the mechanism considered in the present paper for the generation of large-scale vortices is impossible as a matter of principle in two dimensional turbulence as in that case $\mathbf{V} \operatorname{curl} \mathbf{V} = 0$.

1. BASIC EQUATIONS

We shall assume that the initial turbulence is described by the Navier-Stokes equation for the random velocity V_i^t and by the equation of continuity for turbulent pulsations of density ρ'. For the sake of simplicity we consider a polytropic gas with polytropic index equal to two:

$$P = \tfrac{1}{2}\rho_0 c_0^2 (\rho/\rho_0)^2,$$

where P is the pressure and c_0 the sound velocity for $\rho = \rho_0$. In that case the initial equations have the form

$$\frac{\partial V_i^t}{\partial t} + V_k^t \frac{\partial V_i^t}{\partial x_k} = \nu_0 \Delta V_i^t - \frac{c_0^2}{\rho_0} \frac{\partial \rho^t}{\partial x_i}, \tag{1}$$

$$\frac{\partial \rho^t}{\partial t} + \frac{\partial}{\partial x_k}(\rho^t V_k^t) = 0, \tag{2}$$

where Δ is the Laplace operator and ν_0 the kinematic viscosity. Since we assume the turbulence to be homogeneous and isotropic, $\langle \mathbf{V}^t \rangle = 0$, $\langle \rho^t \rangle = \text{const} = \rho_0$.

The appearance of a large-scale vortex structure must mean the existence of an instability leading to the magnification and support of large-scale perturbations at the expense of the small-scale turbulence. We shall understand by large-scale perturbations an average field $\langle \mathbf{V} \rangle$ with characteristic size $L \gg \lambda$.

Let us at $t_0 = 0$ create a small large-scale perturbation $\langle \mathbf{V}_0 \rangle$. When $t > 0$, after $\langle \mathbf{V}_0 \rangle$ has interacted with the turbulent velocity field $\mathbf{V}^t$, the average $\langle \mathbf{V}_0(\mathbf{x},0) \rangle$ is changed to $\langle \mathbf{V}(\mathbf{x},t) \rangle$ and the random part of the velocity has acquired an addition $\tilde{\mathbf{V}}(\mathbf{x},t)$. This means that the total velocity at any time can identically be written in the form

$$V_i = \langle V_i \rangle + \tilde{V}_i + V_i^t, \qquad \langle \tilde{\mathbf{V}} \rangle = \langle \mathbf{V}^t \rangle = 0. \tag{3}$$

Similarly we can write the total density ρ also in the form

$$\rho=\langle\rho\rangle+\bar{\rho}+\rho', \qquad \langle\rho'\rangle=\text{const}=\rho_0, \qquad \langle\bar{\rho}\rangle=0. \qquad (4)$$

Clearly, the total velocity V_i as well as the velocity V_i^t satisfy the Navier-Stokes equation, and the total density the continuity equation. We shall understand by the average value everywhere the result of averaging over an ensemble of realizations of the turbulent pulsations V_i^t and ρ^t. Our problem consists in obtaining closed equations for the average velocity $\langle V_i\rangle$ and density $\langle\rho\rangle$. To do this we use the approach developed in Refs. 9 and 10 and operate on the Novikov-Furutsu formula.[11,12]

Substituting (3) and (4) into the Navier-Stokes and continuity equations and after that taking the ensemble average we get the equations

$$\frac{\partial\langle V_i\rangle}{\partial t}+\langle V_k\rangle\frac{\partial}{\partial x_k}\langle V_i\rangle+\left\langle \bar{V}_k\frac{\partial\bar{V}_i}{\partial x_k}\right\rangle$$
$$+\left\langle \bar{V}_k\frac{\partial V_i^t}{\partial x_k}\right\rangle+\left\langle V_k^t\frac{\partial\bar{V}_i}{\partial x_k}\right\rangle =\nu_0\Delta\langle V_i\rangle-\frac{c_0^2}{\rho_0}\frac{\partial\langle\rho\rangle}{\partial x_i}, \qquad (5)$$

$$\frac{\partial\langle\rho\rangle}{\partial t}+\frac{\partial}{\partial x_k}[\langle\rho\rangle\langle V_k\rangle+\langle\bar{\rho}\bar{V}_k\rangle+\langle\bar{\rho}V_k^t\rangle+\langle\rho'\bar{V}_k\rangle]=0. \qquad (6)$$

