# Nonlinear dynamics of smectic liquid crystals

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(Submitted 6 June 1983)
Zh. Eksp. Teor. Fiz. **85**, 2019–2031 (December 1983)

Nonlinear fluctuation effects in the dynamics of smectic liquid crystals are considered. A diagram technique for interacting long-wavelength modes is utilized. Within the framework of this technique a procedure is constructed for excluding weakly fluctuating variables. An effective action is derived for the smectic mode. It is proved that this mode is strongly damped. The logarithmic behavior of the coefficients describing this mode is derived. The contribution of fluctuations to the spectrum of first sound is considered. It is shown that the fluctuations lead to a divergence $\propto \omega^{-1}$ in the viscosity. The logarithmic behavior of the coefficient in this law is determined.

PACS numbers: 61.30.Cz, 62.10. + s

## INTRODUCTION

Smectic liquid crystals of different kinds are unique physical objects, exhibiting at the same time properties of a two-dimensional fluid and of a one-dimensional solid. The linear theory of elasticity and the hydrodynamics of such systems are well known.[1] For the sake of concreteness we shall consider smectics of the $A$ type, in which the molecules are orthogonal to the layers, although all results obtained below, with appropriate changes of notation, are valid for arbitrary smectics in which there exists a solid-state order in the layers (for instance for smectics of the $C$ type, in which the molecules are inclined relative to the layers, or for the so-called hexatic $B$ smectics, which exhibit a certain orientational order of the bonds in the layer).

Compared with an ordinary isotropic fluid, the only additional hydrodynamic variable necessary in the theory of $A$ type smectics is the displacement $u$ of layers in direction perpendicular to them. The absence of an $A$ type smectic of a Young modulus for changes of $u$ along the layers leads to strong fluctuations of the order parameter, and according to the Landau-Peierls theorem,[2] to an absence of long-range order in such a system. Of course, the order disappears only in an infinite system. In a finite system, or, what amounts to the same, for finite wave vectors, the description of a smectic in terms of a displacement of equidistance layers is correct.

In studying the nonlinear properties of a type $A$ smectic it is more convenient to use in place of the layer displacement a function $W(\mathbf{r}, t)$ which has the interpretation that the equation $W=$ const determines the position of the layer of molecules. The energy density of an $A$ type smectic can be represented as an expansion with respect to the gradients of these functions, for which the leading terms have the form

$$E^{(W)}={}^1/_8\beta[(\nabla W)^2-l^{-2}]^2+{}^1/_2\varkappa(\nabla^2 W)^2. \tag{1}$$

here $\beta$ and $\varkappa$ are the elastic moduli and $l$ is the distance between the layers (the period of density modulation). At equilibrium $W_0=z/l$, with the $z$ axis perpendicular to the layers, since the condition $W_0=$ const defines a plane perpendicular to the $z$ axis.

If one now considers deformations of the molecular layers, one must go over from the variable $W$ to the variable $u$ according to the definition

$$W=(z-u)/l. \tag{2}$$

For small deviations from equilibrium the variable $u$ coincides with the displacement of the layers along the $z$ axis, in agreement with the standard notations of the linear theory.[1] Expanding the expression (1) with respect to $u$ one easily obtains the harmonic part of the energy

$$E^{(2)}=\int d^3r\left[\frac{\beta}{2l^2}(\nabla_z u)^2+\frac{\varkappa}{2}(\nabla^2 u)^2\right]\frac{1}{l^2}. \tag{3}$$

Thus $\beta$ corresponds to the compression modulus of the layers and $\varkappa$ to the Frank modulus for transverse bending in the linear theory.[1] In Refs. 3–5 it was shown that the anharmonic effects (i.e., the interactions of the long-wavelength fluctuations) lead to a substantial logarithmic renormalization of the elastic moduli $\beta$ and $\varkappa$, which in the limit of large values of the logarithmic factors behaves as follows:

$$\beta\propto L^{-4/5},\qquad \varkappa\propto L^{2/5}. \tag{4}$$

Here $L=\ln[\Lambda/\max\{k_z, lk^2\}]$ is the large logarithm, $k$ is the wave vector, $\Lambda$ is a characteristic wave vector cutoff, which determines the limits of applicability of the macroscopic treatment.

It is natural to expect that strong long-wave fluctuations must also manifest themselves in the low-frequency dynamics of an $A$ type smectic. This was first pointed out in Ref. 6. However the authors of Ref. 6 have only calculated the first correction to the viscosity coefficients, correction which diverges as $\omega^{-1}$ for low frequencies. The same divergences are exhibited by the higher corrections of perturbation theory. Therefore, one is not allowed to restrict oneself to first-order perturbation theory in the investigation of the low-frequency dynamics of $A$ type smectics, and one must take into consideration all the leading corrections. It should then be kept in mind that not only the kinetic coefficients (which determine the imaginary part of the oscillation spectrum) are subject to strong renormalizations, but also the coefficients which determine the real part of the spectrum. The higher orders of perturbation theory will be taken into account in the present paper by means of the diagram technique developed by Sukhorukov, Khalatnikov, and one of the present authors (V. L.) in Ref. 7.

  

## INVESTIGATION OF THE WEAKLY FLUCTUATING VARIABLES

The diagram technique developed in Ref. 7 is constructed starting from the generating functional

$$Z(m,y,\nu,\bar\nu)=\int D\varphi\, Dp\, D\psi\, D\bar\psi \exp\left[i\int dt\, d^3r(\mathscr{L}+m_a\varphi_a+y_ap_a+\bar\nu_a\psi_a+\bar\psi_a\nu_a)\right]. \quad (5)$$

Here $\varphi_a(t,\mathbf{r})$ is a complete set of hydrodynamic variables; $P_a$, $\psi_a$, and $\bar\psi_a$ are auxiliary variables, with both $\psi_a$, $\bar\psi_a$ as well as $\nu_a$, $\bar n_a$ anticommuting quantities. Summation over repeated indices is understood. The Lagrange function density has the form

$$\mathscr{L}=p_a\left(\frac{\partial\varphi_a}{\partial t}+F_a\right)-i\bar\psi_a\frac{\partial}{\partial t}\psi_a-i\bar\psi_a\int\frac{\delta F_a}{\delta\varphi_b(\mathbf{r}')}\psi_b(\mathbf{r}')\,d^3r'. \quad (6)$$

The expressions $F_a(\varphi)$ determine the hydrodynamic equations

$$\frac{\partial\varphi_a}{\partial t}+F_a=0 \quad (7)$$

and contain (Ref. 7) both reactive and dissipative terms, as well as random forces over which the functional (5) should be averaged.

