The nonlinear interaction of surface second sound with capillary waves in a superfluid

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The problem of nonlinear combination (Rayleigh) interaction of surface second sound with capillary waves in a weakly superfluid solution of ^3He in ^4He. The equations which describe three-wave interactions are derived. The anharmonic terms of third order are calculated, terms which are responsible for the scattering of surface second sound on a capillary wave, as well as the parametric decay of surface sound into two capillary waves. An estimate is obtained for possible experimental observation of nonlinear interactions of surface waves.

In addition to the capillary waves which are fairly well studied and are proper also to ordinary fluids, in superfluid helium there can exist another type of surface excitations—the so-called surface second sound, which was theoretically predicted in a paper by Andreev and Kompaneets.[1] Surface second sound has been experimentally observed in a weak solution of ^3He in ^4He.[2,3] For sufficiently large intensities of the surface excitations nonlinear interactions between the capillary waves and surface sound may manifest themselves. An experimental investigation of such interactions would be of interest, since it would yield information on the surface thermodynamic functions which determine the nonlinear interactions. Moreover, taking account of the nonlinear interactions would provide us with a more precise picture of what is observed experimentally for sufficiently large wave intensities.

In the present paper we consider the nonlinear combination (Rayleigh) interaction of surface second sound with capillary waves in a weakly superfluid solution of ^3He in ^4He. Approximate equations are derived for the description of three-wave interactions. The effective vertex functions (anharmonicities of the third order) are calculated, vertex functions which determine the scattering of surface second sound on a capillary wave and the decay of surface second sound into two capillary waves. It is shown on the basis of the derived equations that such interactions could be observed experimentally.

As was noted in Ref. 1, the problem of motion of the free surface of a quantum fluid differs substantially from the similar problem for an ordinary liquid. This is related to the fact that in a superfluid such quantities as mass, entropy, etc., can be transported by surface excitations from one place to another, and this radically changes the boundary conditions at the surface compared to an ordinary liquid. In solving the problem we shall start from the equations of surface hydrodynamics of a superfluid, derived in Ref. 1. We shall also assume that for sufficiently low temperatures ($T \leqslant 0.1$ K) one may neglect the normal volume component of the density, and assume that there exists only a normal surface density. For such temperatures practically all of the ^3He is adsorbed at the surface, and its volume density equals zero. In this case the ^3He density at the surface, ν, stops depending on the temperature, a fact which is confirmed experimentally (Refs. 3, 4). We also adopt the condition that the total surface mass vanishes, and therefore at the surface $\nu_n = -\nu_s$ where ν_n and ν_s are respectively, the normal and superfluid surface densities.

As independent variables we choose the temperature T of the surface, the chemical potential μ of the admixture, and the tangential components $\mathbf{v}_{st}$ and $\mathbf{v}_{nt}$ of the superfluid and normal fluid velocity at the surface. Writing the equations (15) and (16) from Ref. 1 accurate to the squares of the deviations of the quantities from their equilibrium values, we obtain the following set of nonlinear boundary conditions:

$$\begin{aligned}
&\nu_n \frac{\partial}{\partial t}(v_{n\beta}-v_{s\beta})+\sigma\frac{\partial T}{\partial x_\beta}+\nu\frac{\partial \mu}{\partial x_\beta}=A_1,\\
&\nu_n \frac{\partial}{\partial x_\alpha}(v_{n\alpha}-v_{s\alpha})-\rho\frac{\partial \varphi}{\partial z}+\rho\frac{\partial \xi}{\partial t}=A_2,\\
&\frac{\partial \sigma}{\partial T}\frac{\partial T}{\partial t}+\sigma\frac{\partial v_{n\alpha}}{\partial x_\alpha}=A_3,\\
&\frac{\partial \nu}{\partial \mu}\frac{\partial \mu}{\partial t}+\nu\frac{\partial v_{n\alpha}}{\partial x_\alpha}=A_4,\\
&p+\alpha\frac{\partial^2 \xi}{\partial x_\alpha^2}=A_5.
\end{aligned} \tag{1}$$