We still need also equations for the quantities $\bar{V}$ and $\bar{\rho}$. To obtain them we again substitute (3) and (4) into the Navier-Stokes and continuity equations and subtract from them the equations which we have obtained for the total velocity and density, (1), (2) and (5), (6). As a result we get

$$\frac{\partial\bar{V}_i}{\partial t}+\bar{V}_k\frac{\partial\bar{V}_i}{\partial x_k}+\langle V_k\rangle\frac{\partial\bar{V}_i}{\partial x_k}+\bar{V}_k\frac{\partial\langle V_i\rangle}{\partial x_k}+\langle V_k\rangle\frac{\partial V_i^t}{\partial x_k}$$
$$+V_k^t\frac{\partial\langle V_i\rangle}{\partial x_k}+\bar{V}_k\frac{\partial V_i^t}{\partial x_k}+V_k^t\frac{\partial\bar{V}_i}{\partial x_k}=\nu_0\Delta\bar{V}_i-\frac{c_0^2}{\rho_0}\frac{\partial\bar{\rho}}{\partial x_i}$$
$$+\left\langle \bar{V}_k\frac{\partial V_i^t}{\partial x_k}\right\rangle+\left\langle V_k^t\frac{\partial\bar{V}_i}{\partial x_k}\right\rangle+\left\langle \bar{V}_k\frac{\partial\bar{V}_i}{\partial x_k}\right\rangle, \qquad (7)$$

$$\frac{\partial\bar{\rho}}{\partial t}+\frac{\partial}{\partial x_k}[\bar{\rho}V_k^t+\langle\rho\rangle V_k^t+\rho'\langle V_k\rangle+\rho'\bar{V}_k+\bar{\rho}\bar{V}_k+\langle\rho\rangle\bar{V}_k+\bar{\rho}\langle V_k\rangle]$$
$$=\frac{\partial}{\partial x_k}[\langle\bar{\rho}V_k^t\rangle+\langle\rho'\bar{V}_k\rangle+\langle\bar{\rho}\bar{V}_k\rangle]. \qquad (8)$$

We shall assume that the turbulence is stationary, homogeneous, isotropic, but not invariant under reflection. In that case the correlators of the random fields V^t and ρ^t have the form

$$\langle V_i^t(\mathbf{x}_1, t_1) V_j^t(\mathbf{x}_2, t_2)\rangle=K_{ij}(\mathbf{x}_1, \mathbf{x}_2)\varphi(t_1-t_2), \qquad (9)$$

$$\langle \rho^t(\mathbf{x}_1, t_1) V_i^t(\mathbf{x}_2, t_2)\rangle=R_i(\mathbf{x}_1, \mathbf{x}_2)\varphi(t_1-t_2), \qquad (10)$$

where K_{ij} and R_i are the spatial parts of the correlators and

$$K_{ik}(\mathbf{x}_1, \mathbf{x}_2)=C(r)\delta_{ik}+B(r)r_ir_k+g(r)\varepsilon_{ikl}r_l, \qquad (11)$$

$$R_i(\mathbf{x}_1, \mathbf{x}_2)=D(r)r_i, \qquad \mathbf{r}=\mathbf{x}_1-\mathbf{x}_2. \qquad (12)$$

The quantities C, B, and D in Eqs. (11), (12) are scalars and g is a pseudoscalar. We note that the last term in (11) is the consequence of the fact that the turbulence is not invariant under reflection (helical), while $g(0)$ has the meaning of the average value of the product of the turbulent velocity and its curl. Indeed, one easily finds from (9), (11) that

$$\langle \mathbf{V}^t(x)\ \text{rot}\ \mathbf{V}^t(x)\rangle=6g(0)\varphi(t_1-t_2).$$

We note also that under the assumptions made about the properties of the turbulence the following relation holds

$$\frac{\partial K_{im}(r)}{\partial r_k}\bigg|_{r=0}=g(0)\varepsilon_{imk}, \qquad (13)$$

as a consequence of which the following average vanishes:

$$\left\langle V_k^t(\mathbf{x})\frac{\partial V_i^t(\mathbf{x})}{\partial x_k}\right\rangle=0.$$

It also follows from (12) that

$$\frac{\partial}{\partial x_k}\langle\rho^t(\mathbf{x}) V_k^t(\mathbf{x})\rangle=0.$$

These averages therefore do not appear in Eqs. (5) and (6). Equations (5), (6) and (7), (8) are the basis of what follows.

