Neutron diffraction by incommensurate magnetic structures

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We investigate the picture of diffraction scattering by an incommensurate structure during a temperature-induced or a magnetic-field-induced phase transition into a commensurate phase. We show that when the field is increased or when the temperature is lowered satellites of higher order are produced in addition to the pair of principal magnetic satellites, and a central peak appears. Subsequently the intensities of all the satellites decrease, while the central peak increases and at the point of transition into the commensurate phase it is transformed into the principal reflection of this phase. In the case of joint action of a magnetic field and of the natural crystalline anisotropy, the nonlinear equation for the distribution of the order parameter can be solved exactly only for second-order anisotropy. At arbitrary anisotropy, asymptotic solutions are obtained near the Néel point and near the phase transition into a commensurate phase, where the soliton-lattice approximation is valid. From the observed diffraction effects one can assess the validity of the soliton picture of the phase transition and of the principal approximation of the theory, viz., that the absolute value of the order parameter is constant.

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1. INTRODUCTION

There are many known substances with helical magnetic structures and with wave vectors that change continuously when the temperature is lowered. In some cases this change halts on reaching the nearest symmetric point of the Brillouin zone, corresponding to the onset of a commensurate structure, either ferromagnetic or antiferromagnetic (see Ref. 1). The physical picture of such a phase transition from an incommensurate to a commensurate phase was first established by Dzyaloshinskiĭ[2] on the basis of a free-energy functional of the following form:

$$\Phi = \frac{1}{V}\int d\mathbf{r}\left\{\gamma\rho^2\left(\frac{d\varphi}{dz}\right)^2 + 2\sigma\rho^2\frac{d\varphi}{dz} + w\rho^n\cos n\varphi\right\}, \qquad (1.1)$$

which includes the exchange energy, the anisotropic interaction described by the Lifshitz invariant, and the energy of the n-th order crystallographic anisotropy. Here ρ and φ are the modulus and phase of the order parameter (DP), and it is assumed that the modulus is independent of the coordinates. The parameter $\gamma > 0$, while σ can have any sign; we assume for the sake of argument $\sigma < 0$.

In the absence of anisotropy, a structure modulated along the z axis is produced, having a temperature-independent wave vector $k_0 = |\sigma|/\gamma$. In the general case the spatial distribution of the DP phase satisfies the mathematical-pendulum equation

$$\frac{d^2(\varphi n)}{dz^2} + v\sin(\varphi n) = 0, \qquad v = \frac{w\rho^{n-2}}{2\gamma}n^2, \qquad (1.2)$$

the solution of which is expressed in terms of elliptic functions[2]

$$n\varphi/2 = \mathrm{am}(qz, \varkappa), \quad q = v^{1/2}/\varkappa, \qquad (1.3)$$

where $\varkappa$ is the modulus of the elliptic functions.[3] The parameter corresponds to the integration constant of Eq. (1.2) and must be obtained from the minimum of the system energy after substituting the solution (1.3) in Φ. The energy is expressed in terms of the first- and second-order complete elliptic integrals K and E (Ref. 2):

$$\frac{\Phi}{\rho^2} = -2|\sigma|\frac{\pi v^{1/2}}{n\varkappa K} + 2\gamma\frac{v}{n^2}\left(\frac{\varkappa^2-2}{\varkappa^2} + \frac{4}{\varkappa^2}\frac{E}{K}\right). \qquad (1.4)$$

Minimization of this expression with respect to $\varkappa$ leads to an equation for the determination of $\varkappa$:

$$\frac{E}{\varkappa} = \left(\frac{v_c}{v}\right)^{1/2}, \quad v_c = \frac{n^2\pi^2\sigma^2}{16\gamma^2}. \qquad (1.5)$$

Analyzing (1.3), Dzyaloshinskiĭ arrived at the following structure of the inhomogeneous state that corresponds to the functional (1.1). A periodic structure with period $L = 4\varkappa K/v^{1/2}$, is produced and evolves with change of the parameter v in the following manner. At small v the phase $\varphi(z)$ differs little from $k_0 z$ and corresponds to a helical structure. With increasing v, the phase is almost constant over a length L of the period, but changes abruptly by $2\pi/n$ at the end points of L. As $v \to v_c$ the relative fraction of the constant-phase section increases. The system can be represented as a periodic structure of domains of a commensurate phase with constant values of φ that are multiples of $2\pi/n$ and are separated by domain walls (solitons). This soliton picture can be easily discerned from the solution (1.3) by using the asymptotic forms of the elliptic integrals 3:

1) $\varkappa \to 0$

$$K = \frac{\pi}{2}\left(1 + \frac{\varkappa^2}{4} + \ldots\right), \quad E = \frac{\pi}{2}\left(1 - \frac{\varkappa^2}{4} + \ldots\right); \qquad (1.6)$$

2) $\varkappa \to 1$ (or $\varkappa' \equiv (1-\varkappa^2)^{1/2} \to 0$):

$$K = \ln\frac{4}{\varkappa'} + \ldots, \quad E = 1 + \frac{1}{2}\varkappa'^2\ln\frac{4}{\varkappa'}. \qquad (1.7)$$

The parameter v varies with temperature because of the temperature dependence of the DO (at $n \geqslant 4$). In the case of uniaxial anisotropy ($n = 2$), v is independent of temperature (if, as usual, all the parameters of the functional (1.1) are assumed constant).

The predicted structure of the incommensurate phase can be verified in experiment by using the neutron scattering method. The aim of the present paper is to investigate the diffraction picture of elastic magnetic scattering by a soliton lattice. Within the framework of one and the same mathematical model we succeed in considering also the question of neutron diffraction by a helical structure placed in a magnetic field, since the field can be regarded formally as first-order anisotropy ($n = 1$). Finally, we solve the problem of the joint action of anisotropy and of an external magnetic field and establish the diffraction picture of the scattering also in this important case.

The cross section for elastic magnetic scattering is given by the relations[4]

$$d\sigma/d\Omega \sim \sum_{\gamma,\gamma'} K_{\gamma\gamma'}(\delta_{\alpha\gamma}-e_\alpha e_\gamma)(\delta_{\beta\gamma'}-e_\beta e_{\gamma'})\mathscr{F}_\alpha^*(\mathbf{Q})\mathscr{F}_\beta(\mathbf{Q}), \quad (1.8)$$

$$\mathscr{F}_\alpha(\mathbf{Q})=\frac{1}{V}\int d\mathbf{r}\, e^{-i\mathbf{Q}\mathbf{r}}M_\alpha(\mathbf{r}), \quad (1.9)$$

$$K_{\gamma\gamma'}=\overline{\sigma_\gamma\sigma_{\gamma'}}=\delta_{\gamma\gamma'}+i\sum_\mu \varepsilon_{\gamma\gamma'\mu}p_\mu^0. \quad (1.10)$$

Here $\mathbf{Q}$ is the scattering vector, $\mathbf{e}$ is the scattering unit vector, $\mathbf{M}_\alpha(\mathbf{r})$ is the magnetic-moment density in the crystal, and σ_γ are Pauli matrices (the neutron spin). The superior bar in (1.10) denotes averaging over the polarizations of the neutrons in the incident beam, $\mathbf{p}^0$ is the vector of the initial polarization of the beam, and $\varepsilon_{\gamma\gamma'\mu}$ is a unit antisymmetric tensor. The main problem reduces thus to finding the Fourier transform of the spin-density function defined by the DP phase (1.3). We consider next several typical physical situations.