By expanding the functional (5) with respect to the variables $m$, $y$, $\nu_a$, $\bar\nu_a$ which are canonically conjugate to the variables $\varphi$, $p$, $\psi_a$, $\bar\psi_a$, one can obtain the correlators (correlation functions) of various powers of the functions $\varphi$, $p$, $\psi_a$, $\bar\psi_a$. It follows from the structure of the density of the Lagrange function (6), that only the following pair correlators do not vanish:

$$D_{ab}(t-t',\mathbf{r}-\mathbf{r}')=-\langle\varphi_a(t,\mathbf{r})\varphi_b(t',\mathbf{r}')\rangle, \quad (8)$$

$$G_{ab}(t-t',\mathbf{r}-\mathbf{r}')=-\langle\varphi_a(t,\mathbf{r})p_b(t',\mathbf{r}')\rangle=i\langle\psi_a(t,\mathbf{r})\bar\psi_b(t',\mathbf{r}')\rangle. \quad (9)$$

The last equality in (9) follows from the invariance of the Lagrange function with respect to the (supersymmetry) transformations

$$\delta\varphi_a=\bar\alpha\psi_a,\qquad \delta\bar\psi_a=-i\bar\alpha p_a.$$

Here the infinitesimal parameter $\bar\alpha$ of the transformation is an element of a Grassmann algebra (an anticommuting quantity). Note that the $D$-function is a correlator of observables, and the $G$-function is the linear response of the system to an external force which has to be added to the right-hand side of Eq. (7), i.e., $G(\omega)$ is holomorphic in the upper half-plane and its poles determine the eigenvalue spectrum of the system.

The exponent of the exponential in the expression (5) is conveniently represented in the form

$$\int dt\, d^3r\, d^2\theta \exp(i\theta\bar\theta)\left[\tilde p_a\frac{\partial\tilde\varphi_a}{\partial t}+\tilde p_aF_a(\tilde\varphi_b)+\tilde p_a\tilde y_a+\tilde m_a\tilde\varphi_a\right]. \quad (10)$$

Here

$$\tilde y_a=y_a+i\bar\theta\nu_a,\qquad \tilde m_a=m_a+i\bar\nu_a\theta,\qquad \tilde p_a=p_a+\bar\psi_a\theta,\qquad \tilde\varphi_a=\varphi_a+\bar\theta\psi_a. \quad (11)$$

The Berezin integral[8] over the Grassmann variables $\theta_a$ and $\bar\theta_a$ is defined so that

$$\int d^2\theta\theta\bar\theta=1.$$

Recall that from this definition follows the fundamental functional relation

$$\int D\tilde p\exp\left[\int d^2\theta\exp(i\theta\bar\theta)\,\tilde p\tilde\varphi\right]=\delta(\tilde\varphi)=\delta(\varphi)\,\delta(\psi).$$

We shall be interested in the effects related to the interaction of long-wavelength fluctuations, i.e., effects determined by the nonlinear terms in the Lagrangian density (6) or the expression (10). However, among the variables of the system studied here only one—the order parameter of the smectic—is strongly fluctuating. This means that the interaction of the various degrees of freedom yields a small contribution to the renormalization of the hydrodynamic equations (a term of the order $\omega^{1/2}$ in the kinetic coefficients) and thus this interaction is negligible. One must take into account only the self-interaction of the order parameter and its interaction with the other degrees of freedom. Accordingly, it is convenient to carry out the mathematical procedure in a general form. Let $\tilde p_a$ and $\tilde\varphi_a$ describe the strongly fluctuating one. Neglecting the interactions of the $\tilde p_a$ and $\tilde\varphi_a$ means that in the expression (10) we must retain only terms quadratic in these quantities, retaining however the full nonlinearity in $\tilde\varphi$. After this one can explicitly integrate with respect to $\tilde p_a$ and $\tilde\varphi_a$ in the expression (5). It is easy to see that the determinant which is obtained this way is equal to one, and a shift in the exponent of the exponential leads to the following expression for the generating functional (5)

$$Z=\int D\tilde\varphi\, D\tilde p\exp\left[\int dt\, d^3r\, d^2\theta\exp(i\theta\bar\theta)\mathscr{L}_1\right], \quad (12)$$

$$\mathscr{L}_1=\tilde p(\partial\tilde\varphi/\partial t+F)+\tilde p\tilde y+\tilde m\tilde\varphi-(\tilde pF_{,a}+\tilde m_a)(\delta_{ab}\,\partial/\partial t+F_{a,b})^{-1}(F_b+\tilde y_b). \quad (13)$$

Here $F_{a,b}$ and $F_{,a}$ denote the differential operators corresponding to the integral operators with the kernels $\delta F_a(\mathbf{r}_1)/\delta\varphi_b(\mathbf{r}_2)$ and $\delta F(\mathbf{r}_1)/\delta\varphi_b(\mathbf{r}_2)$, respectively, in all quantities we have set $\tilde\varphi_a=0$, but have retained the dependence on $\tilde\varphi$. We note that the expression (13) depends explicitly on $\tilde m_a$ and $\tilde y_a$, so that the information on the correlators of weakly fluctuating variables is not lost through the procedure described above.

Until now in this section we have nowhere taken into account the necessity to average over the random forces. As was shown in Ref. 7, this averaging does not change the form of the expression (5), but leads to the appearance of terms quadratic in $\tilde p$ in the exponent of the exponential. Together with the terms coming from the dissipative terms in the hydrodynamic equations, these terms will be written in the form of a dissipative contribution to (13):

$$\mathscr{L}_{1d}=i\tilde p\Sigma\tilde\varphi+{}^1/_2 i\tilde p\Pi\tilde p. \quad (14)$$

The self-energy function $\Sigma$ and the vacuum polarization operator $\Pi$ are related to each other by an expression of the type of the fluctuation-dissipation theorem. Note that a contribution of the same form as (14) to the effective action (12) appears on account of the interaction of the fluctuations.

