# Nonlinear skin effect in metals

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The dependence of the impedance of a metal on the amplitude $\mathscr{H}$ of an incident radio wave under conditions of the anomalous skin effect is investigated theoretically. The analysis is carried out in a broad range of values of the quantity $\mathscr{H}$. It is shown that the impedance varies appreciably as $\mathscr{H}$ increases even when the intensity of the external signal is relatively low. The conductivity of the metal is higher, while the skin thickness and the impedance are smaller, in the case of a well-developed nonlinearity than in the linear approximation. The mechanism underlying the nonlinearity is connected with the existence of a group of electrons localized in the region where the magnetic component of the electromagnetic wave changes sign.

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## 1. INTRODUCTION

Metals are characterized by the fact that they possess a high electrical conductivity, i.e., a low surface impedance. Consequently, an electromagnetic wave of frequency $\omega$ incident on a metal is reflected primarily at the same frequency. All the multiple harmonics are weak, to the extent that the impedance is small. But this does not mean that only the first harmonic exists in the interior of the metallic sample. Under conditions of strong nonlinearity all the harmonics of the electromagnetic field can be of the same order of intensity inside the metal. In this case the surface impedance $Z$ should depend on the amplitude $\mathscr{H}$ of the incident wave.

Experimentally, the dependence of $Z$ on $\mathscr{H}$ in the presence of the anomalous skin effect has been investigated by a number of researchers.[1–7] The present work is the first attempt at the construction of a theory of the nonlinear anomalous skin effect in metals.

Because of the high conductivity, the magnetic component of an electromagnetic wave in a metal is always much stronger than the electric component. This is the essence of the nonlinearity in a metal: the trajectory of the electrons is established by the magnetic field of the wave, the influence of the electric field being negligible. This modification of the electron motion by a variable and inhomogeneous magnetic field is the primary cause of a number of nonlinear effects in pure metals at low temperatures (see, for example, Ref. 8).

Owing to the skin effect, a variable magnetic field in a metal oscillates and attenuates rapidly over a distance of the order of the skin thickness $\delta$. In the case of the anomalous skin effect, only the effective electrons, i.e., those which get into the skin layer and interact with the electromagnetic wave, participate in the conduction. For the indicated nonlinearity mechanism to exist, it is necessary that the trajectory of an effective electron in the inhomogeneous magnetic field in the skin layer have a length

$$L\sim(4cp_F\delta/e\mathscr{H})^{1/2}$$

that is much shorter than the effective mean free path $l = v/|\nu - i\omega|$ (see Fig. 1), i.e., that

$$4cp_F\delta/el^2\ll\mathscr{H}. \tag{1.1}$$

The inequality (1.1) implies that the characteristic magnetic field value $2\mathscr{H}$ in the skin layer should be higher than the field value $h$ at which $L = l$. In other words, the degree of nonlinearity of the anomalous skin effect is determined by the parameter $b = L/l$, or

$$b=(h/2\mathscr{H})^{1/2},\quad h=8cp_F\delta/el^2. \tag{1.2}$$

Here $c$ is the velocity of light, $e$ is the absolute value of the charge, $p_F$ and $v$ are the Fermi momentum and velocity, and $\nu$ is the electron relaxation rate. Let us estimate the quantity $h$. For typical pure metals at low temperatures, $\delta\sim10^{-4}$–$10^{-3}$ cm and $l\sim10^{-1}$ cm, and we find that $h\sim0.5$–$5$ Oe. In experiments the electromagnetic-wave amplitude $\mathscr{H}$ attains values of several tens of oersteds, and, thus, both the case of weak nonlinearity (large $b$) and the case of strong nonlinearity ($b\ll1$) are experimentally realizable.

1. In the case of weak nonlinearity, when the parameter $b$ is large compared to unity, i.e., when

$$b^4\gg1, \tag{1.3}$$

the trajectories of the effective electrons in the skin layer are almost straight lines, being only slightly curved by the wave's magnetic field. Therefore, in the leading approximation in $b^{-4}$, the skin effect is described by a linear theory, and the surface impedance $Z$ does not depend on the amplitude $\mathscr{H}$. The dependence of $Z$ on $\mathscr{H}$ appears only in the next terms of the expansion of the impedance in powers of the parameter $b^{-4}$:

$$[Z(\mathscr{H})-Z(0)]/Z(0)\sim b^{-4}\propto\mathscr{H}^2. \tag{1.4}$$

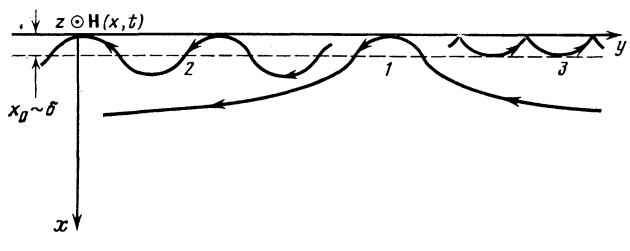


FIG. 1. Trajectories of the effective electrons in the magnetic field of an electromagnetic wave: transient (1), trapped (2), and gliding (3) electrons.

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The series expansion is carried out in the parameter $b^{-4}$, since the impedance of the metal is an analytic function of the magnetic field.

2. In the strong nonlinearity regime the inequality (1.1) is satisfied, i.e.,

$$b \ll 1. \tag{1.5}$$

Since the spatial distribution of the magnetic field $H(x, t)$ is sign-variable, a group of "trapped" electrons occurs in the metal. These electrons move along the sample surface in trajectories that wind about the plane $x = x_0(t)$ in which $H(x_0, t) = 0$ (see Fig. 1). In the direction perpendicular to the boundary (i.e., along the $x$ axis) they execute periodic motion. It must be emphasized that $x_0(t)$ is always of the order of $\delta$. Consequently, the electrons trapped by the Lorentz force conduct all the time in the skin layer, and therefore interact most effectively with the electromagnetic wave. Thus, under the conditions (1.5) of strong nonlinearity, the skin effect is determined by the winding electrons. The relative number of such electrons is of the order of $\delta/L$, and, according to (1.5), is significantly higher than the relative number $\delta/l$ of the effective electrons in the linear theory. For this reason, the metal possesses a significantly higher conductivity, and, hence, a smaller skin thickness $\delta$ and a lower surface impedance in the well-developed non-linearity regime (1.5).