2. SIMPLE HELIX IN AN EXTERNAL FIELD

A phase transition into a simple helical structure with small wave vector along the z axis can be described on the basis of the functional[5]

$$\Phi=\frac{1}{V}\int d\mathbf{r}\left\{\gamma(\nabla M)^2+\beta M_z^2 +2\sigma\left(M_x\frac{dM_y}{dz}-M_y\frac{dM_x}{dz}\right)-\mathbf{MH}\right\}. \quad (2.1)$$

At large $\beta > 0$ the spins lie in the (x, y) plane. They will remain there if the external field H lies in the same plane. Introducing for the magnetization vector the polar coordinates

$$M_x=M\cos\varphi,\quad M_y=M\sin\varphi,\quad M_z=0, \quad (2.2)$$

we reduce the expression (2.1) to (1.1), where

$$n=1,\quad v=h/2\gamma,\quad h=H/M, \quad (2.3)$$

(for the sake of argument, the field was chosen along the $-\mathbf{x}$ direction).

Substituting the solution (1.3) with $n = 1$ in (2.2), we easily obtain the relations

$$\cos\varphi=2\,\mathrm{cn}^2\,qz-1,\quad \sin\varphi=2\,\mathrm{cn}\,qz\,\mathrm{sn}\,qz, \quad (2.4)$$

which show that a field perpendicular to the wave vector of the helix leads to deformation of the initial structure, whereby the spins rotate non-uniformly as they move along the z axis, being more frequently oriented along the applied field.[6] The modulus of the elliptic functions depends on the field via the relations

$$\frac{E}{\varkappa}=\left(\frac{h_c}{h}\right)^{1/2},\quad h_c=\frac{\pi^2}{8}\frac{\sigma^2}{\gamma}. \quad (2.5)$$

We find the expansion of the function (2.4) in Fourier series by a method described in Ref. 3. We obain

$$\mathrm{cn}^2 z=1+\frac{E-K}{\varkappa^2K}+\sum_{p=1}^\infty\frac{\pi^2}{\varkappa^2K^2}\frac{p}{\mathrm{sh}(p\pi K'/K)}\cos\left(p\frac{\pi}{K}z\right),$$
$$\mathrm{sn}\,z\,\mathrm{cn}\,z=\sum_{p=1}^\infty\frac{\pi^2}{\varkappa^2K^2}\frac{p}{\mathrm{ch}(p\pi K'/K)}\sin\left(p\frac{\pi}{K}z\right), \quad (2.6)$$

where K' is a complete elliptic integral of the first kind for the modulus $\varkappa'$. We now calculate from (16) the cross section for elastic scattering

$$\frac{d\sigma}{d\Omega}\sim M^2\sum_{\mathbf{b}}J_0^{x2}(\varkappa)(1-e_x^2)\delta(\mathbf{Q}-\mathbf{b})$$
$$+M^2\sum_{+,-}\sum_{\mathbf{b}}\sum_{p=1}^\infty\{J_p^{x2}(\varkappa)(1-e_x^2)+J_p^{y2}(\varkappa)(1-e_y^2)$$
$$\pm J_p^x(\varkappa)J_p^y(\varkappa)2(\mathbf{p}_0\mathbf{e})e_z\}\delta(\mathbf{Q}\pm p\mathbf{k}-\mathbf{b}), \quad (2.7)$$

where k is the wave vector of the structure:

$$\mathbf{k}=(0,0,k),\quad k=\frac{\pi}{K}q=\frac{\pi}{\varkappa K}\left(\frac{h}{2\gamma}\right)^{1/2}, \quad (2.8)$$

$\mathbf{b}$ is an arbitrary vector of the crystal reciprocal lattice. Besides the fundamental, we obtain thus multiple harmonics with intensities given by the following quantities:

In weak fields we have $\varkappa \ll 1$, and known the asymptotic relations (1.6) and (1.7) we get

$$J_0^x(\varkappa)=1+\frac{2(E-K)}{\varkappa^2K}, \quad (2.9)$$

$$J_p^x(\varkappa)=\frac{\pi^2}{\varkappa^2K^2}\frac{p}{\mathrm{sh}(p\pi K'/K)},\quad J_p^y(\varkappa)=\frac{\pi^2}{\varkappa^2K^2}\frac{p}{\mathrm{ch}(p\pi K'/K)}. \quad (2.10)$$

$$J_1^x(\varkappa)=\frac{1}{2}-\frac{3}{2}\left(\frac{\varkappa}{4}\right)^4,\quad J_1^y(\varkappa)=\frac{1}{2}-\frac{5}{2}\left(\frac{\varkappa}{4}\right)^4,$$
$$J_0^x(\varkappa)=-\frac{1}{8}\varkappa^2,\quad J_p^x(\varkappa)=J_p^y(\varkappa)=\frac{p}{2}\left(\frac{\varkappa}{4}\right)^{2p-2}, \quad (2.11)$$

with $\varkappa^2 = h\,2\gamma/\sigma^2$. It is seen from this that in a zero field there is only a pair of magnetic reflections with $Q = b \pm k$ and with intensities $\sim [1 + e_z^2 \mp 2(\mathbf{p}\cdot\mathbf{e})e_z]$ (Ref. 7). In a field, a new reflection $\mathbf{Q} = \mathbf{b}$ with intensity $\sim h^2$ appears, as well as harmonics with intensities $\sim h^{2p-2}$; the intensity of the fundamental reflections with $p = 1$ decreases with increasing field.