## AN EFFECTIVE ACTION FOR THE ORDER PARAMETER OF SMECTICS

In this section we carry out the described procedure for the elimination of the weakly fluctuating variables for smectics. As will be shown below, the corrections to the dissipative Lagrange function (14) which are due to fluctuations are much larger than the bare quantities, so that we shall be interested only in the nondissipative terms in the nonlinear hydrodynamic equations of type $A$ smectics. We give here the form of these equations discussed in Ref. 9.

The energy density in type $A$ smectics is a function of the following variables:

$$E=E(\rho, \sigma, \mathbf{j}, \nabla_i W, \nabla_i \nabla_k W). \quad (15)$$

Here $\rho$ is the mass density, $\sigma$ is the specific entropy, $\mathbf{j}$ is the momentum density, $W$ is the smectic variable about which we talked in the Introduction. The thermodynamic identity for the energy has the form

$$dE=\mu d\rho+T\rho d\sigma+\mathbf{v}d\mathbf{j}+\psi_k \nabla_k dW+\nabla_k(\psi_{ik}\nabla_i dW). \quad (16)$$

Here $\mu$ is the chemical potential, $\mathbf{v}$ is the velocity, $T$ is the temperature, $\psi_k$ and $\psi_{ik}$ are the variables canonically conjugate to $\nabla_k W$ and $\nabla_k \nabla_i W$. The pressure $P$ is defined as follows:

$$P=\mu\rho+\mathbf{v}\mathbf{j}-E. \quad (17)$$

The thermodynamic identity for the pressure follows from Eqs. (16) and (17):

$$dP=\rho d\mu+\mathbf{j}d\mathbf{v}-T\rho d\sigma-\psi_k\nabla_k dW-\nabla_k(\psi_{ik}\nabla_i dW). \quad (18)$$

Finally, the nondissipative equations of the hydrodynamics of smectics have the form:

$$\frac{\partial \rho}{\partial t}+\nabla \mathbf{j}=0, \quad \frac{\partial \sigma}{\partial t}+\mathbf{v}\nabla\sigma=0,$$
$$\frac{\partial W}{\partial t}+\mathbf{v}\nabla W=0, \quad \frac{\partial j_i}{\partial t}+\nabla_k T_{ik}=0. \quad (19)$$

Here the stress tensor is

$$T_{ik}=[P+\nabla_m(\psi_{mn}\nabla_n W)]\delta_{ik}+\rho v_i v_k+\psi_k\nabla_i W-\nabla_i\psi_{kn}\nabla_n W. \quad (20)$$

We consider the expression for the energy density of the smectic corresponding to the expansion (1):

$$E=\frac{1}{2\rho}\mathbf{j}^2+\varepsilon+\frac{1}{8}\beta\left[(\nabla W)^2-\frac{1}{l^2}\right]^2+\frac{1}{2}\varkappa(\nabla^2 W)^2. \quad (21)$$

Here $\varepsilon$, $\beta$, $l$, and $\varkappa$ are functions of $\pi$ and $\sigma$. Calculating the pressure according to equation (17), we obtain

$$P=\rho\partial\varepsilon/\partial\rho-\varepsilon+P^{(W)}, \quad (22)$$

where the part of the pressure related to the smectic variable is

$$P^{(W)}=\frac{1}{8}\left(\rho\frac{\partial\beta}{\partial\rho}-\beta\right)[(\nabla W)^2-l^{-2}]^2$$
$$+\frac{1}{2}\left(\rho\frac{\partial\varkappa}{\partial\rho}-\varkappa\right)(\nabla^2 W)^2+\frac{\beta\gamma}{2l^2}[(\nabla W)^2-l^{-2}]. \quad (23)$$

In Eq. (23) we have introduced the notation

$$\gamma=(\partial \ln l/\partial \ln \rho)_\sigma. \quad (24)$$

The smectic contribution to the stress tensor has the form

$$T_{ik}^{(W)}={}^1/{}_2\beta[(\nabla W)^2-l^{-2}]\nabla_i W\,\nabla_k W-\varkappa\nabla_k\,\nabla^2 W\,\nabla_i W$$
$$-\varkappa\nabla_i\,\nabla^2 W\,\nabla_k W+[P^{(W)}+\nabla_n(\varkappa\nabla^2 W\,\nabla_n W)]\delta_{ik}. \quad (25)$$

Before proceeding we note that in the linear approximation[1] the equations (19) describe two modes of the acoustic type. One of them is related to the density oscillations and represents ordinary longitudinal sound (in an $A$ type smectic it is called first sound, by analogy with superfluid $He^4$). Its speed of propagation $c_1$ is determined by the compressibility: $c_1^2=\rho\partial^2\varepsilon/\partial\rho^2$. The second mode is related to a compression of the layers (without a change of density). It is called second sound and its velocity of propagation $c_2$ is determined by the elasticity modulus $\beta$, i.e., $c_2^2=\beta/\rho l^4$. In real smectics the order of magnitude of these velocities are in the ratio $c_2^2/c_1^2\sim 10^{-3}$. This means that the spontaneous symmetry breaking in smectics is weak in the parameter $\sim 10^{-3}$.