The dependence of $\delta$ and $Z$ on the amplitude $\mathscr{H}$ is easily derived with the aid of Pippard's ineffectiveness concept. In this model the conductivity of the trapped electrons has the form

$$\sigma = \sigma_0 \frac{\delta}{L} = \sigma_0 \frac{\delta}{l} b^{-1}, \quad \sigma_0 = \frac{Ne^2}{m(\nu - i\omega)}, \tag{1.6}$$

where $N$ and $m$ are respectively the electron density and mass. From the Maxwell equations we find that the relation between the skin thickness and the effective conductivity $\sigma$ is given by the approximate formula

$$\delta^2 = ic^2/4\pi\omega\sigma. \tag{1.7}$$

Substituting (1.6) into (1.7), and solving the resulting equation for $\delta$, we obtain

$$\delta = \left(\frac{c^5 p_F}{4\pi^2 e \mathscr{H} \omega^2 \sigma_0^2}\right)^{1/5}, \quad Z \sim \frac{4\pi\omega\delta}{c^2} \propto \mathscr{H}^{-1/5}. \tag{1.8}$$

Thus, under the conditions (1.5) of strong nonlinearity the surface impedance of the metal depends on the incident-wave amplitude even in the leading approximation. Furthermore, it follows from (1.8) and (1.4) that the effect of the quantity $\mathscr{H}$ on the surface impedance becomes noticeable even earlier: in the transition region where the parameter $b$ is of the order of unity and the intensity of the external signal is relatively low.

The result (1.8) was obtained by us with the aid of the ineffectiveness concept, which, in the linear approximation, yields for the impedance an answer that is correct up to a real constant (see, for example, Ref. 9). But because of the dependence of the conductivity on the time, the ineffectiveness concept does not allow the determination of the true relation between the real and imaginary parts of the impedance (see Sec. 4) in the non-linear case.

## 2. FORMULATION OF THE PROBLEM. THE CURRENT DENSITY

Let us consider a metallic half-space on whose surface is incident a plane monochromatic wave of frequency $\omega$ and amplitude $\mathscr{H}$. Let us orient the $x$ axis along the normal into the interior of the metal ($x = 0$ at the boundary) and the $y$ and $z$ axes parallel to the electric and magnetic components of the electromagnetic field:

$$\mathbf{E}(x, t) = \{0, E(x, t), 0\}, \quad \mathbf{H}(x, t) = \{0, 0, H(x, t)\}. \tag{2.1}$$

Let us, in accordance with the definition, write the surface impedance in the form of a ratio of the first harmonics of the electric and magnetic fields at the boundary of the metal:

$$Z(\mathscr{H}) = \frac{4\pi}{c}\frac{E_\omega(0)}{\mathscr{H}}, \quad E_\omega(x) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\, E(x, t)\exp(i\omega t). \tag{2.2}$$

Here we have taken into account the fact that, up to terms of the order of $\omega\delta/c \ll 1$, the magnetic field $H(0,t)$ on the surface is equal to $2\mathscr{H}\cos(\omega t)$.

To find $E(x, t)$ and $H(x, t)$ in their explicit forms, we must solve the Maxwell equations

$$\frac{\partial H(x, t)}{\partial x} = -\frac{4\pi}{c} j(x, t), \qquad \frac{\partial E(x, t)}{\partial x} = -\frac{1}{c}\frac{\partial H(x, t)}{\partial t}. \tag{2.3}$$

The current density $j(x, t)$ is given by the solution to the kinetic equation linearized in the electric field $E(x, t)$. The nonlinearity is connected with the magnetic field $H(x, t)$, and is contained in the Lorentz force, which determines the electron trajectory. Therefore, the kernel of the integral conductivity operator does not depend on $E(x, t)$, and is a complicated functional of the magnetic-field distribution $H(x, t)$. From this it is clear that, to find the conductivity operator, we must investigate the dynamics of the electron motion in the magnetic field of the wave.

1. Below we shall limit ourselves to the consideration of the quasistatic situation $\omega \ll \nu$. In this case the motion occurs in an inhomogeneous, but constant magnetic field $H(x, t)$, since the phase of the wave remains constant during the entire mean free time. In other words, the "electromagnetic time" $t$ in the equations of motion does not vary, and plays the role of an external parameter.

Let us represent the vector potential $\mathbf{A}(x)$ of the wave's magnetic field in the form

$$\mathbf{A}(x) = \{0, A(x), 0\}, \quad A(x) = \int_0^x dx' H(x', t) \equiv H(0, t)\, a(x). \tag{2.4}$$

In the chosen gauge we have as the integrals of the electron motion the total energy, which is equal to the Fermi energy $\varepsilon_F$, and the generalized momenta $p_z = mv_z$ and $p_y = -eH(0, t)X/c$. We have, for convenience of exposition, introduced the function $a(x)$ and the conserved quantity $X$ (if the magnetic field is equal to $H(0,t)$ everywhere in the space, then $X$ has the meaning of the $x$ coordinate of the center of electron rotation).