In the equations (1) all nonlinear terms have been written in the right-hand side. The quantities A_1–A_5 will be defined below. The surface equations (1) must be complemented with the volume equations of the hydrodynamics of incompressible fluids:

$$p+\rho\frac{\partial \varphi}{\partial t}+\frac{\rho}{2}\left(\frac{\partial \varphi}{\partial x_i}\right)^2=0. \tag{2}$$

The following notations have been used in the listed equations: $\xi(x, y, t)$ is the deviation of the free surface of liquid helium from its equilibrium position $z = 0$; σ is the surface entropy density (entropy per unit area of the projection of the surface onto the xy plane); ρ is the volume density of the fluid; φ is the potential of the superfluid velocity (phase of the wave function): $\mathbf{v}_s = \nabla\varphi$; p is the pressure of the fluid at the surface (we set the pressure above the surface equal to zero); α is the coefficient of surface tension:

$$\alpha=\alpha(T,\mu)-{}^1/_2\nu_n(\mathbf{v}_{nt}-\mathbf{v}_{st})^2;$$

the subscripts α and β run over x and y; we do not take into

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account dissipation. The nonlinear terms have the form

$$A_1=-\frac{\partial\nu_n}{\partial T}\frac{\partial}{\partial t}[\delta T(v_{n\beta}-v_{s\beta})]-\frac{\partial\nu_n}{\partial\mu}\frac{\partial}{\partial t}[\delta\mu(v_{n\beta}-v_{s\beta})]$$

$$-\nu_n\frac{\partial}{\partial t}\left[\frac{\partial\xi}{\partial x_\beta}(v_{nz}-v_{sz})\right]+\frac{1}{2}\frac{\partial^2\alpha}{\partial T^2}\frac{\partial}{\partial x_\beta}(\delta T)^2$$

$$+\frac{1}{2}\frac{\partial^2\alpha}{\partial\mu^2}\frac{\partial}{\partial x_\beta}(\delta\mu)^2$$

$$-\nu_n\frac{\partial}{\partial x_\alpha}(v_{n\alpha}v_{n\beta}-v_{s\alpha}v_{s\beta})-\frac{1}{2}\nu_n\frac{\partial}{\partial x_\beta}(v_{n\alpha}-v_{s\alpha})^2$$

$$+\frac{1}{2}\alpha\frac{\partial}{\partial x_\beta}\left(\frac{\partial\xi}{\partial x_\alpha}\right)^2+\rho v_{s\beta}v_{sz}+\alpha\frac{\partial^2\xi}{\partial x_\alpha^2}\frac{\partial\xi}{\partial x_\beta}-\rho v_{s\beta}\frac{\partial\xi}{\partial t};$$

$$v_{nz}=\frac{\partial\xi}{\partial t},$$

δT and $\delta\mu$ are the deviations of the temperature and the chemical potential from their equilibrium values. Further,

$$A_2=-\frac{\partial\nu_n}{\partial T}\frac{\partial}{\partial x_\alpha}[\delta T(v_{n\alpha}-v_{s\alpha})]-\frac{\partial\nu_n}{\partial\mu}\frac{\partial}{\partial x_\alpha}[\delta\mu(v_{n\alpha}-v_{s\alpha})]$$

$$-\nu_n\frac{\partial}{\partial x_\alpha}\left[\frac{\partial\xi}{\partial x_\alpha}(v_{nz}-v_{sz})\right]-\rho v_{s\alpha}\frac{\partial\xi}{\partial x_\alpha}+\rho\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)\xi;$$

$$A_3=-\frac{\partial\sigma}{\partial T}\frac{\partial}{\partial x_\alpha}(v_{n\alpha}\delta T)-\sigma\frac{\partial}{\partial x_\alpha}\left(v_{nz}\frac{\partial\xi}{\partial x_\alpha}\right)$$

$$-\frac{1}{2}\frac{\partial\nu_n}{\partial T}\frac{\partial}{\partial t}(v_{n\alpha}-v_{s\alpha})^2-\frac{1}{2}\frac{\partial^2\sigma}{\partial T^2}\frac{\partial}{\partial t}(\delta T)^2;$$

$$A_4=-\frac{\partial\nu}{\partial\mu}\frac{\partial}{\partial x_\alpha}(v_{n\alpha}\delta\mu)-\nu\frac{\partial}{\partial x_\alpha}\left(v_{nz}\frac{\partial\xi}{\partial x_\alpha}\right)$$