2. CLOSURE OF THE AVERAGED EQUATIONS

The basic problem consists in the closure of Eqs. (5) and (6). As we consider the case with a small average velocity against the background of initially homogeneous and isotropic turbulence, i.e.,

$$\langle V\rangle\ll\langle (V^t)^2\rangle^{1/2}\ll c_0, \qquad (14)$$

in the initial stage we can neglect in (5) the non-linear terms proportional to $\langle V_i\rangle\langle V_k\rangle$. As $\langle V\rangle$ is small compared to $\langle(V^t)^2\rangle^{1/2}$ we may also assume that the inhomogeneous perturbations of the random field V^t produced by the average velocity $\langle V\rangle$ are also small:

$$(\bar{V}^2)^{1/2}\ll[(V^t)^2]^{1/2} \qquad (15)$$

and neglect in (5) and (6) the non-linear terms proportional to $\langle\bar{V}_i\bar{V}_k\rangle$ and $\langle\bar{\rho}\bar{V}_k\rangle$. The main non-linear terms in (5) thus have the structure $\langle\bar{V}V^t\rangle$. Below we verify that inequality (15) is satisfied. As we shall be interested in a large-scale instability leading to the generation of structures of the average field $\langle\mathbf{V}\rangle$ we shall later consider the case when

$$L\gg\lambda_0, \qquad T\gg\tau, \qquad (16)$$

where L is the characteristic spatial scale of $\langle\mathbf{V}\rangle$, λ_0 the characteristic energy-containing scale of the turbulence, τ the characteristic correlation time of the turbulent velocity, and T the characteristic time scale of $\langle V\rangle$.

Later we shall indicate the criterion for satisfying inequalities (16). The condition $T\gg\tau$ allows us to separate the slow evolution of $\langle V\rangle$ from the fast turbulent pulsations. In that case we can assume the turbulent pulsations V^t to be, to a first approximation, delta-function-correlated in time as compared to the slow evolution of the average velocity $\langle\mathbf{V}\rangle$, i.e., we shall assume that $\varphi(t_1-t_2)=\delta(t_1-t_2)$:

$$\langle V_i^t(\mathbf{x}_1, t_1) V_j^t(\mathbf{x}_2, t_2)\rangle=K_{ij}(\mathbf{x}_1, \mathbf{x}_2)\delta(t_1-t_2),$$
$$\langle \rho^t(\mathbf{x}_1, t_1) V_i^t(\mathbf{x}_2, t_2)\rangle=R_i(\mathbf{x}_1, \mathbf{x}_2)\delta(t_1-t_2), \qquad (17)$$

where K_{ij} and R_i are given by Eqs. (11), (12).

In the present paper we consider the case which is the simplest one from a formal point of view when the random field of the turbulent fluctuations in the velocity can be considered to be Gaussian. Turbulence consisting mainly of vortices with characteristic size λ_0 can serve as an example of such a situation. The velocity field after the lapse of a sufficiently long time is also Gaussian, if the turbulence source is

switched off or even if we do not consider strong turbulence, but small hydrodynamic fluctuations.