The electron motion in the plane perpendicular to the vector $\mathbf{H}(x, t)$ is described by the velocities $v_y(x)$ and $v_x(x)$. For a spherical Fermi surface

$$v_y(x) = -\Omega[X - a(x)] \operatorname{sign} H(0, t),$$
$$|v_x(x)| = \Omega\{R_\perp^2 - [X - a(x)]^2\}^{1/2}. \tag{2.5}$$

Here

$$\Omega = \frac{e|H(0,t)|}{mc}, \quad R_\perp = \frac{v_\perp}{\Omega}, \quad v_\perp = (v^2 - v_z^2)^{1/2}, \tag{2.6}$$

sign $x = 1$ for $x > 0$, sign $x = -1$ for $x < 0$, and sign $x = 0$ for $x = 0$. The possible regions of motion along the $x$ axis can be found from the inequalities

$$X - R_\perp \leqslant a(x) \leqslant X + R_\perp, \tag{2.7}$$

which guarantee the positiveness of the radicand in the formula (2.5) for $|v_x(x)|$. The boundaries of the regions (2.7) for given values of $R_\perp$ and $X$ furnish the turning points.

The distinctive features of the electron motion in the field $H(x, t)$ are determined by the character of the dependence $a(x)$. The electromagnetic field in the metal oscillates and attenuates over a distance of the order of the skin thickness $\delta$. At any moment of time $t$ the spatial distribution of $H(x, t)$ changes sign an infinite number of times. But because of the rapid decrease, only the $H(x, t)$ zero at the point $x = x_0(t)$ closest to the sample boundary turns out to be important. All the remaining points at which the distribution changes sign need not be considered. Indeed, starting from the second point, the magnetic field in the metal is numerically so small that it can simply be considered to be equal to zero. Therefore, we assume that the function $a(x) \approx x$ in the region $x \ll \delta$, attains its maximum value $a(x_0)$ at the point $x = x_0(t)$, and tends to a constant value $a(\infty)$ in the region $x \gg \delta$. The value $a(\infty)$ can be either positive or negative (depending on the moment of time $t$). Furthermore, it follows from the anomaly of the skin effect that $a(x_0) \sim \delta \ll R_\perp$ and $a(\infty) \sim \delta \ll R_\perp$.

Figure 2 shows the region of possible values of the electron coordinate $x$ and the integral $X$ of the motion for $a(\infty) > 0$ (Fig. 2a) and $a(\infty) < 0$ (Fig. 2b). In accordance with the inequalities (2.7), this region is bounded from below by the curve $X = -R_\perp + a(x)$ and from above by the curve $X = R_\perp + a(x)$. It can be seen from Fig. 2 that, depending on the value of $X$, the electrons, by the nature of their motion, split up into four groups.

1) "Surface transient" electrons:

$$\begin{aligned} -R_\perp + a(x_0) \leqslant X \leqslant R_\perp \quad &\text{for} \quad a(\infty) > 0, \\ -R_\perp + a(x_0) \leqslant X \leqslant R_\perp + a(\infty) \quad &\text{for} \quad a(\infty) < 0, \\ 0 \leqslant x < \infty. \end{aligned} \tag{2.8}$$

They execute infinite motion, can reach the surface of the metal, and have only one turning point $x = 0$.

2) "Volume transient" electrons:

$$\begin{aligned} &-R_\perp + a(\infty) \leqslant X \leqslant -R_\perp + a(x_0) \quad \text{and} \quad R_\perp \leqslant X \leqslant R_\perp + a(\infty) \\ &\quad \text{for} \quad a(\infty) > 0, \\ &-R_\perp + a(\infty) \leqslant X \leqslant -R_\perp + a(x_0) \quad \text{for} \quad a(\infty) < 0, \\ &\quad x_1 \leqslant x < \infty. \end{aligned} \tag{2.9}$$

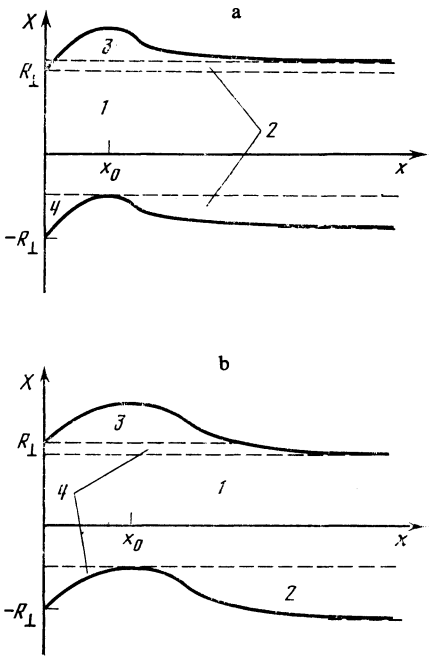


FIG. 2. Plots of the functions $X = \pm R_\perp + a(x)$ and the existence domains of the electron groups that establish the conductivity of the metal (see the text): 1) surface transient electrons; 2) volume transient electrons; 3) whirling (trapped) electrons; and 4) surface electrons.

In contrast to the case (2.8), the volume transient electrons do not get to the sample boundary, but turn around in the magnetic field $H(x, t)$. For the region $-R_\perp + a(\infty) \leqslant X \leqslant -R_\perp + a(x_0)$ the turning point $x_1$ ($x_1 > x_0$) is the larger of the two solutions to the equation

$$a(x) = X + R_\perp. \tag{2.10}$$

For $a(\infty) > 0$ and $R_\perp \leqslant X \leqslant R_\perp + a(\infty)$, the point $x_1$ ($x_1 < x_0$) can be found from the equation

$$a(x) = X - R_\perp. \tag{2.11}$$

3) "Trapped" or "whirling" electrons:

$$\begin{aligned} &R_\perp + a(\infty) \leqslant X \leqslant R_\perp + a(x_0) \quad \text{for} \quad a(\infty) > 0, \\ &R_\perp \leqslant X \leqslant R_\perp + a(x_0) \quad \text{for} \quad a(\infty) < 0, \\ &x_1 \leqslant x \leqslant x_2. \end{aligned} \tag{2.12}$$

Drifting along the $y$ axis, such electrons execute periodic motion along the $x$ axis about the point $x = x_0(t)$. The period of the oscillations of the trapped electrons is equal to $2T_v$, where

$$T_v = \int_{x_1}^{x_2} (dx/|v_x(x)|). \tag{2.13}$$

The turning points $x_1$ and $x_2$ ($x_1 < x_0 < x_2$) can be determined from Eq. (2.11).