In the second limiting case $\varkappa \to 1$ (i.e., $h \to h_c$) the asymptotic form (1.7) shows that the wave vector $\mathbf{k}$ tends logarith-

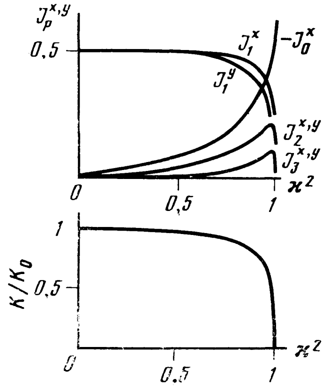

FIG. 1. Dependence of the scattering amplitudes [Eqs. (2.9) and (2.10)] and of the helix wave vector **k** (2.8) on the parameter $\varkappa^2$.

mically to zero and all the reflections approach a central peak whose intensity $J_0^{x2}(\varkappa)\to 1$. It is possible to change in (2.7) from summation to integration, and as a result the scattering cross section is described by

$$\frac{d\sigma}{d\Omega}\sim M^2\sum_{\mathbf{b}}\left(1-\frac{4}{\ln(4/\varkappa')}\right)(1-e_x^2)\,\delta(\mathbf{Q}-\mathbf{b})$$
$$+M^2\sum_{\mathbf{b}}\frac{8}{\pi^2\Delta}\frac{1}{\ln(4/\varkappa')}\left(\frac{Q_z-b_z}{\Delta}\right)^2$$
$$\times\left\{\frac{1-e_x^2}{\mathrm{sh}^2[(Q_z-b_z)/\Delta]}+\frac{1-e_y^2}{\mathrm{ch}^2[(Q_z-b_z)/\Delta]}\right.$$
$$\left.-\frac{2e_z(\mathbf{p}_0\mathbf{e})}{\mathrm{sh}[(Q_z-b_z)/\Delta]\,\mathrm{ch}[(Q_z-b_z)/\Delta]}\right\}\delta(Q_x-b_x)\,\delta(Q_y-b_y),\tag{2.12}$$

where $\Delta=(2/\pi)(h/2\gamma)^{1/2}$. Against a background of a diffuse peak of width $\sim\Delta$ (in the direction of the z axis) there appears an intense central component, into which all the scattering from the diffuse peak is pumped over at $h=h_c$. A phase transition takes place at that instant from the incommensurate into the commensurate (ferromagnetic) structure. The evolution of the scattering picture in the entire field interval $0\leqslant h\leqslant h_c$ is illustrated by Figs. 1 and 2.

The central peak is due to the appearance of magnetization in an external field. Its value is given by

$$M_x=M\frac{1}{V}\int d\mathbf{r}\cos\varphi=MJ_0^{x}(\varkappa),\tag{2.13}$$

whose asymptotic forms have already been established.

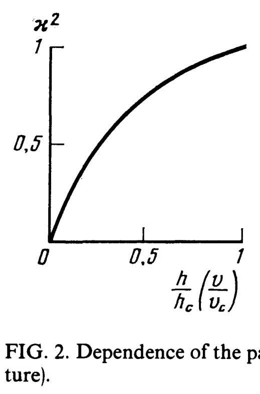

FIG. 2. Dependence of the parameter $\varkappa^2$ on the magnetic field (temperature).

3. SIMPLE HELIX IN AN EXTERNAL FIELD IN THE PRESENCE OF NATURAL FIRST-ORDER ANISOTROPY

We turn now to the simple helix described by the functional (2.1) and take into account the n-th order magnetic anisotropy in the basal plane of the crystal, a plane in which the magnetic field is located. Assuming, as before, that the modulus M of the order parameter does not change in space, we transform to a free-energy functional of the form

$$\Phi=\frac{1}{V}\int d\mathbf{r}\left\{rM^2+uM^4+\gamma M^2\left(\frac{d\varphi}{dz}\right)^2\right.$$
$$\left.+2\sigma M^2\frac{d\varphi}{dz}+MH\cos\varphi+wM^n\cos n\varphi\right\},\tag{3.1}$$

(the field is directed along the $-\varkappa$ axis).

The distribution of the phase is subject to the nonlinear differential equation

$$\frac{d^2\varphi}{dz^2}+\frac{H}{2\gamma M}\sin\varphi+\frac{M^{n-2}w}{2\gamma}n\sin n\varphi=0,\tag{3.2}$$

exact solutions of which can be obtained only for $n=2$ (Refs. 8, 9). We investigate thus uniaxial anisotropy. The case $w>0$ corresponds to a field applied in the hard direction, and $w<0$ to a field applied along the easy axis.

A solution of (3.2) at $n=2$ can be obtained by setting up the first integral of motion and integratng both halves of the equation with respect to φ, after which the variables in the equation separate. It is necessary then to substitute $\cos\varphi=t$ in the resultant equation and introduce a linear-fractional substitution for t. Solutions of this type were in fact obtained in Ref. 8 in connection with a solution of another physical problem, and we shall make use of them.

At $w>0$ there are two solutions corresponding to two incommensurate phases 1s and 2s:

$$1s:$$
$$\cos\varphi=\frac{\mathrm{cn}\,qz-\xi\,\mathrm{dn}\,qz}{\mathrm{dn}\,qz-\xi\,\mathrm{cn}\,qz},\quad q=\left(\frac{1-\xi\alpha}{\xi\alpha}\frac{2w}{\gamma}\right)^{1/2},$$
$$\varkappa^2=\frac{\xi(\xi-\alpha)}{1-\xi\alpha},\quad \alpha\leqslant\xi\leqslant 1,\quad \varkappa^2<\xi^2;\tag{3.3}$$

$$2s:$$
$$\cos\varphi=\frac{\mathrm{cn}\,qz-\xi}{1-\xi\,\mathrm{cn}\,qz},\quad q=\left(\frac{1-\xi^2}{\xi\alpha}\frac{2w}{\gamma}\right)^{1/2},\tag{3.4}$$
$$\varkappa^2=\frac{\xi(\alpha-\xi)}{1-\xi^2},\quad 0\leqslant\xi\leqslant\min\left(\alpha,\frac{1}{\alpha}\right).$$

Here $\alpha=4wM/H$, ξ is a parameter that stems from the integration constant when the integral of motion is set up, and must be obtained by minimizing the energy (in a manner similar to that used to obtain $\varkappa$ in Sec. 2); $\varkappa$ is the modulus of the elliptic functions and is expressed in terms of ξ and α. These intervals of the variation of ξ ensure a change of the modulus $\varkappa$ within the required limits from 0 to 1.

At $w<0$ (i.e., $\alpha<0$) there is only one solution 1s, for which

$$0\leqslant\xi\leqslant 1\quad(\varkappa^2>\xi^2).\tag{3.5}$$

The solutions 1s and 2s describe infiite lattices of solitons of one and two types, respectively.[8]

To find the Fourier components of the functions (3.3) and (3.4) we must find the poles of these functions in the periodicity parallelogram and use the residue theorem.

For the state 1s ($w>0$) we have

$$\cos\varphi = J_0^x(\xi) + \sum_{p=1}^{\infty}\frac{\pi}{K}\left(\frac{1-\xi\alpha}{\xi\alpha}\right)^{1/2}\frac{\mathrm{sh}[(p\pi/2K)(K'-x_0)]}{\mathrm{sh}[p\pi K'/2K]}\cos p\frac{\pi q}{2K}z; \quad (3.6)$$

$\sin\varphi$ is given by an analogous expression, in which there is no free term, the cosine is replaced by a sine, and the denominator contains the hyperbolic cosine rather than the sine. The scattering cross section is determined by the previous expression (2.7) in which, however, J_p^y are functions of ξ:

$$J_0^x(\xi) = \frac{1-\xi^2}{\xi K}\Pi(\xi^2) - \frac{2}{\xi}, \quad (3.7)$$

$$J_p^x(\xi) = \frac{\pi}{2K}\left(\frac{1-\xi\alpha}{\xi\alpha}\right)^{1/2}\frac{\mathrm{sh}[(p\pi/2K)(K'-x_0)]}{\mathrm{sh}[p\pi K'/2K]}, \quad (3.8)$$

and $J_p^y(\xi)$ is obtained from $J_p^x(\xi)$ by replacing the hyperbolic sine in the denominator by the hyperbolic cosine.