$$-\frac{1}{2}\frac{\partial\nu_n}{\partial\mu}\frac{\partial}{\partial t}(v_{n\alpha}-v_{s\alpha})^2-\frac{1}{2}\frac{\partial^2\nu}{\partial\mu^2}\frac{\partial}{\partial t}(\delta\mu)^2;$$

$$A_5=-\rho(v_{sz})^2+\rho v_{sz}\frac{\partial\xi}{\partial t}-\frac{\partial\alpha}{\partial T}\frac{\partial}{\partial x_\alpha}\left(\delta T\frac{\partial\xi}{\partial x_\alpha}\right)$$

$$-\frac{\partial\alpha}{\partial\mu}\frac{\partial}{\partial x_\alpha}\left(\delta\mu\frac{\partial\xi}{\partial x_\alpha}\right)$$

$$+\nu_n\frac{\partial}{\partial t}\left[\frac{\partial\xi}{\partial x_\alpha}(v_{n\alpha}-v_{s\alpha})\right]+\nu_n\frac{\partial}{\partial x_\alpha}\left[\frac{\partial\xi}{\partial t}(v_{n\alpha}-v_{s\alpha})\right].$$

In the equations (1) we have set $\partial\nu/\partial T=\partial\sigma/\partial\mu=0$, according to what was said before. Moreover we have expanded the potential φ (a volume quantity) into a Taylor series about the unperturbed free surface $z=0$.

Eliminating δT, $\delta\mu$, $\mathbf{v}_s$, and p from the equations (1), (2) we obtain the following system of equations:

$$\frac{\partial^2 v_{n\beta}}{\partial t^2}-c^2\frac{\partial^2 v_{n\alpha}}{\partial x_\beta\,\partial x_\alpha}-\frac{\alpha}{\rho}\frac{\partial^2}{\partial t\,\partial x_\beta}\left(\frac{\partial^2\xi}{\partial x_\alpha^2}\right)=N_1,$$

$$\frac{\nu_n}{\rho}\frac{\partial^2 v_{n\alpha}}{\partial t\,\partial x_\alpha}-\frac{\nu_n}{\rho}\frac{\alpha}{\rho}\frac{\partial^2}{\partial x_\beta^2}\left(\frac{\partial^2\xi}{\partial x_\alpha^2}\right)+\frac{\partial^2\xi}{\partial t^2}-\frac{\partial^2\varphi}{\partial z\,\partial t}=N_2, \qquad (3)$$

$$\frac{\partial\varphi}{\partial t}-\frac{\alpha}{\rho}\frac{\partial^2\xi}{\partial x_\alpha^2}=N_3.$$

For the nonlinear parts N_1, N_2, N_3 we obtain

$$N_1=\frac{1}{\nu_n}\frac{\partial A_1}{\partial t}-\frac{\sigma}{\nu_n\partial\sigma/\partial T}\frac{\partial A_3}{\partial x_\beta}-\frac{\nu}{\nu_n\partial\nu/\partial\mu}\frac{\partial A_4}{\partial x_\beta}+\frac{\partial^2 N_3}{\partial x_\beta\,\partial t},$$

$$N_2=\frac{1}{\rho}\frac{\partial A_2}{\partial t}+\frac{\nu_n}{\rho}\frac{\partial^2 N_3}{\partial x_\alpha^2}, \qquad (4)$$

$$N_3=-\frac{1}{\rho}A_5-\frac{1}{2}\left(\frac{\partial\varphi}{\partial x_i}\right)^2-\frac{\partial^2\varphi}{\partial z\,\partial t}\xi.$$

Here we have made use of the relations

$$\sigma=-\frac{\partial\alpha}{\partial T}=\sigma(T)+\frac{1}{2}\frac{\partial\nu_n}{\partial T}(\mathbf{v}_{nt}-\mathbf{v}_{st})^2,$$

$$\nu=-\frac{\partial\alpha}{\partial\mu}=\nu(\mu)+\frac{1}{2}\frac{\partial\nu_n}{\partial\mu}(\mathbf{v}_{nt}-\mathbf{v}_{st})^2;$$

the speed of the surface second sound is

$$c=\left(\frac{\sigma^2}{\nu_n\partial\sigma/\partial T}+\frac{\nu^2}{\nu_n\partial\nu/\partial\mu}\right)^{1/2}.$$