4) "Surface" electrons:

$$\begin{aligned} &-R_\perp \leqslant X \leqslant -R_\perp + a(x_0) \quad \text{for} \quad a(\infty) > 0, \\ &-R_\perp \leqslant X \leqslant -R_\perp + a(x_0) \quad \text{and} \quad R_\perp + a(\infty) \leqslant X \leqslant R_\perp \\ &\quad \text{for} \quad a(\infty) < 0, \\ &\quad 0 \leqslant x \leqslant x_2. \end{aligned} \tag{2.14}$$

The surface electrons oscillate between the metal boundary $x=0$ and the point $x_2$ with period $2T_s$, where

$$T_s = \int_0^{x_2} (dx/|v_x(x)|). \tag{2.15}$$

In the interval $-R_\perp \leqslant X \leqslant -R_\perp + a(x_0)$ the turning point $x_2$ ($x_2 < x_0$) is the smaller of the two solutions to Eq. (2.10). For $a(\infty) < 0$ and $R_\perp + a(\infty) \leqslant X \leqslant R_\perp$, the value of $x_2$ ($x_2 > x_0$) can be found solving (2.11).

Each of the described electron groups makes its own contribution to the electrical conductivity of the metal.

2. To compute the current density $j(x, t)$, we must solve the kinetic equation. The solution should be sought for each group separately. Finally, for the spatial Fourier cosine transform

$$j(k) = 2\int_0^\infty dx \cos(kx) j(x, t) \tag{2.16}$$

we obtain the following result:

$$j(k) = \frac{1}{\pi}\int_0^\infty dk' \mathscr{E}(k') \sum_{\alpha=1}^{4} \sigma_\alpha(k, k'). \tag{2.17}$$

Here $\mathscr{E}(k)$ is the spatial Fourier transform of the electric field $E(x, t)$. The kernel $\sigma_1(k, k')$ of the conductivity operator for the surface transient electrons is given by the formula

$$\sigma_1(k, k') = \frac{3\sigma_0}{\pi l}\int_0^v \frac{dv_z}{v}\int \frac{dX}{R}\int_0^\infty dx \cos(kx) \frac{v_y(x)}{|v_x(x)|} \times \int_0^\infty dx' \cos(k'x') \frac{v_y(x')}{|v_x(x')|} \{\exp[-\nu|\tau(x, x')|] + \rho \exp[-\nu(\tau(0, x) + \tau(0, x'))]\}. \tag{2.18}$$

For the conductivity of the volume transient electrons we have

$$\sigma_2(k, k') = \frac{3\sigma_0}{\pi l}\int_0^v \frac{dv_z}{v}\int \frac{dX}{R}\int_{x_1}^\infty dx \cos(kx) \frac{v_y(x)}{|v_x(x)|} \times \int_{x_1}^\infty dx' \cos(k'x') \frac{v_y(x')}{|v_x(x')|} \{\exp[-\nu|\tau(x, x')|] + \exp[-\nu(\tau(x_1, x) + \tau(x_1, x'))]\}. \tag{2.19}$$

The contribution of the trapped electrons to the current density is given by the term

$$\sigma_3(k, k') = \frac{3\sigma_0}{\pi l}\int_0^v \frac{dv_z}{v}\int \frac{dX}{R} \operatorname{sh}^{-1}(\nu T_v)\int_{x_1}^{x_2} dx \cos(kx) \frac{v_y(x)}{|v_x(x)|} \times \int_{x_1}^{x_2} dx' \cos(k'x') \frac{v_y(x')}{|v_x(x')|} \{\operatorname{ch}[\nu(T_v - |\tau(x, x')|)] + \operatorname{ch}[\nu(T_v - \tau(x, x_2) - \tau(x', x_2))]\}. \tag{2.20}$$

Finally, the surface electrons give rise to the conductivity

$$\sigma_4(k, k') = \frac{3\sigma_0}{\pi l}\int_0^v \frac{dv_z}{v}\int \frac{dX}{R} [\exp(\nu T_s) - \rho \exp(-\nu T_s)]^{-1} \times \int_0^{x_2} dx \cos(kx) \frac{v_y(x)}{|v_x(x)|}\int_0^{x_2} dx' \cos(k'x') \frac{v_y(x')}{|v_x(x')|} \{\exp[\nu(T_s - |\tau(x, x')|)] + \rho \exp[-\nu(T_s - |\tau(x, x')|)] + \exp[\nu(T_s - \tau(x, x_2) - \tau(x', x_2)] + \rho \exp[-\nu(T_s - \tau(x, x_2) - \tau(x', x_2)]\}. \tag{2.21}$$

In the formulas (2.18)–(2.21) we have introduced the notation: $\sigma_0$ is the static conductivity of the metal, $l$ is the electron mean free path ($l = v/\nu$), $R = v/\Omega$, and $\tau(x, x')$ is the time an electron takes to move from the point $x$ to the point $x'$ in the magnetic field $H(x, t)$:

$$\tau(x, x') = \int_x^{x'} (dx''/|v_x(x'')|). \tag{2.22}$$

The interaction of the electrons with the sample surface is characterized by the parameter $\rho$: the probability for specular reflection from the boundary ($0 \leqslant \rho \leqslant 1$). The limits of the integration over $X$ in (2.18)–(2.21) are chosen in accordance with the existence domain of the group of electrons in question (see (2.8)–(2.14)).