To close Eqs. (5) and (6) we must evaluate the averages

$$\left\langle \tilde{V}_k \frac{\partial V_i{}^t}{\partial x_k} \right\rangle, \quad \left\langle V_k{}^t \frac{\partial \tilde{V}_i}{\partial x_k} \right\rangle, \quad \langle \tilde{\rho} V_k{}^t \rangle, \quad \langle \rho^t \tilde{V}_k \rangle.$$

The quantities $\tilde{V}_k$ and $\tilde{\rho}$ are functionals of the random fields V_k^t and ρ^t. The way $\tilde{V}_k$ and $\tilde{\rho}$ depend on V_k^t and ρ^t is given by Eqs. (7), (8). In our statement of the problem the random fields are assumed to be given and decribed by the correlators (9), (10). To evaluate the averages in which we are interested we can use the Novikov-Furutsu formula[11,12] (see also Ref. 13):

$$\langle F[z], z \rangle = \int \langle z(x_1, t_1) z(x_2, t_2) \rangle \left\langle \frac{\delta F[z]}{\delta z(x_2, t_2)} \right\rangle dx_2 \, dt_2, \quad (18)$$

where z is a Gaussian random process and $F[z]$ a functional of this random process.

The application of Eq. (18) gives in our case

$$\langle \tilde{V}_k(x_1, t) V_i{}^t(x_2, t) \rangle = \iint \langle V_i{}^t(x_2, t) V_m{}^t(y, t') \rangle$$

$$\times \left\langle \frac{\delta \tilde{V}_k(x_1, t)}{\delta V_m{}^t(y, t')} \right\rangle dy \, dt'$$

$$+ \iint \langle V_i{}^t(x_2, t) \rho^t(y, t') \rangle \left\langle \frac{\delta \tilde{V}_k(x_1, t)}{\delta \rho^t(y, t')} \right\rangle dy \, dt', \quad (19)$$

$$\langle \tilde{\rho}(x_1, t) V_k{}^t(x_2, t) \rangle = \iint \langle V_k{}^t(x_2, t) V_m{}^t(y, t') \rangle$$

$$\times \left\langle \frac{\delta \tilde{\rho}(x_1, t)}{\delta V_m{}^t(y, t')} \right\rangle dy \, dt'$$

$$+ \iint \langle V_k{}^t(x_2, t) \rho^t(y, t') \rangle \left\langle \frac{\delta \tilde{\rho}(x_1, t)}{\delta \rho^t(y, t')} \right\rangle dy \, dt', \quad (20)$$

$$\langle \rho^t(x_1, t) \tilde{V}_k(x_2, t) \rangle$$

$$= \iint \langle \rho^t(x_1, t) \rho^t(y, t') \rangle \left\langle \frac{\delta \tilde{V}_k(x_2, t)}{\delta \rho^t(y, t')} \right\rangle dy \, dt'$$

$$+ \iint \langle \rho^t(x_1, t) V_m{}^t(y, t') \rangle \left\langle \frac{\delta \tilde{V}_k(x_2, t)}{\delta V_m{}^t(y, t')} \right\rangle dy \, dt'. \quad (21)$$

The integration in Eqs. (19) to (21) is over the whole of space while the integrals over time are taken from zero to t, which reflects the causality principle. (The quantities $\tilde{V}, \tilde{\rho}$ can depend on V^t, ρ^t only at earlier times.) The averages of interest to us which occur in Eq. (5) are found from (19). For instance,

$$\left\langle \tilde{V}_k(x, t) \frac{\partial V_i{}^t(x, t)}{\partial x_k} \right\rangle = \lim_{x_1 \to x_2} \frac{\partial}{\partial x_2} \langle \tilde{V}_k(x_1, t) V_i{}^t(x_2, t) \rangle.$$

[In Eqs. (5), (6) the averages are all taken at one point $x_1 = x_2 = x$.] As we assume the random processes V^t and ρ^t to be delta-function-correlated we can evaluate the functional derivatives which occur in Eqs. (19) to (21) at a single time $t = t'$. Integrating Eqs. (7) and (8) over time and assuming that the functional derivatives of $\tilde{V}$ and $\tilde{\rho}$ with respect to ρ^t and V^t do not contain singularities with respect to the time we let the lower limit of integration tend to the upper one which is equal to t and we obtain the first approximation in τ/T expressions for the functional derivatives:

$$\left\langle \frac{\delta \tilde{V}_i(x, t)}{\delta V_m{}^t(y, t)} \right\rangle = -\frac{1}{2} \delta_{im} \langle V_k(x, t) \rangle \frac{\partial}{\partial x_k} \delta(x-y)$$

$$-\frac{1}{2} \delta(x-y) \frac{\partial}{\partial x_m} \langle V_i(x, t) \rangle, \quad (22)$$