We now eliminate the weakly fluctuating quantities $\rho$, $\sigma$, and $\mathbf{j}$ by means of the nondissipative hydrodynamic equations (19). In the linear approximation with respect to these variables the equation for $\sigma$ separates entirely, so that integration with respect to this variable reduces to replacing $\sigma$ by its (homogeneous) equilibrium value. The integration with respect to $\rho$ reduces to the substitution

$$\delta\rho=\mathbf{k}\mathbf{j}/\omega, \quad (26)$$

where $\mathbf{k}$ is the wave vector and $\omega$ is the frequency; there remains only to eliminate $\varphi_i=j_i$. This leads to the Lagrange density (13). According to Eqs. (19), (22), (23), and (25) the quantities which occur in this expression have the form

$$F=0, \quad F_i=\nabla_k T_{ik}^{(W)}, \quad F_{,i}=\rho^{-1}\nabla_i W, \quad (27)$$

$$(\omega\delta_{ik}+iF_{i,k})^{-1}=-\frac{1}{\omega(1-\omega^2/c_1^2k^2)}\frac{k_ik_k}{k^2}+\frac{1}{\omega}\delta_{ik}. \quad (28)$$

In the expressions (27), (28) we have omitted terms of the order of $c_2^2/c_1^2$; in all coefficients in Eq. (27) $\rho$ and $\sigma$ are equal to their (homogeneous) equilibrium values.

We shall be interested principally in the self-interaction of the strongly fluctuating degrees of freedom. In this case the frequencies have the approximate behavior $\omega\sim c_2k_z$. One may thus omit the second term in the denominator of the expression (28), corresponding to neglecting a quantity of the order $c_2^2/c_1^2$. Omitting in $\mathscr{L}_1$ the dependence on $\tilde{m}$ and $\tilde{y}$, we obtain, upon substitution of Eq. (27) into Eq. (13), the following expression:

$$i\mathscr{L}_{1r}=\tilde{p}\omega\tilde{W}+\tilde{p}'\frac{1}{2\omega}\nabla_k[a(l^2(\nabla\tilde{W})^2-1)\nabla_k\tilde{W}]-\tilde{p}'\frac{1}{\omega}b\nabla^4\tilde{W}. \quad (29)$$

Here

$$a=\frac{\beta}{\rho l^4}, \quad b=\frac{\varkappa}{\rho l^2}, \quad \tilde{p}'=\frac{\nabla_\perp^2}{\nabla^2}\tilde{p}, \quad (30)$$

and $\nabla_\perp$ is the component of the del operator perpendicular to the $z$ axis. In the derivation of Eq. (29) we have omitted terms which contain $\nabla_k(\nabla\tilde{W})^2$, since they contribute little to the long-wavelength properties of smectics.

One must add to the reactive part of the action defined by Eq. (29) the dissipative part (14). Inclusion of the traditional dissipative terms into the hydrodynamic equations (Refs. 1, 9), leads to the following expressions:

$$\Sigma=i\frac{\Gamma}{T}\left(\frac{\beta}{l^2}\nabla_z^2-\varkappa\nabla^4\right), \quad \Pi=2\Gamma. \quad (31)$$

Here $\Gamma$ is a kinetic coefficient. However, already the first fluctuational correction to $\Sigma$ and $\Pi$ has the form:

$$\Sigma=2ig\nabla_\perp^2,\quad \Pi=-\tau\nabla_\perp^4/\omega^2\nabla^2.$$

Here $g$ and $\tau$ are constants. These quantities have a lower power in $k$ than those in Eq. (31), and consequently they exceed (31) for hydrodynamic values of the wave vectors. This circumstance means that the region of applicability of the expressions (31) is absent, and that the dissipative part of the action is determined by the expression

$$\mathscr{L}_{1d}=-2g\tilde{p}'\nabla^2\tilde{W}+{}^1/_2 i\tau\tilde{p}'(\nabla^2/\omega^2)\tilde{p}'. \tag{32}$$

## RENORMALIZATION OF THE ACTION FOR THE SMECTIC VARIABLE

The expressions (29), (32), and (12) define an effective bare action for the smectic variable $W$. As will be shown below, taking into account the interaction of the fluctuations leads to a logarithmic renormalization of the constants $a$, $b$, $g$, and $\tau$. This circumstance allows us to write explicitly the expressions for the Green's functions $D$, Eq. (8) and $G$, Eq. (9) which involve the smectic variable. Retaining the quadratic part of the Lagrange density (29) we are led to the generating functional which in the Fourier representation yields the following expressions for the Green's functions referring to the smectic variable:

$$G(\omega,\mathbf{k})=\frac{\omega}{\omega^2-\eta^2+2igk_\perp^2\omega}, \tag{33}$$

$$D(\omega,\mathbf{k})=-\frac{\tau k_\perp^2}{(\omega^2-\eta^2)^2+4g^2\omega^2k_\perp^4}\frac{k_\perp^2}{k^2}. \tag{34}$$

Here

$$\eta^2=(ak_z^2+bk^4)k_\perp^2/k^2. \tag{35}$$

We point out the following expression which is a consequence of Eqs. (33) and (34)

$$D(\omega,\mathbf{k})=-(\tau k_\perp^2/4g\omega^2k^2)\,[G(\omega,\mathbf{k})-G(-\omega,-\mathbf{k})]. \tag{36}$$

This representation is convenient for the calculation of the intermediate integrals with respect to the frequency. This is due to the fact that, being a linear response function, $G(\omega)$ can have singularities only in the lower half-plane (this corresponds to $g>0$ in Eq. (33)]. It follows from the structure of Eqs. (33) and (35) that for a characteristic wave vector

$$k_z^2/k^2\sim bk^2/a\ll 1.$$

Thus, $k\gg k_z$, and the ratio $k_\perp^2$ may be replaced by unity in all intermediate integrations. At the frequency $\omega$ we have the estimates

$$k_z\sim\omega/a^{1/2},\quad k^2\sim\omega/b^{1/2}. \tag{37}$$

In the action defined by the expressions (29) and (32) the only nonlinear term describing the interaction is the second term in the right-hand side of (29). Based on its structure one can construct a diagram technique for taking into account the higher fluctuational corrections. It turns out that the integrals which correspond to these diagrams are determined by the internal wave vectors $q$ which are much larger than the external wave vectors $k$. This makes it possible to use the one-loop approximation for the renormalization equations for the action under consideration. Moreover, it turns out that the structure of the action defined by (29) and (32) reproduces itself under renormalization, and the corresponding integrals are purely logarithmic. We are thus faced with a typical renormalization group (renormgroup) situation, where the "charges" are the quantities $a$, $b$, $g$, and $\tau$.