Under the conditions ($\delta \ll l$ and $\delta \ll R$) of the anomalous skin effect, the expressions (2.18)–(2.21) for the conductivities can be simplified by replacing them with their asymptotic forms. In this case the dominant contributions to the $X$, $x$, and $x'$ integrals are made by the neighborhoods of those points where the velocity $|v_x(x)|$ is equal to zero. These points are: in the $x$ and $x'$ integrals the turning points; in the $X$ integral, the end points of the integration interval, i.e., the boundaries of the existence domain of each electron group. The conductivity has markedly different asymptotic forms in the cases of small ($b \gg 1$) and large ($b \ll 1$) external-wave amplitudes. Therefore, below we shall consider these two cases separately.

## 3. WEAK NONLINEARITY (SMALL AMPLITUDES $\mathscr{H}$)

Under the conditions (1.3) of weak nonlinearity, the asymptotic form of the current density (2.17) can be written in the form of a sum of two terms:

$$j(k) = j_0(k) + \Delta j(k), \tag{3.1}$$

where $j_0(k)$ is the usual linear anomalous-skin-effect current in zero magnetic field (see, for example, Ref. 10):

$$j_0(k) = \frac{3\pi\sigma_0}{4l}\left[\frac{\mathscr{E}(k)}{k} - \frac{2}{\pi^2}(1-\rho)\int_0^\infty dk' \mathscr{E}(k') \frac{\ln(k/k')}{k^2 - k'^2}\right]. \tag{3.2}$$

It is determined by the surface transient electrons (2.8), whose trajectories in the skin layer are almost straight lines, slightly curved by the wave's magnetic field.

The nonlinear correction $\Delta j(k)$ is small in comparison with $j_0(k)$ ($\Delta j(k)/j_0(k) \ll b^{-4} \ll 1$). It is made up of contributions, which are of the same order of magnitude, from all the electron groups. It turns out that the terms of the expansions of the conductivities (2.18)–(2.21) completely cancel each

other out in the $b^{-1}$, $b^{-2}$, and $b^{-3}$ approximations. As a result, we obtain

$$\Delta j(k)=-\frac{9\sigma_0}{l}\left(\frac{l^2}{8R}\right)^2\left[2-(1-\rho)^2\sum_{n=1}^{\infty}\frac{\rho^{n-1}}{n^2}\right]E(0,t). \quad (3.3)$$

It can be seen that the current $\Delta j(k)$ does not depend on $k$, and that it is due to the electric field $E(0, t)$ at the boundary of the metal. This means that the correction $\Delta j(x)$ to the current has a surface character: $\Delta j(x)\propto\delta(x)$. Indeed, we should, in computing the asymptotic form (3.3), set $\cos(kx)$ and $\cos(k'x')$ equal to unity in the formulas (2.18)–(2.21), since the characteristic domain of the integration over $x$ and $x'$ is of the order of $l^2/R\ll k^{-1}, k'^{-1}$. Therefore, the additional nonlinear corrent $\Delta j(x)$ is concentrated near the metal boundary in a thin layer of thickness $l^2/R\sim b^{-2}\delta\ll\delta$. It is worth noting that the asymptotic expression (3.1)–(3.3) for the current density exactly coincides in form with the one that obtains in the case of a weak constant magnetic field.[11,12]

The Maxwell equations (2.3) with the current density (3.1)–(3.3) are solved by the method of successive approximations. In the zeroth—in the parameter $b^{-4}$—approximation the surface impedance does not depend on the amplitude $\mathscr{H}$, and is the impedance produced in zero magnetic field by the anomalous skin effect[10]:

$$Z(0)=4\pi\sqrt{3}\frac{\omega\delta}{c^2}\frac{\sin^2(\pi z_0/3)}{\sin^2(\pi z_0/2)}\exp\left(-\frac{i\pi}{3}\right). \quad (3.4)$$

Here $\cos(\pi z_0)=\rho$ and $\delta$ is the skin thickness in the linear theory:

$$\delta=(c^2l/3\pi^2\omega\sigma_0)^{1/3}. \quad (3.5)$$

It follows from (2.2) and (2.3) that the nonlinear correction to the impedance (3.4) can be computed from the formula

$$Z(\mathscr{H})-Z(0)=-\frac{16\pi^2}{c^2\mathscr{H}^2}\int_0^{\infty}dx\,E_\omega(x)\Delta j_\omega(x). \quad (3.6)$$

Hence we find

$$\frac{Z(\mathscr{H})-Z(0)}{|Z(0)|^2}=-i\frac{3\sqrt{3}\,c^2}{8\pi^2\omega\delta}\left[2-(1-\rho)^2\sum_{n=1}^{\infty}\frac{\rho^{n-1}}{n^2}\right]b^{-4}. \quad (3.7)$$

It turns out that, in the approximation $b^{-4}\propto\mathscr{H}^2$, the correction (3.7) is purely imaginary. It is not possible to compute the function Re $z(\mathscr{H})$, which appears in the terms of higher order in smallness. But we can conclude from the analyticity condition for $Z(\mathscr{H})$ that

$$\mathrm{Re}[Z(\mathscr{H})-Z(0)]/|Z(0)|\sim b^{-8}\infty\mathscr{H}^4. \quad (3.8)$$

Thus, the real and imaginary parts of the impedance behave quite differently in the region (1.3) of weak nonlinearity when $\mathscr{H}$ is varied.