$$\left\langle \frac{\delta \tilde{\rho}(x, t)}{\delta V_m{}^t(y, t)} \right\rangle = -\frac{1}{2} \frac{\partial}{\partial x_m} [\langle \rho(x, t) \rangle \delta(x-y)], \quad (23)$$

$$\left\langle \frac{\delta \tilde{V}_i(x, t)}{\delta \rho^t(y, t)} \right\rangle = 0, \quad (24)$$

$$\left\langle \frac{\delta \tilde{\rho}(x, t)}{\delta \rho^t(y, t)} \right\rangle = -\frac{1}{2} \frac{\partial}{\partial x_k} [\langle V_k(x, t) \rangle \delta(x-y)]. \quad (25)$$

Substituting (17) and (22) to (25) into Eqs. (19) to (21) and integrating we get for the averages

$$\left\langle \tilde{V}_k \frac{\partial V_i{}^t}{\partial x_k} \right\rangle + \left\langle V_k{}^t \frac{\partial \tilde{V}_i}{\partial x_k} \right\rangle$$

$$= -\frac{1}{2} g(0) \text{rot}_i \langle V \rangle - \frac{1}{4} C(0) \Delta \langle V_i \rangle, \quad (26)$$

$$\frac{\partial}{\partial x_k} \langle \tilde{\rho} V_k{}^t \rangle = -\frac{1}{4} C(0) \Delta \langle \rho \rangle + \frac{1}{4} D(0) \text{div} \langle V \rangle, \quad (27)$$

$$\frac{\partial}{\partial x_k} \langle \rho^t \tilde{V}_k \rangle = \frac{1}{4} D(0) \text{div} \langle V \rangle. \quad (28)$$

We used Eq. (13) in obtaining Eqs. (26) to (28).

Substituting (26) to (28) into Eqs. (5), (6) we finally get

$$\frac{\partial}{\partial t} \langle V_i \rangle - \frac{1}{2} g(0) \text{rot}_i \langle V \rangle = \nu \Delta \langle V_i \rangle - \frac{c_0^2}{\rho_0} \frac{\partial \langle \rho \rangle}{\partial x_i}, \quad (29)$$

$$\frac{\partial}{\partial t} \langle \rho \rangle + \text{div}(\langle \rho \rangle \langle V \rangle) + \frac{1}{2} D(0) \text{div} \langle V \rangle = -\frac{1}{4} C(0) \Delta \langle \rho \rangle, \quad (30)$$

where $\nu = \nu_0 = C(0)/4$. We note that there occurs in Eq. (30) a term proportional to $D(0) \text{div} \langle V \rangle$. The presence of that term leads to a change in the constant part of the average density:

$$\langle \rho \rangle_0 = \rho_0 + D(0)/2$$

and, as a consequence of this, to a change in the sound velocity

$$c_* = c_0 [1 + D(0)/2\rho_0]^{1/2}.$$

For small Mach numbers $M^2 = \langle (V^t)^2 \rangle / c_0^2 \ll 1$ this change in the sound speed is small.

3. LARGE-SCALE INSTABILITY AND APPEARANCE OF STRUCTURE

The solenoidal part of the velocity is split off from the potential part in Eq. (29). As we are interested in the generation of vortex structures we consider the solenoidal part of the velocity. Operating with the curl on Eq. (29) and introducing the notation $\Omega = \text{curl} \langle V \rangle$ we get

$$\frac{\partial \Omega}{\partial t} - \frac{g(0)}{2} \text{rot} \, \Omega = \nu \Delta \Omega. \quad (31)$$

It follows from Eq. (31) that if the helicity of the turbulence is equal to zero there remains in the equation of motion only the turbulent viscosity which corresponds to the damping of large-scale perturbations by turbulence. We consider the problem with large-scale perturbations can be amplified and

supported by turbulence and, hence, lead to the appearance of coherent structures.

