Virial representation of the imaginary part of the Lagrange function of the electromagnetic field

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For the imaginary part of the Lagrange function of a constant homogeneous electric field ε with allowance for radiative self-interaction of the vacuum charges (two-loop approximation) a representation is found in the form of a series in powers of the quasiclassical exponential $\exp(-\pi/\beta)$, $\beta \equiv \hbar e\varepsilon/m^2c^3$, similar to Schwinger's representation for the imaginary part of the single-loop Lagrangian. This representation has the meaning of a virial expansion, and its parameter—the quasiclassical exponential—is the mean number of pairs produced by the field in the 4-volume of pair formation. The leading term of the series is the mean number of pairs in unit 4-volume, and the remaining terms describe quantum-mechanical exchange effects of Fermi repulsion or Bose attraction of the created particles. The radiative corrections to the terms of the series in a weak field reduce for the leading term ($n=1$) to a mass shift of an accelerated charge, and for the following terms ($n\geqslant 2$) to a mass shift of the charge that represents its effective Coulomb interaction with charges of both signs in a group of n coherently created pairs.

INTRODUCTION

It is well known that the interaction of the field of vacuum charged particles with an external electromagnetic field is described by the exact Lagrange function $\mathscr{L}$ of the electromagnetic field. In particular, the probability of vacuum persistence in the volume V during the time T is determined by $\exp(-2\,\mathrm{Im}\mathscr{L}VT)$, i.e., $2\,\mathrm{Im}\mathscr{L}$ is the decay rate of unit 4-volume of the vacuum. For the imaginary part of the Lagrange function of a constant and homogeneous electric field ε in the single-loop approximation Schwinger[1] obtained the representation[1]

$$2\,\mathrm{Im}\,\mathscr{L}^{(1)}=(2s+1)\frac{(e\varepsilon)^2}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{(\pm1)^{n+1}}{n^2}e^{-\pi n/\beta},\quad \beta=\frac{e\varepsilon}{m^2}. \tag{1}$$

Here and below, the upper and lower signs correspond to charged Fermi and Bose particles with spin $s=1/2$ and $s=0$. As can be seen from this representation, the decay rates of the vacua of Fermi and Bose fields differ under otherwise equal conditions by not only the spin statistical weight $(2s+1)$. To elucidate the physical meaning of the individual terms of the representation (1), it is convenient to use Nikishov's representation[2]

$$2\,\mathrm{Im}\,\mathscr{L}^{(1)}VT=\mp\sum_r\int\frac{d^3pV}{(2\pi\hbar)^3}\ln(1\mp\bar n_p),\qquad \bar n_p=\exp\left(-\pi\frac{m^2+p_\perp^2}{e\varepsilon}\right), \tag{2}$$

which relates the imaginary part of the Lagrange function of the field to the mean number $\bar n_p$ of pairs produced by the field in the state with given momentum and spin projection $p=p, r$. The distribution of $\bar n_p$ is degenerate with respect to the spin projection r and the momentum component $p_{\parallel}$ longitudinal with respect to the field with degeneracies $2s+1$ and $L_{\parallel}\Delta p_{\parallel}/2\pi\hbar$, where $\Delta p_{\parallel}=e\varepsilon T$ (see Refs. 2 and 3).

It was pointed out in Ref. 4 that the right-hand side of the representation (2) as a functional of $\bar n_p$ is equal to PV/Θ for an ideal Fermi or Bose gas, where P, V, and θ are the pressure, volume, and temperature. Then $2\,\mathrm{Im}\,\mathscr{L}^1/\hbar$ is analogous to the rate of growth of the pressure in units of Θ.

Expansion of $\ln(1\mp\bar n_p)$ in (2) in a series in powers of $\bar n_p$ and term-by-term integration over $p_\perp$ leads to the series (1). Therefore, the first term of the series (1) is the mean number $\bar n$ of pairs in unit 4-volume,[2] and the approximate expression.

$$2\,\mathrm{Im}\,\mathscr{L}^{(1)}\approx\bar n=(2s+1)\frac{(e\varepsilon)^2}{(2\pi)^3}e^{-\pi/\beta} \tag{3}$$

is analogous to the equation of state of an ideal gas (Clapeyron's equation). The following terms of the series (1) with $n\geqslant 2$ are the quantum-mechanical exchange corrections to the rate of growth of the pressure describing the additional Fermi repulsion or Bose attraction of the particles for the given mean 4-density of their number. They arise because of coherent pair creation, i.e., creation of $n\geqslant 2$ pairs in the same 4-volume of pair formation. Therefore, (1) is a virial expansion for $2\,\mathrm{Im}\mathscr{L}^1$ in powers of the parameter $\exp(-\pi/\beta)\sim\bar n(e\varepsilon)^{-2}$, which is the mean number of pairs in the 4-volume of pair formation.[4] Such a representation is analogous to the virial expansion of the pressure of an ideal Fermi or Bose gas in powers of the degeneracy parameter, i.e., the mean number of particles in the 3-volume determined by the thermal de Broglie wavelength (see §56 in Ref. 5).

In statistical physics, the virial expansion of the pressure proved to be very convenient for describing the deviation of the state of a gas from the ideal state not only due to the effects of quantum-mechanical degeneracy but also the strong interaction of molecules. In our case, the force, i.e., radiative, interaction of the vacuum charges is taken into account by the two-loop correction $\mathscr{L}^2$ to the Lagrange function of the electromagnetic field, which was found in Refs. 6 and 7 in the proper-time representation. In the pres-

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ent paper, we obtain for the imaginary part of the Lagrange function of a constant electric field in the two-loop approximation a representation of the form (1), i.e., for 2 Im$\mathscr{L}$ we obtain with allowance for the radiative interaction of the vacuum charges the virial representation

$$2\,\mathrm{Im}\,(\mathscr{L}^{(1)}+\mathscr{L}^{(2)})$$
$$=(2s+1)\frac{(e\varepsilon)^2}{(2\pi)^3}\sum_{n=1}^{\infty}(\pm1)^{n+1}\left[\frac{1}{n^2}+\alpha\pi K_n(\beta,s)\right]e^{-n\pi/\beta}. \tag{4}$$

The complicated function $K_n(\beta, s)$ simplifies for $n\beta\ll1$ and in this region does not depend on the spin:

$$K_n(\beta,s)=-\frac{c_n}{\beta^{1/2}}+1+\ldots, \tag{5}$$
$$c_1=0,\quad c_n=\frac{1}{2n^{1/2}}\sum_{k=1}^{n-1}[k(n-k)]^{-1/2},\quad n\geqslant2,$$

(the ellipsis denotes terms that vanish as $\beta\to0$). For $n=1$, this result was found earlier by one of the present authors: see Refs. 6 and 7.