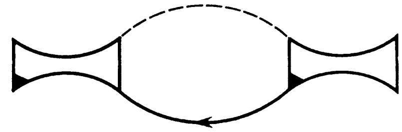

FIG. 1.

The renormgroup procedure consists in separating from Eqs.(29), (32) the "rapid" variables, which have a characteristic wave vector $q\gg k$, and integrate the generating functional (12) with respect to them in the Gaussian approximation, which leaves behind only a dependence on the slow variables with a characteristic wave vector $k$. The renormalization of the interaction vertex is then given by the diagram represented in Fig. 1. In this and the following figures the broken line denotes the $D$ function, the solid line denotes the $G$ function, the quadrilateral denotes the vertex $a/\omega$, where $\nabla\tilde{W}$ is connected to the light corners, and $\nabla\tilde{p}$ is connected to the dark corner; quantities with mutually contracted indices are designated by the straight segments on the quadrilaterals. The renormalization of the vacuum polarization operator $\Pi$ and of the self-energy part $\Sigma$ are described, respectively, by the diagrams represented in Figs. 2 and 3. On the pair of external end points of these diagrams one must substitute the zero values $W_0=z/l$. The diagrams represented in Figs. 1 and 2 are purely logarithmic and yield the renormalizations of $a$ and $\tau$, respectively. The diagram of Fig. 3 diverges for large wave vectors. Consequently, one must carry out a subtraction of an infinite constant from this diagram, and expand the remainder in terms of the external parameters. The expansion with respect to the external frequency yields the renormalization of $g$, and the expansion with respect to the external wave vector leads to the renormalization of $b$.

The technical details of the calculation of the diagrams are relegated to the Appendix; here we list only the final renormgroup equations:

$$a'=-\frac{\tau a^{3/2}}{gb^{3/2}},\quad g'=\frac{1}{4}\left(1+\frac{b}{g^2}\right)\frac{\tau a^{1/2}}{b^{3/2}},$$
$$\tau'=\frac{1}{4}\left(1+\frac{b}{g^2}\right)\frac{\tau^2a^{1/2}}{gb^{3/2}},\quad b'=\frac{1}{2}\frac{\tau a^{1/2}}{gb^{1/2}}. \tag{38}$$

In these equations the accent denotes differentiation with respect to the variable $l^2L/128\pi$, where $L$ is a large logarithm. It follows from the equations (38) that

$$\tau/g=\text{const},\quad dg^2/db=1+g^2/b. \tag{39}$$

The first equality is a reflection of the fluctuation-dissipation theorem. Integrating Eq. (39) as well as Eqs. (38) for $a$ and $b$, taking $\tau/g=$ const into account, we find

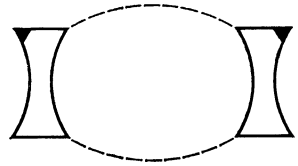

FIG. 2.

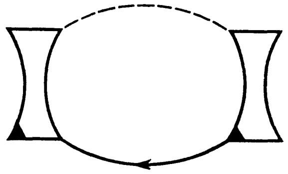

FIG. 3.

$$a\propto L^{-4/5},\quad b\propto L^{2/5},\quad g^2=b\ln(b/b_0). \tag{40}$$

The relations (40) solve the problem of renormalization of the action defined by Eqs. (29), (32). We note that $g$ and $\tau$ vanish at the normalization point $b=b_0$, and then grow with respect to $b^{1/2}$ at a doubly logarithmic scale.

The logarithmic behavior of the quantities $a$ and $b$ in Eq. (40) is equivalent to the logarithmic behavior (4) of the elastic moduli. Thus, the relations (30) remain valid also when the renormalization is taken into account. This guarantees the reproduction of the static limit. Indeed, let us consider the equal-time pair correlator $\langle ww\rangle$, which can be calculated starting from Eq. (1):

$$\langle W(\mathbf{k})W(-\mathbf{k})\rangle=T(\beta l^{-2}k_z^2+\varkappa k^4)^{-1}.$$

On the other hand, the same correlator equals

$$-\int\frac{d\omega}{2\pi}D(\omega,k)=\tau/4g(ak_z^2+bk^4).$$

Taking into account Eq. (30) these two expressions are equivalent, and comparing them one can determine the constant which occurs in Eq. (39):

$$\tau/g=4T/\rho l^2. \tag{41}$$

## FLUCTUATION CORRECTIONS TO THE FIRST-SOUND SPECTRUM

We now consider the fluctuational contribution to the equations for the weakly fluctuating variables. As can be seen from the equations (19) for the mass density $\rho$ and the specific entropy $\sigma$, the fluctuations do not contribute to them, so that we can only find the fluctuation contribution to the equation for the momentum density $\mathbf{j}$, i.e., to the stress tensor $T_{ik}$. In the language of the diagram technique this means that we must find the polarization operator $\Pi_{ik}$ and the self-energy function $\Sigma_{ik}$ corresponding to the variable $\varphi_i=j_i$. For this purpose one must take into account in the Lagrange density $\mathscr{L}_1$ the explicit dependence on $\tilde{m}_i$ and $\tilde{y}_i$, which is reconstructed from Eq. (13), and after that an expansion in terms of these variables yields the Green's functions $G_{ik}$ and $D_{ik}$ for the variable $\varphi_i=j_i$. One has to keep in mind that the expressions for the fluctuational contributions to the first sound spectrum contain terms small of the order $c_2^2/c_1^2$. One must therefore take into account the part of $F_{i,k}$ due to smectic contributions to (25), contribution we have left out in the consideration of the self-interaction:

$$F_{i,k}^{(W)}=-i\frac{\beta\gamma}{\rho l^2}\nabla_n\left(\frac{\gamma}{l^2}\delta_{in}+\nabla_i\tilde{W}\nabla_n\tilde{W}\right)\frac{\nabla_k}{\omega}. \tag{42}$$