In (3.7), $\Pi(\xi^2)$ is a complete elliptic integral of the third kind, and x_0 is determined from the equation

$$\mathrm{dn}(x_0, \varkappa') = \xi, \quad (3.9)$$

which is connected with the procedure of finding the pole of the function (3.3). The wave vector $(0, 0, k)$ of the structure is given by

$$k = q\pi/2K, \quad (3.10)$$

[q is defined in (3.3)]. The expression for the cross section, of the form (2.7), and expression (3.10) for the wave vector remains the same as for the other states.

For the 1s state ($w<0$) we have in place of (3.7)–(3.9)

$$J_0^x(\xi) = \frac{1-\xi^2}{\xi K}\Pi(\xi^2) - \frac{1}{\xi}, \quad (3.11)$$

$$J_p^x(\xi) = \frac{\pi}{2K}\left(\frac{1+\xi|\alpha|}{\xi|\alpha|}\right)^{1/2}\frac{\sin[(p\pi/2K)(K-x_0)]}{\mathrm{sh}[p\pi K'/2K]}, \quad (3.12)$$

$$\mathrm{sn}(x_0, \varkappa) = \xi/\varkappa. \quad (3.13)$$

We present, finally the results for the 2s state:

$$J_0^x(\xi) = \frac{1}{\xi K}\Pi\left(-\frac{\xi^2}{1-\xi^2}\right) - \frac{1}{\xi}, \quad (3.14)$$

$$J_p^x(\xi) = \frac{\pi}{2K}\left(\frac{1-\xi^2}{\xi\alpha}\right)^{1/2}\frac{\mathrm{ch}[(p\pi/2K)(K'-x_0)]}{\mathrm{ch}[p\pi K'/2K]}, \quad p - \text{odd},$$

$$J_p^x(\xi) = \frac{\pi}{2K}\left(\frac{1-\xi^2}{\xi\alpha}\right)^{1/2}\frac{\mathrm{sh}[(p\pi/2K)(K'-x_0)]}{\mathrm{sh}[p\pi K'/2K]}, \quad p - \text{even}, \quad (3.15)$$

$$\mathrm{cn}(x_0, \varkappa') = \xi.$$

The quantity $J_p^y(\xi)$ is obtained from $J_x^z(\xi)$ by interchanging the denominators. The different forms of the reflection intensities of even and odd order are due to the presence of two types of solitons in the 2s state.

To investigate the formulas obtained for the intensities of the reflections as functions of the field and of the temperature, it is necessary to calculate the energy of each state and minimize it with respect to ξ and M. We obtain for the 1s phase:

$$\Phi_{1s} = rM^2 + uM^4 + \frac{HM}{2K\xi}[2(1-\xi^2)\Pi + 2(1-\xi\alpha)E - (3-\xi\alpha-\xi^2)K] - \frac{\pi}{K}\left(\frac{1-\xi\alpha}{2\xi}\frac{\sigma^2}{\gamma}HM^3\right)^{1/2} - wM^2, \quad \Pi \equiv \Pi(\xi^2), \quad (3.16)$$

$$E + \varkappa'^2(\Pi - K) = \pi\left(\frac{\xi}{2(1-\xi\alpha)}\frac{\sigma^2}{\gamma}\frac{M}{H}\right)^{1/2}, \quad (3.17)$$

$$2(r-w)M + 4uM^3 - \frac{H}{K\xi}[(1-\xi^2)\Pi + \xi(\xi-\alpha)K] = 0, \quad (3.18)$$

and for the 2s phase:

$$\Phi_{2s} = rM^2 + uM^4 + \frac{HM}{2K\xi}[2\Pi + 2(1-\xi^2)E - (3-\xi\alpha-\xi^2)K] - \frac{\pi}{K}\left(\frac{1-\xi^2}{2\xi}\frac{\sigma^2}{\gamma}HM^3\right)^{1/2} - wM^2, \quad \Pi \equiv \Pi\left(-\frac{\xi^2}{1-\xi^2}\right), \quad (3.19)$$

$$\Pi + (1-\xi^2)(E-K) = \pi\left(\frac{\xi(1-\xi^2)}{2}\frac{\sigma^2}{\gamma}\frac{M}{H}\right)^{1/2}, \quad (3.20)$$

$$2(r-w)M^2 + 4uM^3 - \frac{H}{K\xi}[\Pi - \xi(\alpha-\xi)K] = 0. \quad (3.21)$$

We note that the expressions for the energies Φ_{1s} and Φ_{2s}, as well as Eqs. (3.17) and (3.20) for the minimzation with respect to ξ, agree fully with the corresponding equations of Ref. 8 (after correcting some misprints). The equations (3.18) and (3.21) for the minimization with respect to M differ from the corresponding equations of Ref. 8, inasmuch an anisotropy of higher order was considered in the latter. It is also easy to verify that all the results of the present section go over into known limiting cases. Thus, the solution (3.4) goes over as $H\to 0$ into $\cos\varphi = \mathrm{cn}\, qz$ (Ref. 10) (there is no solution (3.3) under these conditions). As $w\to 0$ there is no solution (3.4), and (3.3) is Laplace-transformed[11] into $\cos\varphi = 2\,\mathrm{cn}^2(\tilde{q}z, \tilde{\varkappa}) - 1$, i.e., into solution (2.4) with differently defined q and $\varkappa$.

We obtain now the boundary between phases 1s and 2s. It corresponds to $\varkappa = 0$, since reversal of the sign of $\varkappa^2$ continuously transforms one solution into the other. If we take solution 1s, then $\varkappa = 0$ corresponds to $\xi = \alpha$. In this case

$$\Pi(\xi^2) = {}^1/_2\pi(1-\xi^2)^{1/2},$$

and Eqs. (3.17) and (3.18) yield, after eliminating M, the connection between r, w, and H:

$$r = w + \frac{\sigma^2}{\gamma}\left(1 - \frac{2w\gamma}{\sigma^2}\right)^{1/2} - \frac{u\gamma}{4w\sigma^2}H^2. \quad (3.22)$$

The same result is arrived at from the side of the 2s phase.