We consider the nonlinear interaction of three waves which propagate in the xy plane: two second-sound waves $(\omega_0\mathbf{k}_0,\omega_2\mathbf{k}_2)$ and a capillary wave $(\omega_1\mathbf{k}_1)$ with $\omega_0=\omega_1+\omega_2$, $\mathbf{k}_0=\mathbf{k}_1+\mathbf{k}_2$. The spectra of the capillary waves and of the second sound waves have the following form, up to the small parameter $(\nu_n/\rho)k_{1,0,2}$:

$$\omega_1^2=(\alpha/\rho)k_1^3, \quad \omega_{0,2}=ck_{0,2}.$$

In weakly superfluid solutions of ^3He in ^4He the magnitude of the surface tension is $\alpha\simeq 0.3$ erg/cm^2 (Ref. 4), $c\simeq(2-4)\times10^3$ cm/s (Refs. 2,3,5), and therefore for vectors $k_{1,0,2}$ of the same order of magnitude (e.g., $k\sim10^2\,\mathrm{cm}^{-1}$) the frequency of the capillary wave is substantially lower than the frequency of surface sound: $\omega_1\ll\omega_{0,2}$. It follows that the scattering of second sound with $(\omega_0\mathbf{k}_0)$ on a capillary wave with $(\omega_1\mathbf{k}_1)$ can occur in all directions for an almost unchanged frequency of the second sound

$$\omega_0\approx\omega_2, \quad \Delta\omega=\left[\frac{\alpha}{\rho}\left(2k_0\sin\frac{\gamma}{2}\right)^3\right]^{1/2},$$

where λ is the scattering angle, and for an almost unchanged magnitude of the wave vector; the maximal frequency shift occurs for backscattering. The geometric picture of such an interaction is analogous to the Mandel'shtam-Brillouin scattering picture in nonlinear optics.

Before solving the nonlinear system of equations (3) we go over to the corresponding linear system with normal modes $\bar\theta_1$ (the capillary branch) and $\bar\theta$ (the acoustic branch)

$$\xi_k=\bar\theta_1+\frac{\nu_n}{\rho c}\bar\theta=\theta_1\exp(-i\omega_1 t)+\frac{\nu_n}{\rho c}\theta\exp(-i\omega t),$$

$$v_{n\beta k}=-a_k\bar\theta_1+\bar\theta=-a_k\theta_1\exp(-i\omega_1 t)+\theta\exp(-i\omega t),$$

$$a_k=(ck)^{-2}(\alpha k^3/\rho)^{1/2}.$$

Here and in the sequel the parameters $\nu_n k/\rho$ and $(ck)^{-2}\alpha k^3/\rho$ are considered small.

We search for the solution of the system (3) in the form of a sum of the three waves we have described earlier:

$$\xi = \frac{1}{2}\Big\{\theta_1 \exp i(\mathbf{k}_1\mathbf{r}-\omega_1 t) + \sum_{j=0,2}\Big(\frac{v_n}{\rho c}\theta_j+\delta_j\Big)\exp i(\mathbf{k}_j\mathbf{r}-\omega_j t)+\text{c.c.}\Big\},$$

$$v_{n\beta} = \frac{1}{2}\Big[(-a_{k_1}\theta_1+\delta_1)\exp i(\mathbf{k}_1\mathbf{r}-\omega_1 t) + \sum_{j=0,2}\theta_j \exp i(\mathbf{k}_j\mathbf{r}-\omega_j t)+\text{c.c.}\Big]. \quad (5)$$

Here δ_j and δ_1 are the nonlinear additions to the linear relations between the $v_{n\beta k}$ and ξ_k in the second sound wave and in the capillary wave, respectively. The amplitudes of the normal modes $\theta_1,\theta_0,\theta_2$ are assumed to be slowly varying with the distance on account of the nonlinearity of the medium.

