Oscillations of a ferromagnetic liquid

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The oscillations of a liquid ferromagnet are considered. The spectra of small-amplitude oscillations are determined. The simplest nonlinear waves (Riemann waves) are analyzed, as well as stationary discontinuities of two types: a) Alfvén waves and the corresponding rotational discontinuities; these motions are similar to the corresponding motions in an ordinary magnetohydrodynamic medium. b) Extraordinary waves, and the plane-polarized discontinuities corresponding to them. In the extraordinary wave the points with the larger magnetic induction **B** move with a greater velocity. Discontinuities in the Riemann wave arise in sections of increasing **B**. The discontinuity velocity exceeds that of the extraordinary wave in an unperturbed liquid and is less than that of the extraordinary wave behind the discontinuity.

1. EQUATIONS OF MOTION OF A LIQUID FERROMAGNET

At the present time, there is no doubt of the possibility of the existence of liquids that are ferromagnetic at the atomic level. The question of their experimental observation has been covered in detail in Ref. 1; the prospects of the use of liquid ferromagnets as ideally soft magnetic materials was also discussed there. A number of researches [2–4] are known that are devoted to the unperturbed state of a ferromagnetic liquid; the problem of the oscillations of such a liquid has not been considered to date. At the same time, the oscillations of a liquid ferromagnet should be very unusual, differing both from the spin waves in a solid ferromagnet and from the magnetohydrodynamic waves in a conducting liquid in the presence of an external magnetic induction. The present work is devoted to the study of such oscillations. We shall consider oscillations of small amplitude, the simplest nonlinear oscillations—Riemann waves and their evolution, and also ferrohydrodynamic discontinuities.

We emphasize that we deal with a liquid which is a ferromagnet at the microscopic level. Therefore, we use for the magnetization Eq. (1.5), which leads to a significantly anisotropic (depending on the frequency and the wave vector) relation between the magnetic induction **B** and the magnetic field intensity **H**, and also to the possibility of the existence of spin waves. Thus, the considered problem differs from the widely treated problem of oscillations of the so-called magnetized liquid, in which the phenomenological isotropic connection $\mathbf{B} = \mu\mathbf{H}$ has been introduced, where μ is a function of the magnetic field, temperature and density. Without citing the broad literature on a magnetized liquid, we shall indicate only three of the (chronologically) latest researches.[5–7]

We first write out the equations of motion of a magnetohydrodynamic medium with an arbitrary relation between **B** and **H**. Following Ref. 8, we have

$$\operatorname{div}\mathbf{B}=0,\qquad \frac{\partial\mathbf{B}}{\partial t}=\operatorname{rot}[\mathbf{v}\times\mathbf{B}],\tag{1.1}$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0,\tag{1.2}$$

$$\rho\frac{d\mathbf{v}}{dt}=-s^2\nabla\rho+\frac{1}{4\pi}[\operatorname{rot}\mathbf{H}\times\mathbf{B}]+[\mathbf{M}\times\operatorname{rot}\mathbf{H}]+(\mathbf{M}\nabla)\mathbf{H},\tag{1.3}$$

where ρ is the density, **v** is the hydrodynamic velocity, s is the sound velocity and $d/dt\equiv\partial/\partial t+\mathbf{v}\partial/\partial\mathbf{r}$. If the hydrodynamic medium possesses spontaneous magnetization $\mathbf{M}_0$, then we can add the equation of motion of the magnetic moment to these equations:

$$\frac{d}{dt}\left(\frac{\mathbf{M}}{\rho}\right)=g\left[\frac{\mathbf{M}}{\rho}\times\mathbf{H}^{eff}\right]\tag{1.4}$$

where $\mathbf{M}=(\mathbf{B}-\mathbf{H})/4\pi$ is the magnetization, g is the gyromagnetic ratio and $\mathbf{H}^{eff}$ is the effective field. Assuming the liquid to be isotropic, we get for $\mathbf{H}^{eff}$

$$\mathbf{H}^{eff}=\mathbf{H}-\alpha\Delta\mathbf{M},\tag{1.5}$$

where α is the constant of exchange interaction.