The basic equation (3.2) has at $n = 2$ two homogeneous solutions that corresponds to two commensurate phases 1 and 2:

$$1)\ \cos\varphi = -1; \quad (3.23)$$

$$2)\ \cos\varphi = -H/4wM = -1/\alpha. \quad (3.24)$$

The energies of these phases are respectively

$$\Phi_1=rM^2+uM^4+wM^2-MH,\quad \Phi_2=-\frac{(r-w)^2}{4u}-\frac{H^2}{8w}, \tag{3.25}$$

from which it follows that phase 2 is energywise expedient only at $w>0$, i. e., when the field is applied along the hard axis. The two commensurate phases obviously border on the surface

$$r=w-uH^2/8w^2. \tag{3.26}$$

Let us determine the boundaries between commensurate and incommensurate phases. This calls for an astute use of the asymptotic forms of analytic integrals of the third kind (see Ref. 11). It appears that authors of Ref. 8 faced the same problem. Unable to track the laborious mathematical manipulations, we present the final results for boundaries of incommensurate phases corresponding to the limit $\varkappa\to 1$. The results agree in spirit with Ref. 8, although the equations differ somewhat in form because of the lower order of the anisotropy in our case ($n=2$).

The surface that separates the phases 2s and 2 is given by the parametric equations

$$\sin\varepsilon+\frac{\pi}{2}\cos\varepsilon\left(1-\frac{2}{\pi}\varepsilon\right)=\frac{\pi}{2}\left(\frac{\sigma^2}{2w\gamma}\right)^{1/2},$$
$$r=w-\frac{u}{8w^2\cos^2\varepsilon}H^2,\quad \cos\varepsilon=\frac{H}{4wM}. \tag{3.27}$$

From a comparison of the last equation in (3.27) with (3.24) it can be seen that ε is in fact the angle φ in the commensurate phase 2 (more accurately, $\varphi=\pi-\varepsilon$, since the field is directed along the $-\varkappa$ axis). The limits of the interval $0\leqslant\varepsilon\leqslant\pi/2$ define two lines that bound this surface:

$$r=w-\frac{u\gamma^2}{8\sigma^4}H^2,\quad w=\frac{\sigma^2}{2\gamma}, \tag{3.28}$$

and

$$H=0,\quad w=\frac{\pi^2}{8}\frac{\sigma^2}{\gamma}. \tag{3.29}$$

The surface that separates phases 1s and 1 consists of two sections. For $w>0$ it is specified by the equations

$$\sin\varepsilon\cos\varepsilon+\frac{\pi}{2}-\varepsilon=\frac{\pi}{2}\cos^2\varepsilon\left(\frac{\sigma^2}{2w\gamma}\right)^{1/2},$$
$$r-2w\,\mathrm{tg}^2\,\varepsilon=w-\frac{u}{8w^2}H^2\cos^2\varepsilon,\quad \cos^2\varepsilon=\frac{4wM}{H}, \tag{3.30}$$

and is bounded by the lines (3.28) and

$$r-\frac{\pi^2}{16}\frac{\sigma^2}{\gamma}=-\frac{128}{\pi^4}\frac{\gamma^2}{\sigma^4}H^2,\quad w=0. \tag{3.31}$$

For $w<0$ the surface is determined by equations of another type (hyperbolic asymptotic form of π):

$$1+\frac{2\varepsilon}{\mathrm{sh}\,2\varepsilon}=\frac{\pi}{2}\,\mathrm{th}\,\varepsilon\left(\frac{\sigma^2}{2|w|\gamma}\right)^{1/2}, \tag{3.32}$$
$$r-2|w|\,\mathrm{cth}^2\,\varepsilon=w-\frac{u}{8w^2}H^2\,\mathrm{sh}^4\,\varepsilon,\quad \mathrm{sh}^2\,\varepsilon=\frac{4|w|M}{H},$$

and is bounded by the lines (3.31) and

$$H=0,\quad w=-\frac{\pi^2}{8}\frac{\sigma^2}{\gamma}. \tag{3.33}$$

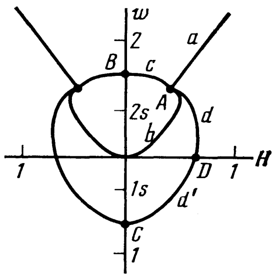

FIG. 3. Phase diagram at constant temperature (r = const < 0). The phase boundaries are defined by the relations: a—(3.26), b—(3.22), c—(3.27), d—(3.30), d'—(3.32). The points A, B, C, and D lie respectively on the lines (3.28), (3.29), (3.31), and (3.33).

A typical planar section r = const of the phase diagram is shown in Fig. 3. On the line that separates the phases 2s and 1s the states go over continuously into one another, and in Ref. 8 are presented arguments that such a boundary is not a phase-transition line at all. As for transitions from incommensurate to commensurate phases, it is known that they are of first order.[12] Our calculations show thus that near the line that separates phases 1s and 1 the quantities M, $\varkappa$, and ξ, which are defined by Eqs. (3.17) and (3.18), are non-single-valued functions of the magnetic field. This leads to the appearance of jumps of these quantities when H is varied (this problem is discussed in the most general form in Ref. 13). The size of the jump decreases rapidly with decreasing temperature.

The relation between the phases is illustrated also by Fig. 4. In the commensurate phase 1 all the magnetic moments are oriented along the field. In phase 1s they rotate around the z axis, but are more frequently in the vicinity of the shaded vector. In phase 2 there are two energywise equivalent orientations of the magnetic moments (two domains). In phase 2s they also rotate arund the z axis, but are more frequently in the vicinity of each of the shaded sectors. There are thus two domain walls (two solitons) with phase shifts φ smaller than and larger than π.

We proceed now to analyze the scattering in the different limiting cases.

In the 2s phase as $H\to 0$ ($\alpha\to\infty$) we have $\xi\to 0$, therefore $x_0\to K'$. In first order in $1/\alpha$ we obtain from (3.14) and (3.15) ($\varkappa^2=\xi\alpha$)

$$J_0^{\varkappa}=-\frac{1}{\alpha}\frac{K-E}{K}$$

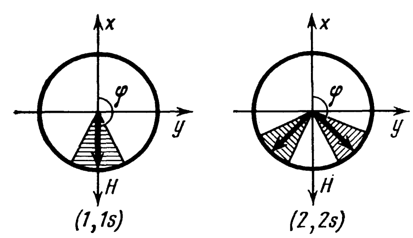

FIG. 4. Structures of commensurate and incommensurate phases with solitons of one and two types.

$$J_p^x=\frac{\pi}{2K\varkappa}1/\mathrm{ch}\left(p\frac{\pi K'}{2K}\right)+O\left(\frac{1}{\alpha^2}\right),\qquad p-\text{even},$$
$$J_p^x=\frac{\pi}{2K\varkappa}\frac{1}{\alpha}\frac{p\pi\varkappa}{2K}\Big/\mathrm{sh}\left(p\frac{\pi K'}{2K}\right),\qquad p-\text{odd};\tag{3.34}$$