In order not to complicate the discussion we shall assume that all three waves propagate along the x axis, and that the wave θ_2 is propagating backwards. Substituting the expressions (5) into the system of equations (3) and eliminating φ, δ_j, and δ_1, we obtain the so-called contracted equations (see Ref. 6) for the amplitudes $\theta_{1,0,2}$:

$$\frac{d\theta_1}{dx}=\frac{i}{2}\frac{k_1}{\omega_1^2}\Big(-N_1\frac{v_n}{\rho}k_1\frac{\omega_1}{c^2k_1^2}+N_{21}\Big),$$
$$\frac{d\theta_0}{dx}=\frac{i}{2}\frac{1}{c^2k_0}\Big(N_1+N_{20}\frac{\alpha}{\rho}\frac{k_0^3}{\omega_0}\Big), \quad (6)$$
$$\frac{d\theta_2}{dx}=-\frac{i}{2}\frac{1}{c^2k_2}\Big(N_1+N_{22}\frac{\alpha}{\rho}\frac{k_2^3}{\omega_2}\Big).$$

Here $N_{2j} = N_2 + k_jN_3$; $j = 1,0,2$. Substituting the expressions (5) into N_1, N_2, and N_3 [see Eqs. (4)] we finally obtain, accurate to the small parameters $(v_n/\rho)k_j$, $\omega_1/\omega_{0,2}$:

$$\frac{d\theta_1}{dx}=\frac{i}{2}\frac{v_n}{\rho}k_1B\theta_0\theta_2^*,$$
$$\frac{d\theta_0}{dx}=\frac{i}{2}k_0\frac{\omega_1^2}{k_1}B\theta_1\theta_2, \quad (7)$$
$$\frac{d\theta_2}{dx}=-\frac{i}{2}k_2\frac{\omega_1^2}{k_1}B\theta_0\theta_1^*,$$

where B is the effective vertex describing the nonlinear interaction:

$$B=\frac{1}{c^2}\Big[2-\frac{\partial v_n}{\partial T}\frac{\sigma}{v_n\partial\sigma/\partial T}+\frac{\partial^2\sigma}{\partial T^2}\frac{1}{v_nc^2}\Big(\frac{\sigma}{\partial\sigma/\partial T}\Big)^3 - \frac{\partial v_n}{\partial\mu}\frac{v}{v_n\partial v/\partial\mu}+\frac{\partial^2 v}{\partial\mu^2}\frac{1}{v_nc^2}\Big(\frac{v}{\partial v/\partial\mu}\Big)^3\Big].$$

It is known (see, e.g., Ref. 7) that the system of equations (7) can be solved in terms of elliptic functions.

We consider the energy relation in this interaction. The energy density of the capillary wave averaged over a period equals

$$\bar{E}_1\approx\frac{\rho}{2S}\int_S\int_{-\infty}^{\xi}\overline{\mathbf{v}_s^2}\,dS\,dz+\frac{\alpha}{S}\int_S\Big\{\Big[\overline{1+\Big(\frac{\partial\xi}{\partial x_\alpha}\Big)^2\Big]^{1/2}}-1\Big\}dS=\frac{\rho}{2}\frac{\omega_1^2}{k_1}|\theta_1|^2; \quad (8)$$

where S is the projection of the surface onto the xy plane.

The second sound wave energy density has the form

$$\bar{E}_{0,2}\approx\frac{1}{S}\int_S\Big\{\frac{v_n\overline{v_{n\alpha}^2}}{2}+\frac{1}{2}(\overline{\delta T\delta\sigma}+\overline{\delta\mu\delta v})\Big\}dS=\frac{v_n}{2}|\theta_{0,2}|^2. \quad (9)$$

Since in the parametric interaction of the waves the energy of one of them is transferred into the others, in the language of numbers of quanta which pass per unit time through a one-dimensional section, the energy conservation law takes on the form

$$dn_0/dx=-dn_1/dx=dn_2/dx. \quad (10)$$

Here n_0, n_1, and n_2 are the numbers of quanta respectively in the waves θ_0, θ_1, and θ_2.