In Sec. 6 we give a physical interpretation of the radiation correction (5) on the basis of the independent physical meaning of the terms in the virial expansion, which gives a picture of coherent pair production. This meaning of the individual terms of the virial series makes it possible to transform the radiative correction into a field-dependent correction to the mass of a charge participating in group tunneling. Such a transformation is equivalent to summation of the radiative corrections of all orders in α in a weak field. Morover, in all terms of the virial expansion the mass of the charge does not acquire finite corrections (corrections that do not vanish with the field), this being due to the correct behavior of the mass renormalization of the charge and the quasiclassical meaning of the nth term of the virial expansion for a weak field (with regard to the quasiclassical nature of the terms of the series (1), see Ref. 8).

Indeed, it was shown in Refs. 6 and 7 that by means of the Lagrange function one can uniquely renormalize the charge and mass of the particles by requiring that in the weak-field limit the real part of the exact Lagrange function be Maxwellian and the imaginary part quasiclassical:

$$\mathrm{Im}\,\mathscr{L}=(e\varepsilon)^2f(\alpha)e^{-\pi m^2/e\varepsilon},\quad \varepsilon\to0. \tag{6}$$

Here, m is the renormalized observable mass of the charged particles. In accordance with this condition, the radiative corrections in the weak field change only the pre-exponential factor $f(\alpha)=f^{(1)}+\alpha f^{(2)}+\alpha^2f^{(3)}+\ldots$, and do not change the argument of the exponential, which by virtue of the quasiclassical situation plays in the mass renormalization a part similar to the part played by the classical Thomson cross section in charge renormalization, namely, in the limit $\varepsilon\to0$ the radiative corrections to m^2 tend to zero like the radiative corrections to the cross section of the Compton effect when the photon frequency tends to zero, $\omega\to0$; see Ref. 9. Thus, the boundary condition (6) imposed on the imaginary part of the exact Lagrange function uniquely fixes the mass counterterm.[6] Because the exponentials with $n\geqslant2$ are quasiclassical, the singularities of the type $C\beta^{-1}$, $n\beta\ll1$, disappear after the mass renormalization in all terms of the virial expansion with $n\geqslant2$. Such singularities would lead after exponentiation to a finite, nonvanishing as $\beta\to0$, mass shift in the terms with $n\geqslant2$, and this would be in contradiction with a quasiclassical nature of the exponentials, where m is defined as the physical of the particle.[6] Thus, to fix the mass counterterm uniquely, a boundary condition of the type (6) could be imposed on any term of the virial series. This emphasizes once more the independent physical meaning of each of these terms.

All calculations in the paper will be made for the spinor case.[10] For the scalar case only the final results will be given (see Ref. 11).

2. PROPER-TIME REPRESENTATION OF THE IMAGINARY PART OF THE LAGRANGE FUNCTION

We proceed from the expression (50) of Ref. 6 for $\mathscr{L}^2$, the two-loop radiative correction to the Lagrange function in spinor electrodynamics. In different notation, $\mathscr{L}^2$ has the form

$$\mathscr{L}^{(2)}=\frac{\alpha(e\varepsilon)^2}{16\pi^3}\int_0^{\infty}\frac{dx}{x^3}e^{-x/\beta}\left\{\frac{2x}{\beta}\int_0^1\frac{d\xi}{\xi(1-\xi)}\right.$$
$$\times\left[\frac{\cos x\xi\cos x(1-\xi)}{a-b}\ln\frac{a}{b}-\frac{\cos x}{b}-\frac{5x^2}{6}\xi(1-\xi)\right]$$
$$-\int_0^1\frac{d\xi}{\xi(1-\xi)}\left[\frac{c\ln(a/b)}{(a-b)^2}-\frac{1-b\cos x(1-2\xi)}{b(a-b)}+\frac{b\cos x+1}{2b^2}+\frac{5x^2}{6}\xi(1-\xi)\right]$$
$$\left.+\left(1+\frac{3x}{\beta}\ln\frac{\bar{\gamma}x}{\beta}\right)\left(\frac{\cos x}{b}-1+\frac{x^2}{3}\right)\right\},$$
$$a=\frac{\sin x\xi\sin x(1-\xi)}{x\xi x(1-\xi)},\quad b=\frac{\sin x}{x},$$
$$c=1-a\cos x(1-2\xi),\quad \ln\bar{\gamma}=\ln\gamma-5/6, \tag{7}$$

where $\ln\gamma$ is Euler's constant. There is a similar expression for the scalar case. With regard to the modifications made in (7) compared with the original expressions of Refs. 6 and 7, see Refs. 10–12.

In the denominators of the expressions, there is the entire function $a-b$, and in what follows its properties will play a fundamental part. The problem is to calculate the imaginary part of the double integral (7) of a function of two variables, this function containing meromorphic terms and a product of the logarithm of the meromorphic function a/b and other meromorphic functions.

If we denote the expression in the curly brackets in (7) together with the coefficient $\alpha(e\varepsilon)^2/16\pi^3$ by $f(x/\beta, x)$, then for appropriate choice of the "physical sheet" for the logarithm we have (see Ref. 12)

$$f^*(x/\beta,x)=f(x^*/\beta,x^*). \tag{8}$$

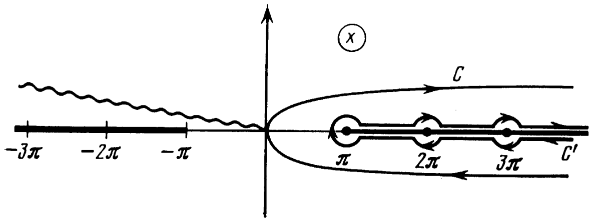

FIG. 1. The cuts beginning at the points $\pm\pi$ due to the first two terms of the function $f(x/\beta,x)$. The wavy line shows a typical cut of the logarithm in the last column. On the lines of the cuts there are also branch points and poles of $f(x/\beta,x)$ at $x=x_n=n\pi$.