Fourier-transforming Eq. (31) leads to

$$-i\omega\Omega_j(\mathbf{k})-{}^1/_2 ig\,[\mathbf{k}\times\mathbf{\Omega}(\mathbf{k})]_j+\nu k^2\Omega_j(\mathbf{k})=0. \qquad (32)$$

Choosing the system of coordinates such that the z-axis is directed along the vector $\mathbf{k}$ and using the relation $\mathbf{k}\times\mathbf{\Omega}=0$ we get the dispersion equation

$$\omega=i(-{}^1/_2 gk-\nu k^2). \qquad (33)$$

From this equation it is clear that $\mathrm{Re}\,\omega=0$. The instability growth rate γ is equal to

$$\gamma(k)=-{}^1/_2 gk-\nu k^2. \qquad (34)$$

We easily find from (34) the largest value of the growth rate γ_{max} and the value k_0 for which the growth rate has its maximum

$$\gamma_{max}=\gamma(k_0)=g^2/16\nu, \qquad (35)$$

$$k_0=|g|/4\nu, \qquad (36)$$

$$g\sim\langle\mathbf{V}^t\,\mathrm{rot}\,\mathbf{V}^t\rangle\tau, \qquad \nu\sim\nu_T\sim\langle(\mathbf{V}^t)^2\rangle\tau\sim E_T\tau. \qquad (37)$$

As a result we get an estimate for the characteristic size of the instability:

$$L^{-1}=\langle\mathbf{V}^t\,\mathrm{rot}\,\mathbf{V}^t\rangle/E_T\sim I_\omega/E. \qquad (38)$$

Formula (38) is interesting because in it there appears only the ratio of two integrals of motion: the topological invariant and the normal invariant. The topological invariant I_ω characterizes the number of linkages of vortex lines with flow lines. The characteristic size of unstable fluctuations is thus determined by the "natural" characteristics of the turbulence, its invariants, and it is an internal property of the turbulence itself.

Since assumed from the outset the instability to be large-scale, we must require that

$$k_0\ll\lambda_0^{-1}. \qquad (39)$$

Inequality (39) is the condition on the correlator parameter of the turbulence velocity field (11):

$$k_0\sim g/C\ll\lambda_0^{-1}, \qquad g/C\sim\langle\mathbf{V}^t\,\mathrm{rot}\,\mathbf{V}^t\rangle/E_T. \qquad (40)$$

Apart from condition (39) we assumed that the large-scale instability was slow, i.e.,

$$T\sim\gamma^{-1}\gg\tau,\ \gamma^{-1}\sim\frac{\nu_T}{g^2}\gg\tau,\ \frac{\langle(\mathbf{V}^t)^2\rangle\tau}{\langle\mathbf{V}^t\,\mathrm{rot}\,\mathbf{V}^t\rangle\tau^2}\gg\tau, \qquad (41)$$

$$\frac{E_T}{\langle\mathbf{V}^t\,\mathrm{rot}\,\mathbf{V}^t\rangle^2}\gg\tau^2\sim\frac{\lambda_0^2}{E_T},\quad \frac{E_T^2}{\langle\mathbf{V}^t\,\mathrm{rot}\,\mathbf{V}^t\rangle^2}\gg\lambda_0^2,\quad L^2\gg\lambda_0^2, \qquad (42)$$

i.e., (42) follows from (40). Thus, a long-wavelength instability is automatically slow, i.e., it can be considered in the delta-function-correlated random process approximation. One can estimate from Eq. (7) the characteristic magnitude of the velocity $\tilde V$:

$$\tilde V_{\mathrm{char}}\sim\langle\tilde V^2\rangle^{1/2}.$$

The main terms in (7) give

$$\frac{d}{dt}\tilde V_i\sim V_k^t\frac{\partial\langle V_i\rangle}{\partial x_k}; \qquad (43)$$

whence we find

$$\langle\tilde V\tilde V\rangle\sim\int dt_1 dt_2\langle V^t V^t\rangle(\nabla\langle V\rangle)^2.$$

The correlator $\langle V^t(\mathbf{x}_1,t_1)V^t(\mathbf{x}_2,t_2)\rangle$ taken in one point $(\mathbf{x}_1=\mathbf{x}_2)$ depends on $t_1-t_2\sim\tau$ and the characteristic time scale of $\langle V\rangle$ is T. Therefore

$$\langle\tilde V^2\rangle\sim T\tau\langle(V^t)^2\rangle(\nabla\langle V\rangle)^2 \quad\text{or}\quad \tilde V\sim(\tau/T)^{1/2}V^t. \qquad (44)$$