J_p^y is obtained by interchanging the hyperbolic sine and cosine. This phase is realized at $w>0$, i.e., when the field is oriented along the hard axis. If the field is oriented along the easy axis ($w<0$), the $1s$ phase is realized at small H, and Eqs. (3.11)–(3.13) yield ($\varkappa^2=\xi\,|\alpha|$):

$$J_0^x=\frac{1}{|\alpha|\varkappa'^2}\left(\frac{K-E}{K}-\varkappa^2\right),$$
$$J_p^x=\frac{\pi}{2K\varkappa}1/\mathrm{sh}\left(p\frac{\pi K'}{2K}\right)+O\left(\frac{1}{\alpha^2}\right),\qquad p-\text{even},\tag{3.35}$$
$$J_p^x=\frac{\pi}{2K\varkappa}\frac{1}{|\alpha|}\frac{p\pi\varkappa}{2K\varkappa'^2}\Big/\mathrm{sh}\left(p\frac{\pi K'}{2K}\right),\qquad p-\text{odd};$$

J_p^y is obtained by replacing the hyperbolic sine with the cosine. In weak fields there are thus two systems of satellites in both cases, and for even orders their intensity is low and $\sim H^2$, just as for the central peak. The satellites of odd order depend on temperature like M^2, and those of even order like M. In the limit $H\to 0$ the central components given by (3.34) and (3.35) vanish for arbitrary $\varkappa$. At the same time on going the limit $\varkappa\to 1$ we should obtain commensurate phases for which the entire scattering intensity goes into just the central peak. The paradox is resolved if the exact equations (3.11) and (3.14) are used for the central components and the limit as $\varkappa\to 1$ is taken first (this corresponds to the thermodynamic limit), and only then letting $H\to 0$. In this case, for example, (3.14) yields $J_0^x(\xi)=1$, which corresponds to one domain of the commensurate phase.

On the boundary of the phases $1s$ and $2s$, the equations (3.14) and (3.15) reduce to the following (it is recognized that on this boundary x_0 satisfies the equation $1/\cosh x_0=\alpha$):

$$J_0^x=\frac{(1-\alpha^2)^{1/2}-1}{\alpha},\qquad J_p^x=J_p^y=\left(\frac{1-\alpha^2}{\alpha^2}\right)^{1/2}J_0^{x\,2p}.\tag{3.36}$$

We note that Eqs. (3.7) and (3.8) for the $1s$ phase lead on this boundary to the same equations. Thus, on this boundary the two systems of satellites (of even and odd order) of the phase $2s$ are transformed into a single system of satellites of the phase $1s$. From this change of the diffractive picture of the scattering we can determine the boundary of two incommensurate phases. On this boundary, the intensities of all the satellites are determined by one angle factor $1+e_z^2\pm 2(\mathbf{p}\cdot\mathbf{e})e_z$, the same as for a pure helix stemming from the paramagnetic phase.

Near the boundaries of the transition to the commensurate phase 1 or 2, the asymptotic values of J_0^x and $J_p^{x,y}$ lead as $\varkappa\to 1$ to a picture of satellites that come closer together, accompanied by an abrupt increase of the central peak, as considered in Sec. 2.

4. ROLE OF n-TH ORDER ANISOTROPY

Although we cannot find the exact solution of the basic equation (3.2) for arbitrary n, we obtain the physical picture by considering the limiting cases of small and large w and H. We rewrite (3.2) in the form

$$\varphi''+v_1\sin\varphi+v_n\sin n\varphi=0,\tag{4.1}$$

where $v_1=H/2\gamma M$, $v_n=nwM^{n-2}/2\gamma$. At small v_1 and v_n we can find asymptotic expansions of the solution by the method of Ref. 14. Accurate to terms quadratic in the field and in the anisotropy, we obtain after excluding the secular terms[14]:

$$\varphi=kz+\frac{v_1}{k^2}\sin kz+\frac{v_n}{(nk)^2}\sin nkz+\frac{v_1^2}{8k^4}\sin 2kz+\frac{nv_n^2}{8(nk)^4}\sin 2nkz$$
$$+\frac{v_1v_n}{2n^2k^4}\left\{\frac{n^3+1}{(n+1)^2}\sin[(n+1)kz]-\frac{n^3-1}{(n-1)^2}\sin[(n-1)kz]\right\}+\dots.\tag{4.2}$$

Here k is an integration constant and should be obtained by minimizing the energy

$$\Phi=rM^2+uM^4+\gamma M^2\left(k^2+\frac{2\sigma}{\gamma}k-\frac{v_1^2}{2k^2}-\frac{v_n^2}{2p^2k^2}\right).\tag{4.3}$$

The equation $\partial\Phi/\partial k=0$ is solved for k by iteration, and yields the following expression for the wave vector (with the same accuracy in v_1 and v_n)

$$k=\frac{|\sigma|}{\gamma}-\frac{v_1^2\gamma^3}{2|\sigma|^3}-\frac{v_n^2\gamma^3}{2n^2|\sigma|^3}.\tag{4.4}$$

It is easy to obtain with the aid of the series (4.1) an expansion for $\cos\varphi$ and $\sin\varphi$ into harmonics and calculate the scattering cross section from Eq. (1.10). The result takes as before the form (2.7), where the amplitudes of the satellites that appear in the first two orders in the field and in the anisotropy are given by

$$J_0^x=-\frac{v_1}{2k^2},\qquad J_1^x=\frac{1}{2}-\frac{3v_1^2}{32k^4}-\frac{v_n^2}{8(nk)^4},\tag{4.5}$$

$$J_1^y=\frac{1}{2}-\frac{5v_1^2}{32k^4}-\frac{v_n^2}{8(nk)^4},\qquad J_2^{x,y}=\frac{v_1}{4k^2},\qquad J_3^{x,y}=\frac{3v_1^2}{32k^4};$$
$$J_n^x=-\frac{v_1v_n(n^2+2)}{4n(n^2-1)k^4},\qquad J_n^y=-\frac{3v_1v_n}{4(n^2-1)k^4},\qquad J_{n\pm1}^{x,y}=\pm\frac{v_n}{4n^2k^2},$$
$$J_{n\pm2}^x=\frac{v_1v_n(n^2+2)}{8n^2k^4(n\pm1)},\qquad J_{n\pm2}^y=\pm\frac{v_1v_n(n^2+2)}{8n^2k^4(n\pm1)},\tag{4.6}$$
$$J_{2n\pm1}^x=\pm\frac{v_n^2(n\pm2)}{32n^4k^4},\qquad J_{2n\pm1}^y=\frac{v_n^2(n\pm2)}{32n^2k^4}.$$

Note that in the absence of anisotropy these expressions coincide with the corresponding asymptotic forms (2.11) of the exact solution of the problem of a helix in a magnetic field. At $n=2$ the expressions obtained coincide with the asymptotic forms (3.34) and (3.35) of the exact solution of the problem of a helix in a field with second-order anisotropy. In the case of arbitrary n for weak fields and weak anisotropy (i.e., in the vicinity of the phase-transition temperature where the helical structure just now sets in) Eqs. (4.5) and (4.6) describe the following diffraction picture.