This choice of the sheet corresponds to reality of $\ln(a/b)$ for real x and $0<x<\pi, 0<\xi<1$. Taking into account (8) we obtain

$$\operatorname{Im}\mathscr{L}^{(2)}=\frac{1}{2i}\int\limits_{C}\frac{dx}{x^3}e^{-x/\beta}f\left(\frac{x}{\beta},x\right),\tag{9}$$

where the contour C is shown in Fig. 1. The upper part of C is the contour of integration in (7). In accordance with the three terms in f, we introduce the notation

$$J=\frac{1}{2i}\int\limits_{C'}dx\,g(x)\int\limits_0^1\frac{d\xi}{\xi(1-\xi)}\times\left[\frac{\cos x\xi\cos x(1-\xi)}{a-b}\ln\frac{a}{b}-\frac{\cos x}{b}\right],\tag{10a}$$

$$J_1=\frac{1}{2i}\int\limits_{C'}dx\,g_1(x)\int\limits_0\frac{d\xi}{\xi(1-\xi)}\times\left[\frac{c\ln(a/b)}{(a-b)^2}-\frac{1-b\cos x(1-2\xi)}{b(a-b)}+\frac{1+b\cos x}{2b^2}\right],\tag{10b}$$

$$J_2=\frac{1}{2i}\int\limits_{C'}dx\,g_2(x)\left(1+\frac{3x}{\beta}\ln\frac{\gamma x}{\beta}\right)\frac{\cos x}{b},\tag{10c}$$

$$g(x)=\frac{\alpha(e\varepsilon)^2}{16\pi^3}\frac{2x}{\beta}\frac{\exp(-x/\beta)}{x^3},$$

$$g_1(x)=-g_2(x)=-\frac{\alpha(e\varepsilon)^2}{16\pi^3}\frac{\exp(-x/\beta)}{x^3}.\tag{10d}$$

In Eqs. (10), we have used the analyticity of f in the strip $0<\operatorname{Re}x<\pi$ and replaced the contour C by C' (see Fig. 1). After this, the terms $-5/6x^2\xi(1-\xi)$, $5/6x^2\xi\ (1-\xi)$, $-1+x^2/3$ in (7) can be omitted. The integral J_2 can be readily calculated, and the result is obviously

$$J_2=-\frac{\alpha^2\varepsilon^2}{2\pi}\sum_{n=1}^{\infty}e^{-x_n/\beta}\left[\frac{1}{2x_n^2}+\frac{3}{2\beta x_n}\left(\ln\frac{\gamma x_n}{\beta}-\frac{5}{6}\right)\right],\quad x_n=n\pi.\tag{11}$$

It can be seen that series (11) has coefficients that increase as $\beta\to0$ for all powers of $\exp(-\pi/\beta)$. In accordance with the principle of charge and mass renormalization of Refs. 6 and 7, such a dependence on β in the total sum $J+J_1+J_2$ must vanish in the limit $\varepsilon\to0$ for at least the coefficient of the first exponential.

3. ANALYTIC PROPERTIES OF THE INTEGRANDS

The integrals J and J_1 are nonvanishing only if their integrands $f(x/\beta,x)$ have poles or branch points within the contour of integration C'. We shall show that f has poles and branch points only at the points $x=x_n=n\pi$.

The analytic properties of the functions in the square brackets in (10a) and (10b) are of the same kind. It is sufficient to consider the first of them. It can be seen that all singularities could consist of branch points of the function $\ln(a/b)$, zeros of the function $a-b$, zeros of the function b, and points $\xi=0$, $\xi=1$. To determine the branch points of $\ln(a/b)$, we go over to the variables $u=x(1-\xi)$, $v=x\xi$. Then

$$\frac{a}{b}=\frac{\sin u}{u}\frac{\sin v}{v}\frac{u+v}{\sin(u+v)}.\tag{12}$$

Since u and v are positive, on the u, v plane (see Fig. 2) the function a/b is positive in the regions where one or all three of the sines are positive and is negative where one or all three sines are negative. Since $\sin u$, $\sin v$, and $\sin(u+v)$ are, respectively, positive in the odd vertical, horizontal, and inclined strips and negative in the corresponding even strips, the function a/b is negative only in the regions of the intersection of one even strip with two odd or two even strips; in Fig. 2, these triangular regions are hatched. On a straight line parallel to an inclined strip the variable $u+v=x$ is constant, and the variable ξ increases monotonically from zero (u axis) to unity (v axis). It can be seen from Fig. 2 that if x is fixed and lies within the nth inclined strip, i.e., $(n-1)\pi<x<n\pi$, then as ξ increases from zero to unity the function a/b becomes negative $n-1$ times. Moreover, the points at which a/b changes from a positive to a negative sign are characterized by increasing values $\xi=\xi_{-k}$, where

$$\xi_{-k}=1-(n-k)\pi/x,\quad k=1,\ 2,\dots,n-1,\tag{13}$$

and they alternate with points $\xi=\xi_{+k}$ at which a/b changes from negative to positive sign:

$$\xi_{+k}=k\pi/x,\quad k=1,\ 2,\dots,n-1,\tag{14}$$

i.e., $\xi_{-1}<\xi_{+1}<\xi_{-2}<\xi_{+2}<\dots<\xi_{+n-1}$.

In the integral (7), $x=\operatorname{Re}x+i0$, i.e., x has a vanishingly small positive imaginary part. Therefore, in accordance with (13) and (14) the points ξ_{-k} are shifted from the interval $0<\xi<1$ into the complex plane of ξ upward, while the points ξ_{+k} are shifted downward. As ξ moves along the

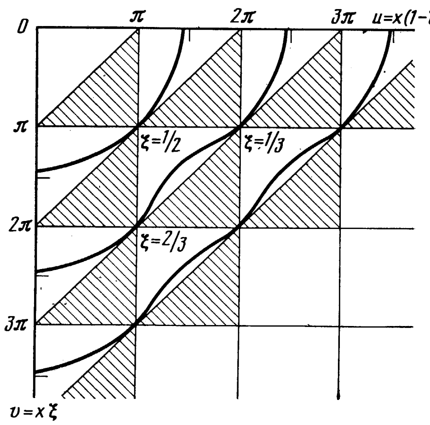

FIG. 2. The regions in which $a/b<0$ are hatched. Between them are curves on which $a=b=0$.