The terms left out of Eq. (42) are small on account of their proportionality to the quantity $(\nabla\tilde{W})^2-l^{-2}$. In addition, the frequency of the first sound is no longer small compared to $c_1k_1$, so that both terms are now important in the denominator of Eq. (28), and the expression (28) ceases to be a transverse projector. Thus, one must now take into account in $F_i$ the longitudinal part omitted in our discussion of the self-interaction of the smectic mode, so that in the leading approximation with respect to the difference $(\nabla\tilde{W})^2-l^{-2}$, the nonlinear part of $F_i$ becomes

$$F_i=\frac{\beta}{2}\nabla_k\left\{\left(\nabla_k\tilde{W}\nabla_i\tilde{W}+\frac{\gamma}{l^2}\delta_{ik}\right)[(\nabla\tilde{W})^2-l^{-2}]\right\}. \tag{43}$$

We now consider the Lagrange density (13). In this case, taking account of Eq. (27), the nonlinear part of this density has the form

$$i\mathscr{L}_{1\,int}=\rho^{-1}\tilde{p}\nabla_i\tilde{W}B_{ik}^{-1}F_k. \tag{44}$$

Here $F_k$ is defined by Eq. (43), and

$$B_{ik}=\omega(\delta_{ik}+c_1^2\nabla_i\nabla_k/\omega^2)+iF_{i,k}. \tag{45}$$

The bare value of $F_{i,k}$ is given by Eq. (42). Separating in the Lagrange density (13) the terms which depend on $\tilde{m}_i$ and $\tilde{y}_i$, we find

$$i\mathscr{L}_{1s}=\rho^{-1}\tilde{p}\nabla_i\tilde{W}B_{ik}^{-1}\tilde{y}_k+\tilde{m}_iB_{ik}^{-1}F_k+\tilde{m}_iB_{ik}^{-1}\tilde{y}_k+{}^1/_2\tilde{m}_iB_{in}^{-1}\Pi_{nm}B_{km}^{-1}\tilde{m}_k. \tag{46}$$

The last term in (46) appears for renormalization.

The renormalization of the action defined by the Lagrange density (44) is carried out similarly to the renormalization of the coefficient $a$ in the preceding section. We must separate the two rapid variables $\tilde{W}$ in $F_k$ [from the last factor of Eq. (43)] and pair them with the rapid variables $\tilde{p}$ and $\tilde{W}$ [$\tilde{W}$ is taken from $F_k$, the first term in Eq. (43)]. This yields a diagram of the type of the one represented in Fig. 1, which gives the renormalization of the coefficient $\beta$ in the expression (43). If the two rapid variables $\tilde{W}$ in $F_k$ are paired with the rapid variables $\tilde{p}$ and $\tilde{W}$ [with $\tilde{W}$ taken from $F_{ik}$, the last term in Eq. (42)], there appears a diagram yielding the renormalization of the coefficient $\beta$ in the expression (42). Similarly one renormalizes the first two terms in the Lagrange density (46); the renormalization of the third term in (46) is given by a diagram which is obtained by pairing the first and second terms in Eq. (46). All the mentioned diagrams are of the same type and yield a synchronous renormalization of the coefficients $\beta\propto L^{-4/5}$ in all the expressions under consideration (so that, even taking into account the renormalization, $\beta$ coincides with the appropriate elasticity modulus, as expected). Thus, the structure of the bare action is reproduced by the renormalization.

When one considers the acoustic lines one must take into account the corrections determined by diagrams with an intermediate smectic $G$-line. Such diagrams occur when one pairs the factor $\tilde{p}$ from (44) or (46) with the factor $\tilde{w}$ from $F_i$ [the last term in Eq. (43)] and they can be collected in the following correction to $B_{ik}$ which occurs in the acoustic lines:

$$\delta B_{ik}=i\beta\rho^{-1}\nabla_n[(\nabla_i\tilde{W}\nabla_n\tilde{W}+\gamma l^{-2}\delta_{in})\nabla_k\tilde{W}\nabla_m\tilde{W}]\omega^{-1}\nabla_m. \tag{47}$$

In this expression we have taken into account the fact that the sound frequency $\omega\sim c_1k$ satisfies the inequality $\rho\gg c_2\mathrm{k}$, $b^{1/2}k^2$, so that the smectic Green's function may be set approximately $D\simeq\omega^{-1}$. The expression (47) yields an additional contribution to the self-energy function corresponding to the variable $\varphi_i=j_i$.

The renormalization of the last term in Eq. (46) is determined by the diagram obtained by pairing two factors coming from the second term of Eq. (46). In each of these factors one must separate a pair of rapid variables $\tilde{w}$ in $F_k$ [from the last term of Eq. (43)]. Their pairing leads to a diagram of the type represented in Fig. 2. As a result of this we find the following expression for the polarization operator $\Pi_{ik}$:

$$\Pi_{ik}=(\gamma k_i+n_ik_z)(\gamma k_k+n_kk_z)$$

$$\times\frac{1}{2l^2}\int\frac{d\nu\, d^3q}{(2\pi)^4}\beta^2D(\nu,\mathbf{q})D(\omega+\nu,\mathbf{q})q^4. \qquad (48)$$

(Recall that $n_i$ are the components of a unit vector along the $z$ axis). We make the following remarks regarding the expression (48). First, for the acoustic mode $k_z\sim k$, so that in (48) it is legitimate to retain the factors with $k_z$. Secondly, the frequency for the acoustic mode satisfies $\omega\sim c_1k\gg c_2k, b^{1/2}k^2$, hence one may neglect the $k$ dependence in the integrand of Eq. (48). The integral (48) is investigated in the Appendix.

The structure of the Lagrange density (46) allows one to write down directly the expressions for the Green's functions $G_{ik}$ and $D_{ik}$ obtained by expanding the generating functional sandwiched respectively between $m_i, y_k$, and $m_i, m_k$. Setting $W$ in the third and fourth terms of the expression (46) equal to its equilibrium value $z/l$, we obtain, taking account of Eqs. (45) and (47)

$$G_{ik}^{-1}=\omega(\delta_{ik}-c_1^2k_ik_k/\omega^2)-\Sigma_{ik}, \qquad (49)$$

$$D_{ik}=G_{in}\Pi_{nm}G_{km}, \qquad (50)$$

$$\Sigma_{ik}=-(\beta/\rho l^2\omega)(\gamma k_i+k_zn_i)(\gamma k_k+k_zn_k). \qquad (51)$$

We shall be interested particularly in the imaginary part of the self-energy function $\Sigma_{ik}$. In order to determine it one must take the diagram which yields the renormalization of $\beta$ and separate in it the imaginary part. As a result of this we obtain

$$\operatorname{Im}\Sigma_{ik}=-\Pi_{ik}/2\rho T. \qquad (52)$$

The expression (52) is a form of the fluctuation-dissipation theorem.