## 4. STRONG NONLINEARITY (LARGE AMPLITUDES $\mathscr{H}$)

1. The electron dynamics in the inhomogeneous magnetic field of a wave are fundamentally distinguished by the fact that some of the electrons are trapped by this field, and are localized near the sign-reversal plane for $H(x, t)$. The group of winding electrons thus produced plays the most important role in the strong-nonlinearity case (1.1). Therefore, to obtain the asymptotic form of the current density $j(k)$, we must first of all compute the conductivity $\sigma_3(k, k')$.

Let us set $v_y(x)v_y(x')=v_\perp^2$ in the expression (2.20), and expand $|v_x|$ in the neighborhood of $X=R_\perp$, using the fact that $a(x)$ is small compared to $R_\perp$. Going over to the integration variable $\bar{x}=X-R_\perp$, we obtain

$$\sigma_3(k,k')=\frac{3\sigma_0}{\pi l}\int_0^v\frac{v_\perp\,dv_z}{v^2}\int_{a(\infty)\theta(a(\infty))}^{a(x_0)}d\bar{x}\,\mathrm{cth}(\nu T_v)\times\int_{x_1}^{x_2}\frac{dx\cos(kx)}{[a(x)-\bar{x}]^{1/2}}\int_{x_1}^{x_2}\frac{dx'\cos(k'x')}{[a(x')-\bar{x}]^{1/2}}, \quad (4.1)$$

where $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$. The $\bar{x}$, $x$, and $x'$ integrals can be evaluated with the use of the quadratic expansion of the "vector potential" $a(x)$ about the point $x=x_0(t)$. As a result, the asymptotic expression for the conductivity $\sigma_3(k, k')$ of the trapped electrons assumes the form

$$\sigma_3(k,k')=\frac{6\pi\sigma_0}{l}\int_0^v\frac{v_\perp\,dv_z}{v^2}\mathrm{cth}(\nu T)\cos(kx_0)\cos(k'x_0)\times[\varkappa_0kJ_1(k\varkappa_0)J_0(k'\varkappa_0)-\varkappa_0k'J_0(k\varkappa_0)J_1(k'\varkappa_0)]\frac{1}{k^2-k'^2}. \quad (4.2)$$

Here $J_0(x)$ and $J_1(x)$ are the Bessel functions of zeroth and first order; $2T$ is the limiting period of the winding electrons with velocity $v_x$ and turning-point separation $x_2-x_1$ equal to zero, i.e.,

$$T=\pi(mc/ev_\perp|H'(x_0,t)|)^{1/2}, \quad (4.3)$$

where the prime denotes differention with respect to $x$. The quantity $\varkappa_0(0\leqslant\varkappa_0\leqslant x_0)$ is determined from the equation

$$a(x_0-\varkappa_0)=a(\infty)\theta(a(\infty)) \quad (4.4)$$

and is the maximum distance from the point $x_0$ to the turning point closest to the metal surface. For $a(\infty)\leqslant 0$ the quantity $\varkappa_0=x_0$.

The structure of the asymptotic expression (4.2) indicates that the first of the Maxwell equations (2.3) with current density (2.17) is an integrodifferential equation with a kernel that is a functional of the sought function $H(x, t)$. It is not possible to solve such an equation analytically. Therefore, we are forced to carry out further simplifications of the current density, retaining the main characteristics of the mechanism underlying the nonlinearity.

It is not difficult to understand that the main source of the nonlinearity is the dependence of the period (4.3) of the trapped-electron motion on the magnetic field $H(x, t)$. It is precisely because of this dependence that the conductivity of the metal is a function of the amplitude $\mathscr{H}$ of the incident wave. Therefore, let us write the current density associated with the trapped electrons in the form

$$j_3(k)=\frac{3\sigma_0}{2l}\frac{\mathscr{E}(k)}{k}\int_0\frac{v_\perp\,dv_z}{v^2}\mathrm{cth}(\nu T). \quad (4.5)$$

In this model expression we have retained all the major char-

acteristics of the conductivity: we have in the denominator the wave number $k$, which is characteristic of current generated under the conditions of the anomalous skin effect, and there is the function $\cot(\nu T)$, which is responsible for the nonlinear dependence of the conductivity on the field $H(x, t)$. The current density (4.5) differs from the true asymptotic expression, given by the formulas (2.17) and (4.2), by only a real function equal in order of magnitude to unity. Consequently, the replacement of the current (2.17), (4.2) by (4.5) can only lead to the appearance of an additional numerical real factor in the impedance. With the same degree of accuracy, the current density associated with all the electron groups in the case of diffuse scattering from the metal boundary is

$$j(k)=\frac{3\sigma_0}{l}\frac{\mathscr{E}(k)}{k}\int_0^v\frac{v_\perp\,dv_z}{v^2}\,[1-\exp(-2\nu T)]^{-1}. \tag{4.6}$$

2. The Maxwell equations (2.3) with the current density written in the model (4.6) can be solved exactly. The distribution of the electromagnetic field can be represented in the form

$$\begin{aligned} H(x,t)&=\sum_{n=-\infty}^{\infty}H_n\left(\frac{x}{\delta_n}\right)\exp[-in\xi(\omega t)],\\ E(x,t)&=\frac{1}{\tilde\sigma(\omega t)}\sum_{n=-\infty}^{\infty}E_n\left(\frac{x}{\delta_n}\right)\exp[-in\xi(\omega t)]. \end{aligned} \tag{4.7}$$

Here

$$\begin{aligned} H_n\left(\frac{x}{\delta_n}\right)&=H_n(0)\,\frac{2}{\pi}\int_0^\infty\frac{q^2dq\sin(qx/\delta_n)}{q^3-in/|n|},\\ E_n\left(\frac{x}{\delta_n}\right)&=-\frac{in\omega\delta_n}{\mu c}H_n(0)\,\frac{2}{\pi}\int_0^\infty\frac{qdq\cos(qx/\delta_n)}{q^3-in/|n|}. \end{aligned} \tag{4.8}$$