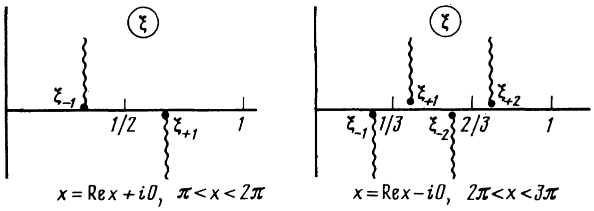

FIG. 3.

interval (0,1), the path passes below the points $\xi = \xi_{-k}$ and above the points $\xi = \xi_{+k}$, and the phase of the function a/b changes at these points from zero to π and from π to zero. In the integrals (9) and (10) on the lower part of the contour of integration $x = \operatorname{Re} x - i0$, and in this case the points ξ_{-k} and ξ_{+k} are displaced from the interval $0 < \xi < 1$ downward and upward, so that the phase of the function a/b changes at these points from zero to $-\pi$ and from $-\pi$ to zero. The position of the branch points (13) and (14) near the real axis ξ for $x = \operatorname{Re} x \pm i0$ is shown in Fig. 3. Thus, $\ln(a/b)$ has imaginary part equal to $\pm i\pi$ if

$$x = \operatorname{Re} x \pm i0, \quad (n-1)\pi < x < n\pi, \quad \xi_{-k} < \xi < \xi_{+k}. \tag{15}$$

Outside these regions, $\operatorname{Im} \ln(a/b) = 0$. We emphasize that in the region $0 < x < \pi, 0 < \xi < 1$ the function $\ln(a/b)$ does not have branch points. But when x varies in the complex plane and $0 < \operatorname{Re} x < \pi$, the branch points ξ_{-k} and ξ_{+k} do not intersect the interval of integration over ξ, and this in conjunction with the absence of other singularities with respect to ξ leads to analyticity of the function $f(x/\beta, x)$ for $0 < \operatorname{Re} x < \pi$.

We now consider the zeros of the function a–b. In the variables u,v

$$a - b = \frac{\sin u \sin v}{u+v}\left(\frac{1}{v} - \operatorname{ctg} v + \frac{1}{u} - \operatorname{ctg} u\right). \tag{16}$$

On the u,v plane, the line of zeros a–b is determined by the equation

$$1/v - \operatorname{ctg} v + 1/u - \operatorname{ctg} u = 0. \tag{17}$$

The coordinate axes $u = 0$ and $V = 0$ are obviously also lines of zeros. We denote the line of zeros lying in the nth inclined strip by $x = x_n^0(\xi)$. These lines of zeros are shown in Fig. 2. The points $x_n^0(0) = x_n^0(1)$ at which the line of zeros $x_n^0(\xi)$ meets the coordinates axes u and v are the roots of the equation $\cot z = z^{-1}$. These roots lie to the left of the numbers $(n - 1/2)\pi$, $n = 2, 3 \ldots$, but with increasing n tend to them. At the points $\xi = m/n$, $m = 1, 2, \ldots, n-1$, the line of zeros $x_n^0(\xi)$ reaches its maxima, which are equal to $n\pi$.

It is clear that a line of zeros of the function a–b avoids the region where $a/b < 0$. It can also be seen from the figure within the interval $0 < \xi < 1$ it is not for all x that points at which $a - b = 0$ are present. If we describe the straight line $u + v = x$ and then increase x from $x_n + \delta$ ($\delta > 0$ and σ is small) to $x_{n+1} - \delta$, then for $x > x_{n+1}^0(0) = x_{n+1}^0(1)$ in the interval (0,1) of the variable ξ points $\xi = \xi_{0i}(x)$ begin to appear in pairs (in Fig. 2, the points of intersection of the straight line $u + v = x$ and the line of zeros (17) would correspond to them). As x approaches x_{n+1} from the left, the zeros $\xi_{0i}(x)$ are displaced toward the points $m(n+1)^{-1}$ ($m = 1, 2, \ldots, n$)—two zeros for each rational point of this type. Note that in the neighborhood of the point $x = x_n$, $\xi = m/n$ the line of zeros of the function a–b has the form of a parabola touching the straight line $x = x_n$:

$$x_n^0(\xi) = x_n - \frac{x_n^3}{x_m x_{n-m}}\left(\xi - \frac{m}{n}\right)^2 + \ldots.$$

The zeros of the function a–b are not poles of the integrand if $b \neq 0$, since for $a - b \to 0$ but $b \neq 0$ we have $\ln(a/b) \to 0$, and then

$$\frac{\cos x\xi \cos x(1-\xi)}{a-b} \ln\frac{a}{b} - \frac{\cos x}{b} \tag{18}$$

$$= \frac{1}{b}\Big[\sin x\xi \sin x(1-\xi) + \cos x\xi \cos x(1-\xi) \times\left(\frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{4} + \ldots\right)\Big], \quad z = 1 - \frac{a}{b}.$$

Thus, the function in the square brackets is finite for $a - b \to 0$, $b \neq 0$. The additional singularities associated with the factor $[\xi(1-\xi)]^{-1}$ in (10a) and (10b) are also unimportant, since as $\xi \to 0$ or $\xi \to 1$ the variable z and the first term in the square brackets tend to zero as ξ or $1 - \xi$, and $b \neq 0$ if $x \neq n\pi$. As a result, the function (18) compensates these singularities. Thus, there remain the zeros of the function b, i.e., the points $x = x_n = n\pi$.

We arrive at the necessity of splitting the integral (10a) into two groups of terms:

$$J = \sum_{n=1}^{\infty} \frac{1}{2i} \int_{K_n} dx g(x) \int_0^1 \frac{d\xi}{\xi(1-\xi)} \times\left[\frac{\cos x\xi \cos x(1-\xi)}{a-b} \ln\frac{a}{b} - \frac{\cos x}{b}\right]$$
$$+ \sum_{n=1}^{\infty} \frac{1}{2i} \operatorname{disc} \int_{x_n+\delta_n}^{x_{n+1}-\delta_{n+1}} dx g(x) \int_0^1 \frac{d\xi}{\xi(1-\xi)} \times\left[\frac{\cos x\xi \cos x(1-\xi)}{a-b} \ln\frac{a}{b} - \frac{\cos x}{b}\right]. \tag{19}$$

The first group of terms is formed by integrals around small semicircles K_n of radius δ_n surrounding the points $x = x_n$ above and below the real axis, the contour passing around them clockwise. The second group is formed by integrals along the two edges of the cut, which are traversed in the indicated limits along the upper edge in the positive direction and along the lower in the negative (δ_n are small parameters that tend to zero in the final expression. In general, they depend on n). In the limit $\delta \to 0$, both terms in (19) contain δ-dependent singular terms. But in the complete expression for J they cancel each other, as is necessary by virtue of the finiteness of the physical quantity $\operatorname{Im} \mathscr{L}^2$.

All that we have said above concerning the analytic properties of the integrand in (10a) also applies to the integrand in (10b).