## CONCLUSION

We have thus shown that if one takes into account the fluctuational corrections, one must change radically one's conceptions on the spectrum of smectics. As far as the second sound is concerned, its dispersion law is determined by the poles of the Greens functions $G$ of Eq. (33) and has the form

$$\omega=(-igk^2\pm\xi k_\perp/k), \qquad (53)$$

$$\xi=[ak_z^2+bk^4-g^2k^2k_\perp^2]^{1/2}. \qquad (54)$$

Taking into account the large magnitude of the logarithm $L$ we have $\ln(b/b_0)>1$, so that Eq. (40) implies $b-g^2<0$. Thus, for

$$k_z>[(g^2-b)/a]^{1/2}k^2$$

the frequency of second sound has a real part, i.e., it describes the propagation of a wave, whereas for the opposite inequality the corresponds to two purely diffusional modes, and as the frequency decreases the diffusion region increases.

In a realistic experimental situation[10-12] $k_z$ is of the order of $k_\perp$, and therefore, on account of the inequality $ak_z^2\gg bk_\perp^4$ implied by it, the damping of the second sound described by Eq. (32) becomes weak. Thus the case $k_z\sim k_\perp$ requires separate discussion. The expressions for $\Pi$ and $\Sigma$ in this limit can again be obtained from the diagrams of Fig. 2 and Fig. 3, respectively, in which one must consider terms proportional to $k_z^2$:

$$\operatorname{Im}\Sigma(\omega,k)=-\frac{\rho l^2}{2T}\omega^2\frac{k^2}{k_\perp^2}\Pi(\omega,k)=-\rho l^4k_z^2k_\perp^2$$

$$\times(4Tk^2)^{-1}\int\frac{d\nu\, d^3q}{(2\pi)^4}a^2q^4D(\nu,\mathbf{q})D(\omega+\nu,\mathbf{k}+\mathbf{q}).$$

This integral cannot be calculated explicitly, however, for $k_z\sim\omega k/a^{1/2}k_\perp$ one can obtain an estimate for it, similar to Eq. (A12):

$$\operatorname{Im}\Sigma\sim-10^{-2}T|\omega|\beta^{1/2}/\varkappa^{3/2}l\propto|\omega|L^{-1}.$$

The fact that this expression is linearly proportional to the frequency shows that for $k_z\sim k_\perp$ too the imaginary part of the spectrum has a purely fluctuational origin.

As already mentioned, the first sound spectrum is determined by the poles of the Green's functions $G_{ik}$, Eq. (49). The real part of the self-energy function $\Sigma_{ik}$, Eq. (51) gives rise to an anisotropic correction to the speed of sound.

$$\delta c_1=-\frac{\beta}{2\rho l^2c_1}\left(\gamma+\frac{k_z^2}{k^2}\right)^2\propto L^{-4/5}. \qquad (55)$$

This correction is small of order $c_2^2/c_1^2$ compared to $c_1$.

The imaginary part of the self-energy function determines the damping of first sound. In traditional terminology[1,9] the presence of $\operatorname{Im}\Sigma_{ik}$, Eq. (52), is equivalent to the existence of a fluctuational contribution to the following viscosity coefficients:

$$\delta\eta_4\gamma^{-2}=\delta\eta_5\gamma^{-1}=\delta\eta_1, \qquad (56)$$

$$-\operatorname{Im}\Sigma_\parallel=2(\gamma k+k_z)^2\delta\eta_1\rho^{-1}. \qquad (57)$$

The imaginary part of the longitudinal self-energy function $\Sigma_{ik}$ which appears in Eq. (57) determines directly a damping decrement of first sound related to the contribution of fluctuations.

The expression of $\delta n_1$ can be obtained from Eq. (48) and is listed in the Appendix. In the limit $g^2/b\gg1$ the integration can be carried out to the end (see the Appendix), and yields

$$\delta\eta_1=T\beta^{3/2}/64|\omega|l^3\varkappa^{3/2}. \qquad (58)$$

For values of $g^2/b$ which are not too large and have a doubly logarithmic scale (40), the proportionality is preserved:

$$\delta\eta_1\propto|\omega|^{-1}L^{-9/5}. \qquad (59)$$

The expression (58) remains a good approximation to $\delta\eta_1$. We note that, in distinction from $\operatorname{Re}\Sigma_{ik}$, $\operatorname{Im}\Sigma_{ik}$ does not contain the small parameter $c_2^2/c_1^2$. However, the contribution of the fluctuations to the imaginary part of the spectrum is small, on account of the smallness of the thermal energy $T$ compared to the elastic energy $\rho c_1^2l^3$. For typical values of the parameters $T\sim10^{-14}$ erg, $l\sim10^{-7}$ cm, $\rho\sim$ g/cm$^3$, $c_1\sim10^5$ cm/s, we obtain the estimate $\operatorname{Im}\Sigma_\parallel\sim10^{-6}$.

As can be seen from Eq. (59) the contribution of the fluctuations to the viscosity coefficient (and accordingly the damping of sound) increases as the frequency decreases and at low frequencies it exceeds substantially the bare constants. This allows one to explain the experimentally observed[10–12] deviations from the $\omega^2$ law for the damping of first sound in smectics. However, in interpreting experimental data one should keep in mind that for the frequencies used in the experiments the bare constant viscosity coefficients may compete with the fluctuational contribution (which contains the smallness indicated above), so that in handling the experimental data both terms must be taken into account.

A preliminary communication about the phenomena discussed above has been previously published in Ref. 13.