The function $\xi(\varphi)$ is given by the formulas

$$\begin{aligned} \xi(\varphi)&=\frac{1}{\mu}\int_0^\varphi\frac{d\varphi'}{\tilde\sigma(\varphi')},\qquad \mu=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{\tilde\sigma(\varphi)},\\ \tilde\sigma(\varphi)&=\frac{4}{\pi}\int_0^v\frac{v_\perp\,dv_z}{v^2}\,[1-\exp(-2\nu T(\varphi))]^{-1}. \end{aligned} \tag{4.9}$$

Since $\xi(\varphi+2\pi)=\xi(\varphi)+2\pi$, the solutions (4.7) are periodic in time, with period equal to that, $2\pi/\omega$, of the incident wave. The penetration depth $\delta_n$ of the $n$ component of the field is

$$\delta_n=\delta|n|^{-1/3},\qquad \delta^3=c^2l\mu/3\pi^2\omega\sigma_0. \tag{4.10}$$

It must be emphasized that the period $2T(\varphi)$ of the motion of the trapped electrons and, consequently, $\tilde\sigma(\varphi)$ and $\mu$ depend on the quantity $\delta$. This means that the last relation in (4.10) should be regarded as an equation for the determination of $\delta$.

The coefficients $H_n(0)$ in (4.8) are determined from the boundary condition imposed on the metal surface:

$$H(0,t)=2\mathscr{H}\cos(\omega t),$$

which yields

$$H_n(0)=\frac{\mathscr{H}}{\pi\mu}\int_0^{2\pi}\frac{d\varphi}{\tilde\sigma(\varphi)}\cos\varphi\exp[in\xi(\varphi)]. \tag{4.11}$$

Using the results (4.7)–(4.11), we can represent the surface impedance (2.2) of the metal in the following form:

$$Z(\mathscr{H})=\frac{16\pi\omega\delta}{3\sqrt3c^2}(\zeta'-i\zeta''),$$

$$\begin{aligned} \zeta'&=\frac{1}{2}\sum_{n=0}^{\infty}(2n+1)^{2/3}\left|\frac{2}{\pi}\int_0^\pi\frac{d\varphi\cos\varphi}{\mu\tilde\sigma(\varphi)}\exp[i(2n+1)\xi(\varphi)]\right|^2,\\ \zeta''&=\frac{4}{\pi^2}\sum_{n=0}^{\infty}(2n+1)^{2/3}\int_0^\pi\frac{d\varphi\sin\varphi}{\mu\tilde\sigma(\varphi)}\int_0^\pi\frac{d\varphi'\cos\varphi'}{\mu\tilde\sigma(\varphi')}\\ &\quad\times\sin\left\{(2n+1)[\xi(\varphi)-\xi(\varphi')]-\frac{\pi}{6}\right\}. \end{aligned} \tag{4.12}$$

In the expression (4.12) the main dependence on the incident-wave amplitude $\mathscr{H}$ is contained in $\delta$. The ratio $\zeta''/\zeta'$ of the imaginary part of the impedance to the real part is not equal to $\sqrt3$, as in the linear anomalous skin effect, but, generally speaking, depends on $\mathscr{H}$. The deviation of the quantity—Im $Z$/Re $Z$ from $\sqrt3$ can be used in experiment as a criterion for nonlinearity of the skin effect.

3. To analyze the dependence of the penetration depth $\delta$ and the impedance $Z$ on the amplitude $\mathscr{H}$, let us explicitly separate out in the expression $2\nu T(\varphi)$ the nonlinearity parameter $b$:

$$2\nu T(\varphi)=b\left(\frac{v}{v_\perp}\right)^{1/2}\alpha(\varphi),\quad \alpha(\varphi)=\pi\left(\frac{\mathscr{H}\delta^{-1}}{|H'(x_0,t)|}\right)^{1/2}. \tag{4.13}$$

The function $\alpha(\varphi)$, being periodic with period $\pi$, attains a maximum value $\alpha_{\max}$ in the interval $(0,\pi)$. In the case of extremely strong nonlinearity, when

$$b\alpha_{max}\ll1, \tag{4.14}$$

the trapped electrons play the major role in the conductivity, and

$$\tilde\sigma(\varphi)=\frac{24\sqrt{2\pi}}{5\Gamma^2(1/4)}b^{-1}\alpha^{-1}(\varphi). \tag{4.15}$$

Accordingly, the quantity $\mu$ entering into the equation (4.10) for $\delta$ is:

$$\mu=\frac{5\Gamma^2(1/4)}{24\sqrt{2\pi}}\bar\alpha b,\qquad \bar\alpha=\frac{1}{\pi}\int_0^\pi d\varphi\alpha(\varphi). \tag{4.16}$$

It should be noted that in the case (4.14) the function $\alpha(\varphi)$ does not contain any parameters, and, in particular, does not depend on $\delta$ and $\mathscr{H}$. This holds true for the product $\mu\tilde\sigma(\varphi)$ and the function $\xi(\varphi)$. We can easily verify this by normalizing the field $H(x, t)$ and the variable $x$ to the amplitude $\mathscr{H}$ and the quantity $\delta$ respectively. As a result, the formulas (4.7)–(4.11) and (4.13) will furnish a closed system of universal equations for the determination of $\alpha(\varphi)$. Thus, $\zeta'$ and $\zeta''$ in (4.12), $\alpha_{\max}$ in (4.14), and $\bar\alpha$ in (4.16) are numbers ($\bar\alpha\approx\alpha_{\max}\sim50$), and the entire dependence of $\mu$ on $\delta$ is concentrated in the parameter $b$. Substituting (4.16) and (1.2) into (4.10), we obtain

$$\delta=\left(\frac{25\Gamma^4(1/4)\,\bar\alpha^2}{81(2\pi)^5}\frac{c^5p_F}{e\mathscr{H}\omega^2\sigma_0^2}\right)^{1/5}. \tag{4.17}$$

The expressions (4.12) and (4.17) determine the dependence of the impedance of the metal on the amplitude $\mathscr{H}$, the frequency $\omega$, and the mean free path $l$. Simple estimates show

that the numerical ratio $\zeta''/\zeta'$ is of the order of 1.2, and differs from $\sqrt{3}$.