It is known that the unperturbed hydrodynamic medium is characterized by two quantities with dimension of velocity: the sound velocity s and the Alfvén velocity

$$v_A=(BH/4\pi\rho)^{1/2}.\tag{1.6}$$

Even in very strong fields ($H\sim10^3$kOe) at $\rho\sim10$ g·cm,$^{-3}$ we have $v_A\sim10^4$cm/sec, while $s\sim3\times10^5$ cm/sec. Therefore, the sound (more precisely, the fast magnetosonic) wave does not actually interact with the Alfvén wave (see Ref. 9). Therefore, we shall not linger on this subject. In the study of the remaining waves, we can set $s\to\infty$. The equation

$$s^2\mathbf{n}\operatorname{grad}\rho=\frac{1}{4\pi}\mathbf{n}[\operatorname{rot}\mathbf{H}\times\mathbf{B}]+\mathbf{n}(\mathbf{M}(\mathbf{n}\nabla)\mathbf{H})\tag{1.7}$$

(**n** is a unit vector along the direction of the propagation of the wave) is then separated, while Eqs. (1.2)—(1.4) reduce to the form

$$\rho=\text{const},\quad \operatorname{div}\mathbf{v}=0.\tag{1.8}$$

$$\rho\frac{d\mathbf{v}_\perp}{dt}=\frac{1}{4\pi}[\operatorname{rot}\mathbf{H}\times\mathbf{B}]_\perp,\tag{1.9}$$

$$\frac{d\mathbf{M}}{dt}=g\ \mathbf{M}\times\mathbf{H}-\alpha\Delta\mathbf{M}],\tag{1.10}$$

where the index $\perp$ denotes the components of the vectors

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perpendicular to the **n** direction. The relations (1.1) and (1.8)–(1.10) form a complete set of equations for the description of the oscillation of a ferromagnetic liquid in the range of frequencies and wavelengths that are remote from the region of sound (and the associated magnetosonic) waves. We shall call such oscillations ferrohydrodynamic.

2. FERROHYDRODYNAMIC WAVES OF SMALL AMPLITUDE

In an unperturbed ferromagnetic liquid,

$$\mathbf{B}=\mathbf{B}_0=\mathbf{H}_0+4\pi\mathbf{M}_0, \quad \mathbf{M}_0\|\mathbf{H}_0,$$

where $\mathbf{H}_0$ is the external magnetic field. We shall use below a system of coordinates in which the unperturbed value of the hydrodynamic velocity $\mathbf{v}_0 = 0$.

For the determination of the spectra of the ferrohydrodynamic oscillations, we should linearize Eqs. (1.1), (1.8)–(1.10) and seek small perturbations of the quantities in the form $\exp\{i\mathbf{k}\cdot\mathbf{r} - i\omega t\}$. Without giving the corresponding linearized equations, we immediately write down the dispersion equation

$$\omega^6-\omega^4(\Omega+\omega_g)^2+\omega^2 v_B^2 k^2(\Omega+\omega_g)[\sin^2\theta(\Omega+\omega_g)+(1+\cos^2\theta)\Omega]-v_B^4 k^4\Omega(\Omega+\omega_g\sin^2\theta)=0, \quad (2.1)$$

where

$$v_B^2=B_0^2\cos^2\theta/4\pi\rho, \quad \omega_g=4\pi g M_0, \quad \Omega=g(H_0+M_0\alpha k^2) \quad (2.2)$$

(θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{M}_0$).

Equation (2.1) defines three branches of oscillations. In the absence of spontaneous magnetization ($\omega_0 = 0$), these are the spin wave, the Aflvén wave, and the slow magnetosonic wave.