## APPENDIX

Making use of the expressions for the vertex and Green's functions from the main text we obtain for the diagrams represented in Figs. 1–3

$$a=-l^2\int\frac{d^3q\,d\nu}{(2\pi)^4}a^2q^4G(\nu,\mathbf{q})D(\nu,\mathbf{q})\frac{1}{\nu}, \quad \text{(A1)}$$

$$\tau=l^2\int\frac{d^3q\,d\nu}{(2\pi)^4}a^2q_z^2q^2D(\nu,\mathbf{q})D(\nu,\mathbf{q}), \quad \text{(A2)}$$

$$-bk^4+2igk^2\omega\leftarrow l^2\int\frac{d^3q\,d\nu}{(2\pi)^4}a^2\frac{1}{\omega+\nu}$$
$$\times(2\mathbf{kq}+\mathbf{k}^2)^2q_z^2G(\omega+\nu,\mathbf{k}+\mathbf{q})D(\nu,\mathbf{q}). \quad \text{(A3)}$$

The arrow in Eq. (A3) signifies that in the integral of the right-hand side one must subtract an infinite constant, as described in the body of the paper.

The integrals with respect to the frequency are easily calculated by the residue method and yield:

$$a=-\frac{l^2}{8}\int\frac{d^3q}{(2\pi)^3}\frac{q^4}{\eta^4}\frac{\tau a^2}{g}, \quad \text{(A4)}$$

$$\tau=\frac{l^2}{32}\int\frac{d^3q}{(2\pi)^3}q_z^2\frac{\eta^2+4g^2q^4}{\eta^6}\frac{a^2\tau^2}{g^3}, \quad \text{(A5)}$$

$$-bk^4+2igk^2\omega\leftarrow i\frac{l^2}{16}\int\frac{d^3q}{(2\pi)^3}\frac{\tau a^2}{g}q_z^2(2\mathbf{kq}+\mathbf{k}^2)^2\frac{1}{\xi}$$
$$\times\Big[\frac{1}{\nu_+}\{g\nu_+[\mathbf{q}^2+(\mathbf{q}+\mathbf{k})^2]$$
$$-i\omega\nu_+-gq^2\omega+i\frac{b}{2}[\mathbf{q}^4-(\mathbf{q}+\mathbf{k})^4]\}^{-1}-\dots\Big], \quad \text{(A6)}$$

where

$$\xi=[\eta^2-g^2q^4]^{1/2},\quad \nu_\pm=-igq^2\pm\xi,$$

and the ellipsis (...) denotes an expression obtained from the one explicitly written by substituting $v_-$ for $v_+$. Separating in (A6) the terms linear in $\omega$ and of fourth degree in $k$, we obtain

$$g=\frac{l^2}{32}\int\frac{d^3q}{(2\pi)^3}\frac{\tau a^2}{g^2}q_z^2\frac{1}{\eta^6}(\eta^2+4g^2q^4), \quad \text{(A7)}$$

$$b=-\frac{l^2}{8}\int\frac{d^3q}{(2\pi)^3}\frac{\tau a^2}{g}\frac{q_z^4}{\eta^4}$$
$$\times\Big[-\frac{1}{2}+\frac{3}{4}\frac{b}{g^2}+6\frac{b^2q^8}{\eta^4}-3\frac{bq^4}{\eta^2}-3\frac{b^2q^4}{g^2\eta^2}\Big]. \quad \text{(A8)}$$

Introducing polar variables in the $q_z$, $q^2$ plane:

$$q_z=\eta a^{-1/2}\sin\chi,\quad q^2=\eta b^{-1/2}\cos\chi \quad \text{(A9)}$$

and taking into account that

$$\int\frac{d^3q}{(2\pi)^3\eta^4}=\frac{1}{16\pi^2}\int\frac{d\eta}{\eta^3}\frac{d\chi}{(ab)^{1/4}}, \quad \text{(A10)}$$

we obtain for (A4), (A5), (A7), and (A8)

$$a=-\frac{l^2}{128\pi}\int\frac{d\eta}{\eta}\frac{\tau a^{3/4}}{gb^{5/4}},\quad \tau=\frac{l^2}{128\pi}\int\frac{1}{4}\frac{d\eta}{\eta}\frac{\tau^2a^{1/4}}{gb^{3/4}}\Big(1+\frac{b}{g^2}\Big),$$

$$g=\frac{l^2}{128\pi}\int\frac{d\eta}{\eta}\frac{\tau a^{1/4}}{b^{3/4}}\frac{1}{4}\Big(1+\frac{b}{g^2}\Big),\quad b=\frac{l^2}{128\pi}\int\frac{1}{2}\frac{d\eta}{\eta}\frac{\tau a^{1/4}}{gb^{1/4}}. \quad \text{(A11)}$$

By differentiation these expressions yield the equations (38) of the text.

We now calculate the integral which occurs in the expressions (48) and (56) for the fluctuational corrections to the viscosity:

$$\delta\eta_1=\frac{1}{4Tl^2}\int\frac{d\nu\,d^3q}{(2\pi)^4}\beta^2q^4D(\nu,\mathbf{q})D(\omega+\nu,\mathbf{q}). \quad \text{(A12)}$$

Making use of the expression for the Green's function and calculating the integral over the frequencies, we obtain

$$\delta\eta_1=\frac{\rho^2l^6a^{3/2}\tau^2}{1024\pi^2T|\omega|b^2g}\int_{-\infty}^{+\infty}dx\,x^2\int_{-\pi/2}^{\pi/2}d\chi\cos^3\chi$$
$$\times\frac{1+x^2+4g^2b^{-1}x^2\cos^2\chi}{(1+x^2g^2b^{-1}\cos^2\chi)[(1-x^2)^2+4g^2b^{-1}x^2\cos^2\chi]}. \quad \text{(A13)}$$

Making use of the ratio $\tau/g$ from Eq. (41) and the relations of $b$ with $\varkappa$ and $a$ with $\beta$ from Eq. (30), we obtain in the limit $g^2b^{-1}\gg1$ we obtain from (A13) the equation (58) of the text.


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Translated by Meinhard E. Mayer