As $\tau\ll T$ we have $\tilde V\ll V^t$ as we assumed earlier. We can rewrite the estimate (44) as

$$\langle\tilde V^2\rangle\sim T\tau\langle(V^t)^2\rangle\frac{\langle V\rangle^2}{L^2}\sim\frac{T}{\tau}\Big(\frac{\lambda}{L}\Big)^2\langle V\rangle^2$$

$$\tilde V\sim\frac{\lambda}{L}\Big(\frac{T}{\tau}\Big)^{1/2}\langle V\rangle. \qquad (45)$$

We note that from the estimates (44), (45) it follows that $V^t\sim(\lambda/L)(T/\tau)\langle V\rangle$ and, as we consider the case $\langle V\rangle\ll V^t$, hence

$$(\lambda/L)(T/\tau)\gg1. \qquad (46)$$

From (42) it follows that $T/\tau\sim(L/\lambda_0)^2$ and hence condition (46) is equivalent to $L\gg\lambda_0$ and is satisfied automatically. Then (45) gives

$$\tilde V\sim\langle V\rangle. \qquad (47)$$

We can introduce the spectrum of the quantity $\tilde V$:

$$\langle\tilde V^2\rangle=\int\tilde E(k)\,dk.$$

We then get from (44) an estimate for the spectral density:

$$\tilde E(k)\sim(\tau/T)E_T(k).$$

Hence it follows that the characteristic scale of the fluctuations $\tilde V$ is the same as the external scale of turbulence λ_0. The characteristic time of these fluctuations is of order τ because of the transfer of $\tilde V$ by the velocity V^t. In the long-wavelength part of the spectrum we have $\tilde V(L)\ll\langle V\rangle$ as the energy-containing part of $\tilde V$ is concentrated at a scale λ_0 and satisfies the estimate (47). We note that we can use Eq. (31) to find the actual configuration of the structure in $\mathbf{x}$-space. To do this it is necessary to assign to it an actual boundary value problem.

CONCLUSION

We note that in magnetohydrodynamics helical turbulence has the capacity to magnify initially small magnetic fields.[14,15] The equation of the "dynamo" theory which is then obtained has the same structure as Eq. (31). The growth rates of the two instabilities then turn out to be of the same order of magnitude.

The results obtained show that there occurs in helical turbulence a large-scale instability leading to the appearance of vortex structures. It follows from Eq. (31) that these structures themselves are helical and it is natural to call them "g-structures." The evolution of the instability found here is accompanied by the transfer of energy from small-scale to large-scale sizes. We emphasize that the g-structures are not relics of the average flow such as the von Karman vortices and the instability found in this paper is not a modification of the instability of shear flows. The magnification of a coherent structure occurs merely due to the fact that the turbulence is not invariant under reflection.

It is important that the characteristic size of the struc-

tures which occur is determined by the internal characteristics of the turbulence: the ratio of the topological invariant to the energy of the turbulence. In the present paper we considered the Gaussian case which, apparently, contains the basic physical traits of the process of the appearance of a large-scale structure in turbulence. It is nevertheless interesting that an isotropic velocity field is able to magnify large-scale regular motions.

The case of non-Gaussian velocity field is more complicated from a formal point of view. However, also in that case it is possible to obtain closed averaged equations of motion. The basic features of the large-scale instability are apparently retained. We note merely that the moments $\langle V_1^t V_2^t \ldots V_n^t \, \text{curl} \, V^t \rangle$ which characterize higher order helicity contribute to the growth rate of the instability. We shall consider the case of non-Gaussian turbulence in another paper.

Note added in proof (31 October 1983). In the present paper we considered a model in which the velocity of the turbulent pulsations is assumed to be a random process, δ-correlated in time. In the case of a small, but finite correlation time instead of $g(0)$ there occurs in Eq. (31) $Mg(0)$ where $M = \langle (V^t)^2 \rangle^{1/2}/c_0$ is the Mach number, and correspondingly, the growth rate γ [see (35)] is multiplied by M^2.

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