Around the principle pair of reflections with $p=1$ there are higher-order satellites due to the external field. The distance between the nearest satellites is **k**. The anisotropy gives rise to a pair of satellites about the vectors $n\mathbf{k}$, $2n\mathbf{k}$, etc. About each of these satellites appear weaker ones, due to the superposition of the field and the anisotropy. The distance between the nearest satellites is again **k**. Superposition of two satellites systems with periodicities **k** and $n\mathbf{k}$ thus sets in. In the particular case $n = 2$ the periodicities **k** and 2**k** correspond to different behaviors of the even and odd satellites, as we have seen from the exact solution of the problem.

In the vicinity of the transition to the commensurate phase, the solution of Eq. (4.1) can be represented in the form of a lattice of solitons (see Ref. 1). We obtain a single-soliton solution of (4.1) for low anisotropy using perturbation theory in the parameters $v_n/v_1 \ll 1$. This solution is provided by the asymptotic form $\cos\varphi = -1$ as $z \to \pm\infty$ (then $\cos\varphi = (-1)^n$). We write the first integral of the equation in the form

$$(2v_1)^{1/2} z = \int \frac{d\varphi}{(1+\cos\varphi)^{1/2}} \left[1 + \frac{v_n}{n v_1} \frac{(-1)^{n+1} + \cos n\varphi}{1+\cos\varphi} \right]^{-1/2}. \tag{4.7}$$

Hence, e.g., for the case $n = 4$ we obtain in first order in v_n/v_1 the solution

$$\sin\frac{1}{2}\varphi(z) = \operatorname{th}\Delta z + \frac{2}{5}\frac{v_n}{v_1}\frac{(4\operatorname{th}^2\Delta z + 5)\operatorname{th}\Delta z}{\operatorname{ch}^2\Delta z}, \tag{4.8}$$

where $\Delta = v_1^{1/2}(1 - 2v_1/v_n)$ determines the reciprocal width of the soliton localized at the point $z_0 = 0$. For a soliton lattice with period L we can write:

$$\sin\frac{1}{2}\varphi_L(z) = \sum_{l=-\infty}^{\infty} \sin\frac{1}{2}\varphi(z - lL). \tag{4.9}$$

With aid of (1.9) and (1.10) we arrive at the following expression for the cross section near T_c

$$\begin{aligned} \frac{d\sigma}{d\Omega} \sim M^2 \sum_{\mathbf{b}} (1-e_x^2)\,\delta(\mathbf{Q}-\mathbf{b}) + M^2 \\ \times \sum_{\mathbf{b}} \frac{32}{\pi^3 L \Delta^2} \left\{ (1-e_x^2) F_x^2\left(\frac{Q_z-b_z}{\Delta}\right) \right. \\ \left. + (1-e_y^2) F_y^2\left(\frac{Q_z-b_z}{\Delta}\right) - 2e_z(\mathbf{p}_0\mathbf{e}) F_x\left(\frac{Q_z-b_z}{\Delta}\right) F_y\left(\frac{Q_z-b_z}{\Delta}\right) \right\} \\ \times \delta(Q_x-b_x)\,\delta(Q_y-b_y), \end{aligned} \tag{4.10}$$

where $F_x(t)$ and $F_y(t)$ are the soliton form factors:

$$\begin{aligned} F_x(t) &= \frac{t}{\operatorname{sh} t}\left[1 - \frac{v_n}{v_1} P_x\left(\frac{4t^2}{\pi^2}\right)\right], \\ F_y(t) &= \frac{t}{\operatorname{ch} t}\left[1 + \frac{v_n}{v_1} P_y\left(\frac{4t^2}{\pi^2}\right)\right], \end{aligned} \tag{4.11}$$

and P_x and P_y are cubic polynomials:

$$\begin{aligned} P_x(u) &= \frac{36}{5} + \frac{34}{15}u + \frac{3}{50}u^2 + \frac{32}{5\cdot 7!}u^3, \\ P_y(u) &= \frac{4\cdot 226}{5\cdot 6!} - \left(1 + \frac{16}{6!}\right)u - \frac{36}{5\cdot 5!}u^2 + \frac{16}{5\cdot 5!}u^3. \end{aligned}$$

In the absence of anisotropy Eq. (4.10) goes over into (2.12) if account is taken of the expression for $L = (4/\pi\Delta)\ln(4/\varkappa')$. Fourth-order anisotropy leads to a larger scattering asymmetry in the x and y directions and to a larger width of the diffuse scattering near the central peak. The results (4.10) and (4.11) pertain to the case $n = 4$. They remain valid also for arbitrary n; all that changes is the form of the polynomials P_x and P_y of degree $n - 1$.

5. "ANTIFERROMAGNETIC" HELIX IN AN EXTERNAL FIELD

So far we have considered helical structures with near-zero wave vectors (the corresponding commensurate phase was ferromagnetic). In this section we consider a two-sublattice antiferromagnetic structure with the spins located in the (x, y) plane and with helical modulation in the direction of the z axis. The action of the magnetic field on a helical antiferromagnetic structure was first investigated by Dzyaloshinskiĭ[2] on the basis of an invariant of the form $(\mathbf{l}\cdot\mathbf{H})^2$, where $\mathbf{l}$ is the antiferromagnetism vector. It was shown that the problem reduces to Eq. (1.2) with second-order effective anisotropy. In contrast to Ref. 2, where high temperatures (the vicinity of the phase transition) were considered, we investigate another case, that of low temperatures, when account must be taken of the constancy of the modulus of the magnetic moment at each point. It is convenient to specify the magnetic-moment density by the relation

$$\mathbf{M}(\mathbf{r}) = M(\mathbf{m} + \mathbf{l}e^{i\mathbf{q}_A\mathbf{r}}), \tag{5.1}$$

where q_A is the antiferromagnetism wave vector and is equal to half of some reciprocal-lattice vector of the crystal. The vectors **m** and **l** satisfy the conditions

$$\mathbf{m}^2 + \mathbf{l}^2 = 1, \quad (\mathbf{ml}) = 0, \tag{5.2}$$

which ensure constancy of the modulus of the magnetic moment at each point.

The energy of such a structure is described by the functional

$$\begin{aligned} \Phi/M^2 = \frac{1}{V}\int d\mathbf{r} \left\{ r\mathbf{l}^2 + \gamma(\nabla\mathbf{l})^2 + \beta l_z^2 + \beta' m_z^2 \right. \\ \left. + 2\sigma\left(l_x\frac{\partial l_y}{\partial z} - l_y\frac{\partial l_x}{\partial z}\right) + m_x h \right\}, \end{aligned} \tag{5.3}$$

where $r < 0$ in a magnetically ordered phase.