In the longwave limit ($k\to 0$), in accord with (2.1), the following three types of oscillations are possible:

a) uniform precession of the magnetization with velocity

$$\omega=gB_0; \quad (2.3)$$

b) an Alfvén wave,

$$\omega=kv_A\cos\theta, \quad (2.4)$$

where v_A is determined by Eq. (1.5), and

c) an estraordinary wave,

$$\omega=kv_N\cos\theta, \quad v_N^2=\frac{B_0^2}{4\pi\rho}\left(1-\frac{4\pi M_0}{B_0}\cos^2\theta\right). \quad (2.5)$$

We note that in the ground state $\mathbf{H}_0 = 0$ and $\mathbf{k}\|\mathbf{M}_0$, the Alfvén and the extraordinary waves vanish. In this case, in place of these waves, two waves appear with quadratic dispersion laws:

$$\omega=Ak^2, \quad A=\frac{v_B^2}{2\omega_g^2}\left\{1\pm\left[1-\left(\frac{2gM_0\alpha\omega_g}{v_B^2}\right)^2\right]^{1/2}\right\}. \quad (2.6)$$

If at $H_0 = 0$ the vectors $\mathbf{k}$ and $\mathbf{M}_0$ are not parallel, then the extraordinary wave is characterized as before by the phase velocity $\mathbf{v}_N$, while in place of the Alfvén wave, a wave appears with the quadratic dispersion law

$$\omega=A_1k^2, \quad A_1=M_0\left(\frac{\alpha}{\rho}\right)^{1/2}. \quad (2.7)$$

3. SIMPLE (RIEMANN) WAVES

We now consider the simplest nonlinear ferrohydrodynamic waves—Riemann waves, in which all the quantities characterizing the medium are functions only of one of them, and this quantity in turn depends on the time t and one of the coordinates x. Riemann waves are of interest for two reasons: first, they allow us to follow the evolution of the form of the initial perturbation and, second, only such waves can abut (in the absence of discontinuities) to the unperturbed medium. For each of the variable quantities X in the Riemann wave, we have the following relation:

$$\dot{X}=-VX', \quad (3.1)$$

where V is a variable depending on the quantities characterizing the liquid (the dot indicates differentiation with respect to time, while the prime indicates differentiation with respect to a spatial coordinate).

According to (1.1) and (1.8), we have

$$B_x=\text{const}, \quad v_x=\text{const}. \quad (3.2)$$

Transforming to the system of coordinates in which $v_x = 0$, and using (1.1), (1.9), we obtain

$$\dot{\mathbf{v}}_\perp=-V\dot{\mathbf{B}}_\perp/B_x, \quad V^2\dot{\mathbf{B}}_\perp=(B_x^2/4\pi\rho)\dot{\mathbf{H}}_\perp. \quad (3.3)$$

So far as Eq. (1.10) is concerned, in the low-frequency case of interest to us ($\omega \ll gM_0$), it is equivalent to the following connection between $\dot{\mathbf{B}}$ and $\dot{\mathbf{H}}$:

$$\dot{\mathbf{B}}=\frac{B}{H}\dot{\mathbf{H}}-\frac{4\pi\mathbf{M}}{MH}(\mathbf{M}\dot{\mathbf{H}}). \quad (3.4)$$

The velocity V as a function of the quantities characterizing the medium is determined from the condition of the compatibility of the resultant equations. It turns out here that Riemann waves of two types are possible in a ferromagnetic liquid.

a) *Alfven waves*, in which M_x, H_x, B_x, do not change, while the vectors $\mathbf{M}_\perp$, $\mathbf{H}_\perp$, $\mathbf{B}_\perp$ return to the yz plane, remaining mutually parallel and not changing in value. Thus, the Alfvén waves in the ferromagnetic liquid are actually in no way different from the Alfvén waves in an ordinary magnetohydrodynamic medium. In particular, the velocity of these waves

$$V=(H_xB_x/4\pi\rho)^{1/2} \quad (3.5)$$

turns out to be constant. Therefore, Alfvén waves of arbitrary amplitude propagate without change in shape.

b) *Extraordinary waves.* They are plane polarized in a plane passing through the direction of the sound propagation, and

$$\dot{B}_y=\frac{\dot{H}_y}{H}\left\{(B-4\pi M\sin^2\theta)-\frac{(4\pi M\sin\theta\cos\theta)^2}{B-4\pi M\cos^2\theta}\right\}, \quad (3.6)$$

$$B_z=H_z=0, \quad (3.7)$$

where θ is the angle between the vectors $\mathbf{M}$, $\mathbf{H}$, $\mathbf{B}$ and the x axis (the z axis is directed along the normal to the plane of polarization of the wave). We see that in the extraordinary wave, both the angle θ and the length of the induction vector $\mathbf{B}$ change. The velocity of this wave is determined by the formula

$$V^2=\frac{B_x^2}{4\pi\rho}\left(1-\frac{4\pi M}{B}\cos^2\theta\right). \tag{3.8}$$