4. CALCULATION OF THE INTEGRALS AND REPRESENTATION OF Im $\mathscr{L}^2$ AS A SERIES WITH RESPECT TO EXPONENTIALS

Lack of space does not permit detailed calculations here of the integrals J and J_1 in accordance with the scheme (19). These calculations are given in the preprints of Refs. 10 and 11. The final expression for J and J_1 are two-dimensional residues of the functions in the square brackets of (10a) and (10b), i.e.,

$$J=\frac{\alpha^2\varepsilon^2}{2\pi}\sum_{n=1}^{\infty}e^{-x_n/\beta}\varphi_n(\beta), \tag{20}$$

$$J_1=\frac{\alpha^2\varepsilon^2}{2\pi}\sum_{n=1}^{\infty}e^{-x_n/\beta}\varphi_{1n}(\beta),$$

$$\varphi_n(\beta)=\frac{2}{\beta x_n}(\ln n-1)+\frac{2}{\beta}\sum_{m=1}^{n}\int_{x_n}^{x_{n+1}}dx\left[\frac{1}{x^2}\exp\left(-\frac{x-x_n}{\beta}\right)\right.$$
$$\left.\times\int_{\xi_{-m}}^{m(n+1)^{-1}}d\xi\frac{\cos x\xi\cos x(1-\xi)}{\xi(1-\xi)(a-b)}+\frac{\delta_{m1}}{x_n(x-x_n)}\right], \tag{21}$$

$$\varphi_{1n}(\beta)=\frac{n+1}{2x_n^2}+\frac{1-2\ln n}{4\beta x_n}+\sum_{m=2}^{n}\frac{\theta(n-2)\pi^{1/2}}{2(x_nx_{m-1}x_{n-m+1})^{1/2}}$$
$$-\sum_{m=1}^{n}\int_{x_n}^{x_{n+1}}dx\left[\frac{1}{x^3}\exp\left(-\frac{x-x_n}{\beta}\right)\right.$$
$$\left.\times\int_{\xi_{-m}}^{m(n+1)^{-1}}\frac{d\xi c}{\xi(1-\xi)(a-b)^2}-\text{c. t.}\right] \tag{22}$$

Here, δ_{m1} is the Kronecker delta, $\xi_{-m}=1-(n+1-m)\pi x^{-1}$ in accordance with (13) for the $(n+1)$th pole, $\theta(z)=1$ for $z\geqslant0$ and $\theta(z)=0$ for $z<0$, and c.t. denotes

$$\text{c. t.}=\frac{(x-x_n)^{-2}}{2x_n}-\frac{(x-x_n)^{-1}}{2\beta x_n}-\frac{(x-x_{n+1})^{-1}}{x_1x_{n+1}}e^{-\pi/\beta},\quad m=1,$$
$$\text{c. t.}=\frac{\pi(x-x_n)^{-3/2}}{4(x_nx_{n-m+1}x_{m-1})^{1/2}}-\frac{(x-x_n)^{-1}}{2x_{m-1}x_{n-m+1}}-\frac{(x-x_{n+1})^{-1}}{x_mx_{n+1}}e^{-\pi/\beta},$$
$$m=2,\dots,n. \tag{23}$$

Adding the expressions (11) and (20), we finally obtain for $s=1/2$

$$\operatorname{Im}\mathscr{L}^{(2)}=\frac{\alpha^2\varepsilon^2}{2\pi}\sum_{n=1}^{\infty}\exp(-x_n/\beta)\left[\varphi_n(\beta)+\varphi_{1n}(\beta)\right. \tag{24}$$
$$\left.-\left(\frac{1}{2x_n^2}+\frac{3}{2\beta x_n}\ln\frac{\tilde{\gamma}x_n}{\beta}\right)\right].$$

The imaginary part of the two-loop correction to the Lagrange function in scalar electrodynamics ($s=0$) is[11]

$$\operatorname{Im}\mathscr{L}^{(2)}=\frac{\alpha^2\varepsilon^2}{8\pi}\sum_{n=1}^{\infty}\exp(-x_n/\beta)\left[\varphi_n(\beta)+\varphi_{1n}(\beta)\right.$$
$$\left.+(-1)^{n+1}\left(\frac{1}{2x_n^2}-\frac{3}{2\beta x_n}\ln\frac{\tilde{\gamma}x_n}{\beta}\right)\right]. \tag{25}$$

Here

$$\varphi_n(\beta)=\frac{(-1)^{n+1}}{\beta x_n}(\ln n-1)$$
$$-\frac{1}{\beta}\sum_{m=1}\int_{x_n}^{x}dx\left[\frac{1}{x^2}\exp\left(-\frac{x-x_n}{\beta}\right)\right.$$
$$\left.\times\int_{\xi_{-m}}^{m(n+1)^{-1}}\frac{d\xi}{\xi(1-\xi)(a-b)}-\frac{(-1)^{n+1}\delta_{m1}}{x_n(x-x_n)}\right], \tag{26}$$

$$\varphi_{1n}(\beta)=(-1)^{n+1}\left(\frac{2\ln n-1}{4\beta x}+2-\frac{n+1}{2x^2}\right)$$
$$+\sum_{m=2}^{n}\frac{(-1)^n\theta(n-2)\pi^{1/2}}{2(x_nx_{m-1}x_{n-m+1})^{1/2}}$$
$$+\sum_{m=1}^{n}\int_{x_n}^{x_{n+1}}dx\left[\frac{1}{x^3}\exp\left(-\frac{x-x_n}{\beta}\right)\right.$$
$$\left.\times\int_{\xi_{-m}}^{m(n+1)^{-1}}\frac{d\xi(a-c)}{\xi(1-\xi)(a-b)^2}-\text{c. t.}\right], \tag{27}$$

a and b are the same as in (7),

$$c=\cos x\xi\cos x(1-\xi)+3\sin x\xi\sin x(1-\xi),$$
$$\ln\tilde{\gamma}=\ln\gamma-7/6,$$

and c.t. differs from (23) by reversal of the sign of the last term and by the common factor $(-1)^{n+1}$.