Making use of the equations (7), as well as of the expressions (8) and (9), one can see that the equations (10) are satisfied. Indeed, from the equations (7) we obtain

$$\frac{dn_1}{dx}=\frac{1}{2}\rho\frac{\omega_1^2}{k_1^2}\frac{d|\theta_1|^2}{dx}=\frac{i}{2}v_n\frac{\omega_1^2}{k_1}B\,\text{Im}(\theta_0\theta_1^*\theta_2^*),$$
$$\frac{dn_0}{dx}=\frac{1}{2}\frac{v_n}{k_0}\frac{d|\theta_0|^2}{dx}=-\frac{i}{2}v_n\frac{\omega_1^2}{k_1}B\,\text{Im}(\theta_0\theta_1^*\theta_2^*), \quad (11)$$
$$\frac{dn_2}{dx}=\frac{1}{2}\frac{v_n}{k_2}\frac{d|\theta_2|^2}{dx}=-\frac{i}{2}v_n\frac{\omega_1^2}{k_1}B\,\text{Im}(\theta_0\theta_1^*\theta_2^*).$$

From Eq. (11) one can see that the equations (10) are indeed valid.

Energy and momentum conservation also allow for the interaction of two capillary waves ($\omega_1\mathbf{k}_1$, $\omega_2\mathbf{k}_2$) and a surface second-sound wave ($\omega_0\mathbf{k}_0$), so that accurate to the ratio c_1/c of the speeds of the capillary waves and the second-sound wave we have

$$\omega_1\approx\omega_2\approx\omega_0/2, \quad k_0=k_1-k_2\ll k_1,\ k_2$$

(for the case of interaction along one line; the θ_2 wave is propagating backward).

The vertex function A describing such an interaction can be determined similarly to the vertex B:

$$A=\frac{1}{c}\Big[1-\frac{\partial v_n}{\partial T}\frac{\sigma}{v_n\partial\sigma/\partial T}+\frac{\partial^2\sigma}{\partial T^2}\frac{1}{v_nc^2}\Big(\frac{\sigma}{\partial\sigma/\partial T}\Big)^3\Big(\frac{c_1}{c}\Big)^2 - \frac{\partial v_n}{\partial\mu}\frac{v}{v_n\partial v/\partial\mu}+\frac{\partial^2 v}{\partial\mu^2}\frac{1}{v_nc^2}\Big(\frac{v}{\partial v/\partial\mu}\Big)^3\Big(\frac{c_1}{c}\Big)^2\Big].$$

The contracted equations for the amplitudes have the form

$$\frac{d\theta_1}{dx}=\frac{i}{2}k_1\frac{v_n}{\rho}k_1A\theta_0\theta_2^*,$$
$$\frac{d\theta_2}{dx}=-\frac{i}{2}k_2\frac{v_n}{\rho}k_2A\theta_0\theta_1^*, \quad (12)$$
$$\frac{d\theta_0}{dx}=\frac{i}{2}k_0\frac{\omega_0^2}{4}A\theta_1\theta_2.$$

The system of equations (12) as well as the system (7) admits solutions in terms of elliptic functions. The discussion can easily be extended to the case of waves which propagate under an angle relative to each other.

Let us estimate the possibility of experimentally observing the nonlinear interactions of the surface waves. For instance, let us determine the order of magnitude of the distance a over which the surface second sound θ_2 can be amplified on account of fluctuations to observable levels, if at the boundary one prescribes a sufficiently strong surface sound θ_0 and a capillary wave θ_1. For a numerical estimate the amplitudes θ_0 and θ_1 can be considered to be approximately constant. Then from the last equation of the system (7) we get

$$|\theta_2(0)|-|\theta_2(a)|\approx|\theta_2(0)|\approx{}^1/_4\omega_1{}^2B|\theta_0||\theta_1|a,$$

$$|\theta_2(0)|\gg|\theta_2(a)|.$$

Let

$$|\theta_2(0)|\approx10^{-2}|\theta_0|,\quad \omega_1\approx2\pi\cdot10^3\,\mathrm{s}^{-1};$$

$$B\approx2/c^2\approx2\cdot(2\cdot10^3\ \mathrm{cm/s})^{-2},$$

$$\theta_1\approx0.1\lambda_1=0.1\cdot2\pi/k_1\approx0.3\cdot10^{-2}\ \mathrm{cm}.$$

(Some values typical of experiments with capillary waves have been taken from Ref. 8.) We obtain $a<1$ cm, which from an experimental point of view is a realistic magnitude.

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