Using (3.6) and (3.8), we obtain

$$V\frac{dV}{dB}=\frac{3M}{2\rho}\cos^4\theta. \tag{3.9}$$

Thus, in the propagation of the wave, points with larger values of B move with a greater velocity so that discontinuities appear on sections where in the magnetic induction increases.

4. FERROHYDRODYNAMIC DISCONTINUITIES

Equations (1.1), (1.8) and (1.9) (together with the relations $M^2=\text{const}$, $\mathbf{M}\|\mathbf{H}$) allow us to analyze the stationary one-dimensional ferrohydrodynamic discontinuities. Integrating these equations over the infinitely small range x surrounding the discontinuity, we find the relations that connect the jumps in the variable quantities at the discontinuity:

$$\{M^2\}=0,\qquad \{B_x\}=0,\qquad \{v_x\}=0, \tag{4.1}$$

$$V\{\mathbf{B}_\perp\}+B_x\{\mathbf{v}_\perp\}=0,\qquad V^2\{\mathbf{B}_\perp\}=(B_x^2/4\pi\rho)\{\mathbf{H}_\perp\}, \tag{4.2}$$

where V is the velocity of the discontinuity (in the direction of the x axis) relative to the liquid. Taking it into account that $\mathbf{M}\|\mathbf{H}$ on both sides of the discontinuity, we can show that two types of discontinuities are possible.

a) *Rotational discontinuities*, in which

$$\{M_\perp\}=\{B_\perp\}=\{H_\perp\}=0, \tag{4.3}$$

while the vectors $\mathbf{M}_\perp$, $\mathbf{B}_\perp$, $\mathbf{H}_\perp$ are rotated, remaining parallel to one another. The velocity of this discontinuity is identical with the velocity of the Alfvén wave (3.5).

b) *Plane polarized discontinuity* (the magnetization wave) at which the magnetic vectors vary in the xy plane (the coordinate axes are chosen in the same manner as in Sec. 3), whence

$$\begin{aligned} B_{1y}&=B_x\operatorname{tg}\theta_1, & B_{2y}&=B_x\operatorname{tg}\theta_2,\\ M_{1y}&=M\sin\theta_1, & M_{2y}&=M\sin\theta_2,\\ M_{1x}&=M\cos\theta_1, & M_{2x}&=M\cos\theta_2, \end{aligned} \tag{4.4}$$

where θ_1, θ_2 are the angles between the magnetic vectors and the direction of propagation of the discontinuity in front of and behind the discontinuity, respectively. The velocity of such a discontinuity is determined by the relation

$$V^2=\frac{B_x^2}{4\pi\rho}\left\{1-\frac{4\pi M}{B}\cos^2\theta_1 f(\theta_1,\theta_2)\right\}, \tag{4.5}$$

$$f(\theta_1,\theta_2)=\frac{\sin\theta_2-\sin\theta_1}{\cos^3\theta_1(\operatorname{tg}\theta_2-\operatorname{tg}\theta_1)}.$$

The general analysis of Eq. (4.5) is a complicated one. We therefore limit ourselves to the case of small but finite amplitude. Here

$$f(\theta_1,\theta_2)=1-{}^3/_2\operatorname{tg}\theta_1\cdot\delta\theta,$$

where $\delta\theta=\theta_2-\theta_1$. In the zeroth approximation in $\delta\theta$ the velocity V is identical with the velocity of the extraordinary wave, which is determined by Eq. (3.8). In the first approximation, $\partial V/\partial\delta\theta>0$. Since $\partial B/\partial\theta>0$ and the discontinuity sets in accord with Sec. 3 on segments of increase in B, we have

$$V_2>V>V_1,$$

where V_1, V_2 are the velocities of the extraordinary wave in front of and behind the discontinuity.

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