5. ASYMPTOTIC BEHAVIOR OF Im $\mathscr{L}^2$ FOR A WEAK FIELD

We find the asymptotic behavior of the functions $\varphi_n(\beta)$ and $\varphi_{1n}(\beta)$ ad $\beta\to0$. It can be shown that these functions have the general structure (the index of φ is omitted)

$$\varphi(\beta)=k_1(\beta)+e^{-\pi/\beta}k_2(\beta), \tag{28}$$

i.e., at the point $\beta=0$ the function $\varphi(\beta)$ has an essential singularity, which is separated by the representation (28), in which $k_{1,2}(\beta)$ do not have an essential singularity at $\beta=0$ but may have square-root and logarithmic singularities there. For $\beta\ll1$, the functions $k_{1,2}(\beta)$ can be represented by asymptotic expansions of the form

$$k(\beta)=\beta^{-1/2}\sum_{r=0}^{\infty}a_r\beta^r+\beta^{-1}(\ln\beta+c)\sum_{s=0}^{\infty}b_s\beta^s. \tag{29}$$

The following simple example illustrates the expression (28):

$$\int_0\frac{dxe^{-x/\beta}}{(a+x)^2}=\beta\int_0^{\infty}\frac{dye^{-y}}{(a+\beta y)^2}-e^{-\pi/\beta}\beta\int_0^{\infty}\frac{dze^{-z}}{(a+\pi+\beta z)^2}$$
$$=k_1(\beta)+e^{-\pi/\beta}k_2(\beta). \tag{30}$$

The functions $k_{1,2}(\beta)$, which are determined in (30) by integrals, can be represented for $\beta\ll1$ by asymptotic series:

$$k_1(\beta,a)=\frac{\beta}{a^2}-\frac{2\beta^2}{a^3}+\dots,\quad -k_2(\beta,a)=k_1(\beta,a+\pi), \tag{31}$$

the terms of which are obtained from the original integral by successive integrations by parts. This representation can also be generalized to the case when $(a+x)^{-2}$ in the integral is replaced by the functions $\ln x$, $x1/2$, $x\ln x, x^{3/2}, \ldots$ encountered in the calculation.

In contrast to the integral (30), for the functions $\varphi(\beta)$ it is difficult to obtain simple closed expressions for the coefficients $k_{1,2}(\beta)$ in the representation (28). However, the expansion (28), which is a complete asymptotic expansion (see Ref. 13) of the functions φ, can be uniquely constructed by successive integration by parts. The finding of the functions $k_{1,2}(\beta)$ in (28) makes it possible to rewrite the expressions (24) and (25) in the form

$$\operatorname{Im}\mathscr{L}^{(2)}=(2s+1)\frac{(e\varepsilon)^2}{16\pi^3}\sum_{n=1}^{\infty}(\pm 1)^{n+1}\alpha\pi K_n(\beta,s)e^{-n\pi/\beta}, \tag{32}$$

where the function $\alpha\pi K_n(\beta,s)$ no longer has an essential singularity at $\beta=0$ and determines the radiative correction to the nth term of the virial series [see Eq. (4)].

We calculate $K_n(\beta,s)$ for $\beta\ll 1$.

We consider the asymptotic behavior of the integral term of the function $\varphi_n(\beta)$ in (21) as $\beta\to 0$. Using the expression (A5), we obtain for the integrals in the sum over m and corresponding to $m\neq 1$

$$\frac{1}{\beta}\int_0^{\pi}dx'\frac{e^{-x'/\beta}F_m(x')}{(x_n+x')^2}=-e^{-x'/\beta}\left[\frac{F_m(x')}{(x_n+x')^2}-\frac{A}{x'^{1/2}}\right]\Bigg|_0^{\pi}$$
$$+\int_0^{\pi}dx'e^{-x'/\beta}\left[\frac{F_m(x')}{(x_n+x')^2}-\frac{A}{x'^{1/2}}\right]'+\frac{A\pi^{1/2}}{\beta^{1/2}}-\frac{A}{\pi^{1/2}}e^{-\pi/\beta}(1+\ldots),$$
$$A=\frac{-\pi}{2(x_nx_{m-1}x_{n-m+1})^{1/2}}. \tag{33}$$

The integration by parts can be performed arbitrarily many times, the singular terms in the expansions of the integrands at the origin being subtracted each time (as was done for F_m). It follows from the expansion of $F_m(x')$ at the origin that such singularities will give power, square-root, and logarithmic contributions multiplied by positive powers of β,

We give here the results of the final calculation, only the terms that do not vanish as $\beta\to 0$ being given for k_1 and k_2; the intermediate expressions are given for reference in Appendix D of the preprint Ref. 10. We have

$$\varphi_n(\beta)=\frac{2}{\beta x_n}\left(\ln\frac{\gamma x_n}{\beta}-1\right)+\theta(n-2)$$
$$\times\sum_{m=2}^{n}\left[\frac{-\pi^{3/2}\beta^{-1/2}}{(x_nx_{m-1}x_{n-m+1})^{1/2}}+\frac{2}{x_nx_{m-1}}\right] \tag{34}$$
$$+1+\ldots-e^{-\pi/\beta}\sum_{m=1}^{n}\frac{2}{x_mx_{n+1}}+\ldots,$$

$$\varphi_{1n}(\beta)=-\frac{1}{2\beta x_n}\left(\ln\frac{\gamma x_n}{\beta}-\frac{3}{2}\right)$$
$$+\theta(n-2)\sum_{m=2}^{n}\left[\frac{\pi^{3/2}\beta^{-1/2}}{2(x_nx_{m-1}x_{n-m+1})^{1/2}}\right. \tag{35}$$
$$\left.-\frac{\ln(\gamma\pi/\beta)}{2x_{m-1}x_{n-m+1}}\right]+\frac{1}{2x_n^2}+\ldots+e^{-\pi/\beta}\sum_{m=1}^{n}\frac{\ln(\gamma\pi/\beta)}{x_mx_{n+1}}+\ldots$$

The asymptotic behavior of (35) for $\varphi_{1n}(\beta)$ was also calculated in accordance with (33). Integration by parts and the possibility of differentiating infinitely many times the integrands $F_{1m}(x')$ with known expansions in the neighborhood of the points $x'=0$ and $X'=\pi$ make it possible to construct asymptotic expansions of the type (28). As can be seen from (A4) and the calculations indicated in (21) and (22), the asymptotic expressions (34) and (35) are obtained when

$$x'n\sim\beta n\ll 1. \tag{36}$$

Replacing φ_n and φ_{1n} in (24) by their asymptotic expressions (34) and (35), we obtain an asymptotic representation for the function in the square brackets (24):

$$[\;]=\theta(n-2)$$
$$\times\sum_{m=2}^{n}\left(\frac{-\pi^{3/2}\beta^{-1/2}}{2(x_nx_{m-1}x_{n-m+1})^{1/2}}-\frac{\ln(\gamma\pi/\beta)}{2x_{m-1}x_{n-m+1}}+\frac{2}{x_nx_{m-1}}\right)+1 \tag{37}$$
$$+\ldots+e^{-\pi/\beta}\sum_{m=1}^{n}\frac{\ln(\gamma\pi/\beta)-2}{x_mx_{n+1}}+\ldots.$$