At positive and large anisotropy constants β and β' the spins lie in the basal plane; this plane contains also a field directed along the $-x$ axis ($h = H/M$). Under these conditions we shall specify the vectors **l** and **m** by the relations

$$m_x = m\cos\varphi, \quad m_y = m\sin\varphi; \quad l_x = l\sin\varphi, \quad l_y = -l\cos\varphi, \tag{5.4}$$

with $m_z = l_z = 0$. If, as in the preceding section, we assume the moduli m and l to be constant, the problem of a phase transition in a field reduces to an analysis of the functional

$$\begin{aligned} \Phi/M^2 = \frac{1}{V}\int d\mathbf{r} \left\{ r l^2 + \gamma l^2 \left(\frac{d\varphi}{dz}\right)^2 \right. \\ \left. + 2\sigma l^2 \frac{d\varphi}{dz} + mh\cos\varphi \right\} - \lambda(m^2 + l^2 - 1). \end{aligned} \tag{5.5}$$

Variation with respect to the phase φ leads to Eq. (1.2), where

$$n = 1, \quad v = mh/2\gamma l^2. \tag{5.6}$$

Variation with respect to l and m yields two addtional equations

$$l\left\{r+\frac{1}{V}\int dr\left[\gamma\left(\frac{d\varphi}{dz}\right)^2+2\sigma\frac{d\varphi}{dz}\right]-\lambda\right\}=0, \tag{5.7}$$

$$h\frac{1}{V}\int d\mathbf{r}\cos\varphi-2m\lambda=0. \tag{5.8}$$

There are four solutions of these equations:

1) $\varphi=k_0z,\ l=1,\ m=0\ \ (h<h_2)$;

2) $\cos\varphi=-1,\ l=[1-(h/2r)^2]^{1/2},\ m=-h/2r\ \ (h_1<h<h_3)$;

3) $\cos\varphi=-1,\ l=0,\ m=-1\ \ (h>h_3)$;

4) $\varphi=2\,\mathrm{am}\,(v^{1/2}z/\varkappa,\varkappa)$.

The parentheses contain the stability limits of the solutions:

$$h_1=\left(\frac{16\sigma^2}{\pi^2\gamma|r|}+1\right)^{-1/2},\quad h_2=\left[2\frac{\sigma^2}{\gamma}\left(\frac{\sigma^2}{\gamma}-r\right)\right]^{1/2},\quad h_3=2|r|. \tag{5.9}$$

The fourth solution has no stability region. For the remaining solutions the stability regions can overlap, and this leads to first-order phase transitions. Comparison of the free energies of the phases shows that at $T>T^*$ (T^* is determined from the condition $|r|=\sigma^2/\gamma$), with increasing field, a first-order transition takes place from a simple helix (solution 1) to a ferromagnetic structure (solution 3). The transition takes place in a field $h=|r|+\sigma^2/\gamma$. At $T<T^*$, with increasing field, a first-order transition takes place from a simple helix to a spin-flop phase (solution 2) at $h=2(|r|\sigma^2/\gamma)^{1/2}$, and then a second-order transition at $h=h_3$ into a ferromagnetic state.

If there is no field, it can be easily seen on the basis of the first section of the article that the natural anisotropy leads to a distortion of the helical structure in accord with expression (1.3)–(1.5). Thus, the neutron-diffraction pattern will reveal the satellite system described in Sec. 2, clustered around the "antiferromagnetic" site $\mathbf{b}+\mathbf{q}_A$ of the reciprocal lattice.

6. CONCLUSION

The analyzed physical situations are perfectly sufficient to set up a complete picture of neutron diffraction by incommensurate structures and its evolution with change of field and temperature. This will permit an analysis of numerous experimental data when neutron-diffraction investigations of magnetic structures reveal higher-order satellites and their variation on approaching the point of transition to a commensurate phase (see, e.g., Refs. 1 and 15). Direct application of our results to a quantitative comparison with experiment is not quite feasible, since we have restricted ourselves in the first three sections of the article to simplest cases, when the crystal has no inversion center, although in a real situation this is most frequently not the case. Moreover, incommensurate magnetic structures are usually described by four-component OP (see Ref. 4), whereas here we confine ourselves to a two-component OP. For a detailed comparison of the theory with experiment the calculation must be for a concrete magnetic structure in accordance with the schemes developed here. Such an analysis is most timely, since it will permit a verification of the phase-transition soliton picture itself and a check on the validity of one of the principal approximations of the theory—the constancy of the OP modulus (see Ref. 12). On the other hand, a correct understanding of the nature of satellite reflections is quite essential for neutron-diffraction interpretation of noncommensurate structures. A detailed analysis of the experiment will be presented elsewhere.

[1] P. Bak, Rept. Progr. Phys. **45**, 587 (1982).
[2] I. E. Dzyaloshinskiĭ, Zh. Eksp. Teor. Fiz. **47**, 992 (1984) [Sov. Phys. JETP **20**, 665 (1965)].
[3] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, 4th ed., Cambridge U. P., 1927.
[4] Yu. A. Izyumov, V. E. Naish, and R. P. Ozerov, Neitronografiya magnetikov (Neutron Diffraction Analysis of Magnets), Atomizdat, 1981.
[5] L. D. Landau and E. M. Lifshitz, Elektrodinamika sploshnykh sred. (Electrodynamics of Continuous Media), Nauka, 1982 [Pergamon].
[6] T. K. Soboleva, E. P. Stefanovskii, and V. V. Tarasanko, Fiz. Tverd. Tela (Leningrad) **22**, 2353 (1980) [Sov. Phys. Solid State **22**, 1313 (1980)].
[7] Yu. A. Izyumov, Zh. Eksp. Teor. Fiz. **42**, 1673 (1962) [Sov. Phys. JETP **15**, 1162 (1962)].
[8] V. A. Golovko and D. G. Sannikov, Zh. Eksp. Teor. Fiz. **82**, 959 (1982) [Sov. Phys. JETP **55**, 562 (1982)].
[9] C. A. Condat, R. A. Guyer, and M. D. Miller, Phys. Rev. **B27**, 474 (1983).
[10] T. A. Soboleva and E. P. Stefanovskiĭ, Fiz. Met. Metallov, **54**, 186 (1982).
[11] M. Abramovitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, 1964.
[12] D. G. Sannikov, Fiz. Tverd. Tela (Leningrad) **23**, 953 (1981) [Sov. Phys. Solid State **23**, 553 (1981)].
[13] T. A. Poston and I. Stuart, Catastrophe Theory (Russ. transl.), Mir, 1980.
[14] N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon & Breach, 1964.
[15] W. C. Koehler, In: Magnetic Properties of Rare Earth Metals [Ed. by R. J. Elliot], Plenum Press, 1972.

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