Let us recall that, in the linear anomalous skin effect, the absorption of the electromagnetic energy by the metal is due to the collisionless Landau damping, and that the surface impedance does not depend on the collision rate $\nu$. Under the condition (1.1), because of the strong nonlinearity, the mechanism underlying the absorption changes. It can be seen from (4.17) that the Landau damping disappears, and the absorption has a collisional character.

The formulas (4.6)–(4.17) pertain to the case of diffuse electron reflection from the metal surface ($\rho = 0$). In the case of specular reflection ($\rho = 1$) the function $Z(\mathscr{H})$ will not change, since the whirling electrons, which determine the characteristics of the nonlinear skin effect, do not interact with the sample boundary at all. Unimportant differences will be connected with the appearance of a group of gliding electrons (Fig. 1) whose contribution to the conductivity in the specular case is of the same order of magnitude as the contribution of the trapped electrons.

In conclusion, let us compare the nonlinear skin effect with another well-known phenomenon: the nonlinear Landau damping,[13–15] which occurs during the propagation of natural electromagnetic waves in a metal. Common to the mechanisms underlying these effects is the phenomenon of electron capture by the field of the electromagnetic wave, a phenomenon which sets in at the same $\mathscr{H}$ values satisfying the inequality (1.1). But it turns out that this phenomenon plays different roles in these effects. In the case of the nonlinear Landau damping the electromagnetic properties of the medium (the conductivity) and the wave dispersion are determined by the transient electrons, while a relatively small number of "resonance" electrons moving in phase with the wave are responsible for the wave absorption. The capture leads to the decrease of the number of resonance electrons, as a result of which the absorption also decreases. The dynamics of the captured particle does not then play any role. The nonlinear skin effect considered here, like the "current states" in metals,[8,16–18] is a case of strong nonlinearity, when the electromagnetic properties of the medium are determined precisely by the dynamics of the captured electrons. The effect of the wave field on the conductivity turns out to be so strong that there occurs a qualitative change in the picture of the skin effect.

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[1] V. F. Gantmakher, Pis'ma Zh. Eksp. Teor. Fiz. **2**, 557 (1965) [JETP Lett. **2**, 346 (1965)].
[2] J. F. Cochran and C. A. Shiffman, Bull. Am. Phys. Soc. **10**, 110 (1965).
[3] J. F. Cochran and C. A. Shiffman, Phys. Rev. A **140**, 1678 (1965).
[4] V. T. Dolgopolov and L. Ya. Margolin, Pis'ma Zh. Eksp. Teor. Fiz. **17**, 233 (1973) [JETP Lett. **17**, 167 (1973)].
[5] V. T. Dolgopolov, S. S. Murzin, and P. N. Chuprov, Zh. Eksp. Teor. Fiz. **78**, 331 (1980) [Sov. Phys. JETP **51**, 166 (1980)].
[6] V. V. Boĭko, L. V. Ovchinnikova, and G. N. Landysheva, Pis'ma Zh. Eksp. Teor. Fiz. **32**, 432 (1980) [JETP Lett. **32**, 405 (1980)].
[7] V. V. Vas'kin, V. Ya. Demikhovskiĭ, I. F. Voloshin, V. M. Dzugutov, and L. M. Fisher, Rezisy dokladov 22 Vsesoyuznogo soveshchaniya po fizike nizkikh temperatur (Abstracts of Papers Presented at the Twenty-Second All-Union Conference on Low Temperature Physics), Kishinev, 1982, Vol. 2, p. 180.
[8] V. T. Dolgopolov, Usp. Fiz. Nauk **130**, 241 (1980) [Sov. Phys. Usp. **23**, 134 (1980)].
[9] É. A. Kaner, N. M. Makarov, V. L. Fal'ko, and V. A. Yampol'skiĭ, Zh. Eksp. Teor. Fiz. **73**, 1400 (1977) [Sov. Phys. JETP **46**, 737 (1977)].
[10] L. E. Hartmann and J. F. Luttinger, Phys. Rev. **151**, 430 (1966).
[11] É. A. Kaner, O. I. Lyubimov, N. M. Makarov, and I. B. Snapiro, Zh. Eksp. Teor. Fiz. **77**, 1183 (1979) [Sov. Phys. JETP **50**, 597 (1979)].
[12] O. I. Lyubimov, N. M. Makarov, and V. A. Yampolski, Solid State Commun. **39**, 815 (1981).
[13] R. K. Mazitov, Zh. Prikl. Mekh. Tekh. Fiz. **1**, 27 (1965).
[14] T. O'Neil, Phys. Fluids **8**, 22 (1965).
[15] G. A. Vugal'ter and V. Ya. Demikhovskiĭ, Zh. Eksp. Teor. Fiz. **70**, 1419 (1976) [Sov. Phys. JETP **43**, 739 (1976)].
[16] G. I. Babkin and V. T. Dolgopolov, Solid State Commun. **18**, 713 (1973).
[17] N. M. Makarov and V. A. Yampol'skiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 421 (1982) [JETP Lett. **35**, 520 (1982)].
[18] N. M. Makarov and V. A. Yampol'skiĭ, Zh. Eksp. Teor. Fiz. **85**, 614 (1983) [Sov. Phys. JETP **58**, 357 (1983)].

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