Note that the terms of the integral J_2 have canceled completely against the corresponding terms in the asymptotic behaviors of the functions φ_n and φ_{1n}. In the infinite sum (24) there is a further canceling of the terms with equal powers of $\exp(-\pi/\beta)$, so that for the function $K_n(\beta,s)$ from (32) we obtain the simple asymptotic behavior (5), and for the imaginary part of the Lagrange function the expression

$$\operatorname{Im}(\mathscr{L}^{(1)}+\mathscr{L}^{(2)})$$
$$=(2s+1)\frac{(e\varepsilon)^2}{16\pi^3}\sum_{n=1}^{\infty}(\pm 1)^{n+1}\left[\frac{1}{n^2}\right. \tag{38}$$
$$\left.+\alpha\pi\left(\frac{-c_n}{\beta^{1/2}}+1+\ldots\right)\right]e^{-n\pi/\beta},$$

where the ellipsis denotes terms that vanish as $\beta\to 0$. The expression (38) also contains the case of scalar electrodynamics ($s=0$), lower sign).

The appearance of the characteristic square-root singularity in (38) for the terms $n\geqslant 2$ is due to the square-root asymptotic behavior of the integrals $F_m(x')$ and $F_{1m}(x')$, $m=2,3,\ldots,n$ as $x'\to 0$ [see (A5) and (A6)]. This behavior of these integrals is, in turn, due to the function $a-b$ having lines of zeros that, although not entering the regions of integration over the hatched triangles in Fig. 2, do touch them, which makes it impossible to retain in the expansion of the denominator in (A4) only the first term [cf. (A2)].

6. INTERPRETATION

The radiative correction to the n-th term of the virial expansion for $n\beta\ll 1$ obtained in Sec. 5 has the following remarkable property [see (5)]: 1) it does not depend on the spin s; 2) for $n\geqslant 2$ it has a term singular in the limit $\beta\to 0$, in contrast to the correction to the leading term $n=1$; 3) the singular and the constant term have different signs; 4) for $n\gg 1$ the correction depends on n and β only through the product $n\beta$, since $c_n \approx (\pi/2)n^{-1/2}$ for $n \gg 1$. All these properties obtain a simple physical explanation.

Since the terms of the virial expansion have an independent physical meaning, the radiative corrections to them must not increase unboundedly with decreasing field. This suggests that allowance for the further corrections of order $\alpha^2, \alpha^3, \ldots$, will lead to exponentiation of the correction in (38):

$$\left[\frac{1}{n^2}+\alpha\pi\left(\frac{-c_n}{\beta^{1/2}}+1+\ldots\right)\right]\exp(-n\pi m^2/e\varepsilon) \approx \frac{1}{n^2}\exp(-n\pi m_*^2/e\varepsilon), \tag{39}$$

where

$$m_*(n)=m+\frac{1}{2}\alpha n c_n(e\varepsilon)^{1/2}-\frac{1}{2}\alpha n\frac{e\varepsilon}{m}, \quad n\beta\ll 1, \tag{40}$$

i.e., $2\,\mathrm{Im}(\mathscr{L}^1+\mathscr{L}^2+\ldots)$ acquires the form of the virial series (1) with replacement of the mass m in the weak field by the mass $m_*(n)$ determined by the expression (40).

The properties of the radiative correction listed above are not transformed into properties of the mass shift (40) and can be explained by means of a picture of the coherent production of n pairs.[4] The coherent tunneling of n particles separated from each other by the longitudinal distance $\Delta r_{\|} \sim m^{-1}$ takes place as the tunneling of one particle with charge $Q = ne$ and mass $M = mm$ through a barrier of width $2M/Q\varepsilon = 2m/e\varepsilon$. Therefore, to coherent tunneling of n particles there corresponds the quasiclassical exponential $\exp(-n\pi/\beta)$, and their distribution with respect to the transverse momenta has the form $(\bar{n}_p)^n$. Therefore, the effective value of the transverse momentum of a particle in the group is $p_\perp \sim (e\varepsilon/n)^{1/2}$, and the effective transverse distance between the particles is $r_\perp \sim (n/e\varepsilon)^{1/2}$. As a result of the coherent tunneling two narrow groups of oppositely charged particles are formed during the time $\Delta t \sim m/e\varepsilon$ with distance $r_{\|} \approx 2m/e\varepsilon$ between them:

$$\Delta r_{\|} \sim n/m \ll r_\perp \sim (n/e\varepsilon)^{1/2} \ll 2m/e\varepsilon. \tag{41}$$

The conditon (41) is equivalent to the condition $n\beta \ll 1$ (36), under which the asymptotic behavior (38) is valid.

The negative term $-1/2\alpha n(e\varepsilon/m)$ of the mass shift (40) does not depend on $\hbar$ and can be interpreted as the energy of effective attraction of an individual charge to the n charges of the opposite sign in the coherent group at distance $r_{\|} \approx 2m/e\varepsilon$ from it:

$$-\frac{ne^2}{4\pi r_{\|}}\bigg|_{r_{\|}=2m/e\varepsilon} = -\frac{1}{2}\alpha n\frac{e\varepsilon}{m}. \tag{42}$$

For $n = 1$, the negative term of the shift (40) exhausts the entire shift and is equal to the classical mass shift of an accelerated charge found in Ref. 14. This agreement confirms the validity of the exponentiation of not only the singular but also the constant term of the radiative correction to the nth term of the virial expansion.

The positive term $1/2\alpha n c_n(e\varepsilon)^{1/2}$ of the shift (40) depends on $\hbar$ and is the energy of the Coulomb repulsion of an individual charge by the remaining $n-1$ like charges, the partners in the coherent group, which are at distance $r_\perp \sim (n\hbar c/e\varepsilon)^{1/2}$ from it [see (41)].

$$\frac{(n-1)e^2}{4\pi r_\perp}\bigg|_{r_\perp\sim(n\hbar c/e\varepsilon)^{1/2}} \sim \alpha\frac{n-1}{n^{1/2}}(\hbar c e\varepsilon)^{1/2}. \tag{43}$$

The estimates (42) and (43) reproduce all the qualitative features of the shift (40), including the fact that it does not depend on the spin.

Particular attention should be drawn to the agreement between the mass shift for the leading term $n = 1$ and the mass shift of a uniformly accelerated charge.[14] This casts light on the reason for the radiative enhancement of pair production by a weak electric field—the accelerated charges are lighter than the unaccelerated charges, and therefore they can be more readily produced. In addition, this also makes it possible to give a new interpretation of the shift itself: Whereas in the scattering channel "e + field $\to e$ + field" it can be interpreted as a reactive energy[15] or as a manifestation of the clock paradox,[16] in the cross channel "field $\to e^+e^-$ + field" it can be regarded as an effective attraction between the e^+e^- in their region of formation.

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APPENDIX

We find the asymptotic behavior with respect to $x' = x - x_n$ of the integrals

$$F_m(x')=\int_{\xi_{-m}}^{m(n+1)^{-1}} \frac{d\xi\cos x\xi\cos x(1-\xi)}{\xi(1-\xi)(a-b)} \tag{A1}$$

$$F_{1m}(x')=\int_{\xi_{-m}}^{m(n+1)^{-1}} \frac{d\xi c}{\xi(1-\xi)(a-b)^2}, \quad \xi_{-m}=\frac{x_{m-1}+x'}{x_n+x'},$$

assuming that x lies in the $(n+1)$st inclined strip, i.e., $0 < x' < \pi$. As $x' \to 0$, expanding the integrand with respect to x' we obtain

$$F_1(x') = -x_n^2\int_{x'(x_n+x')^{-1}}^{(n+1)^{-1}} \frac{d\xi(\cos^2 x_n\xi+\ldots)}{\sin^2 x_n\xi}\left(1-\frac{x'(S_n-1)}{x_nR_n}+\ldots\right)$$
$$= -\frac{x_n}{x'}-2+x_n\operatorname{ctg}\frac{x_n}{n+1}+\frac{x_n^2}{n+1}+O(x', x'\ln x'). \tag{A2}$$

In the complete range of integration, the second term in the expansion is of order x' relative to the first. We have used the expressions

$$a-b=(-1)^n\left[R_n+\frac{x'}{x_n}(S_n-1)+\ldots\right],$$

$$c=1-R_n\cos 2x_n\xi-\frac{x'}{x_n}S_n\cos 2x_n\xi+\ldots,$$

where R_n and S_n are functions of ξ with the following behavior near the origin and the point m/n:

$$R_n=-\xi-\xi^2+\ldots, \quad S_n=1+\xi+\ldots, \quad \xi\to 0,$$

$$R_n = -\frac{n^2\xi'^2}{m(n-m)}\left[1-\frac{n(n-2m)}{m(n-m)}\xi'+\dots\right],$$

$$S_n = \frac{n(n-2m)}{m(n-m)}\xi'\left[1-\frac{n(n-2m)}{m(n-m)}\xi'+\dots\right],$$

$$\xi'=\xi-\frac{m}{n}\to 0,$$

which is important for calculating the correction terms of the asymptotic behaviors.

Similarly, for $F_{11}(x')$ we obtain

$$F_{11}(x')=\frac{1}{2}\left(\frac{x_n}{x'}\right)^2+\frac{3}{2}\frac{x_n}{x'}+O(1,\ln x'). \tag{A3}$$

The asymptotic behaviors of the integrals $F_m(x')$,$F_{1m}(x')$, $m = 2,\dots,n$ as $x'\to 0$ can be found similarly if instead of ξ we introduce the variable

$$u=x_n\xi'=x_n\left(\xi-\frac{m-1}{n}\right).$$

Then

$$F_m(x')=-x_n\int\limits_{x'x_{n-m+1}(x_n+x')^{-1}}^{x_{n-m+1}(n+1)^{-1}} du\,\frac{1-\sin^2 u+\dots}{\sin^2 u+f+\dots},$$

$$f=\frac{x'x_{m-1}x_{n-m+1}}{x_n}. \tag{A4}$$

The ellipsis in the denominator denotes all the remaining terms, which are $\sim x'u^2, x'^2$, i.e., in the complete range of integration they are of order x' relative to the leading terms that we have given explicitly. The ellipsis in the numerator denotes terms $\sim x'u$, which for the given accuracy can also be ignored (like terms $\sim x'$ if there were any). The integral determined by the leading terms can be calculated exactly [see 1.5.9 (27) in Ref. 17)]:

$$F_m(x')$$

$$=-x_n[f(1+f)]^{-1/2}\tan^{-1}\left[\left(\frac{1+f}{f}\right)^{1/2}\tan u\right]\Bigg|_{x'x_{n-m+1}(x_n+x')^{-1}}^{x_{n-m+1}(n+1)^{-1}}$$

$$+x_n\frac{x_{n-m+1}}{n+1}+\dots=x_n\left[\frac{-\pi}{2f^{1/2}}+\frac{1}{x_{m-1}}+\cot\frac{x_{n-m+1}}{n+1}+\frac{x_{n-m+1}}{n+1}\right]$$

$$+O(x'^{1/2},x'\ln x'). \tag{A5}$$

The difference between the integrals $F_1(x')$ and $F_m(x')$, $m\neq 1$, as $x'\to 0$ is due to the different behavior of the second term of the expansion with respect to x' of the integrand near the lower limits of integration, as a result of which the integral (A2) is basically determined by the first term of the expansion in the region $\xi\sim x'$, while the integral (A4) is determined by both the first and the second term in the region $\xi'=\xi-(m-1)/n\sim x'^{1/2}$. A similar calculation for $F_{1m}(x')$ gives as $x'\to 0$

$$F_{1m}(x')=\frac{\pi x_n^2 x_n^{1/2}x'^{-1/2}}{4(x_{m-1}x_{n-m+1})^{1/2}}-\frac{x_n^3x'^{-1}}{2x_{m-1}x_{n-m+1}}$$

$$+O(x'^{-1/2},\ln x'). \tag{A6}$$

The asymptotic behaviors of the functions F_m and F_{1m} as $X'\to\pi$ are

$$F_m(x')=\frac{x_{n+1}}{x_m}+O(x'-\pi),\quad F_{1m}(x')=-\frac{x_{n+1}^2}{(x'-\pi)x_m}+O(1),$$

$$m=1,\dots,n. \tag{A7}$$

We have used expansion of the denominator $a-b$ in the neighborhood of the point $x=x_{n+1}$. The simplification of the asymptotic behavior is due to the limits of integration with respect to ξ getting closer together: As $X'\to\pi$, we have $\xi_m\to m(n+1)^{-1}$.

[1]We use a system of units in which $\hbar=c=1$, $\alpha=e^2/4\pi\hbar c$, except when it is desirable to emphasize the quantum or relativistic nature of the quantities.

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