Exact nonlinear theory of breakup of an electron beam into individual bunches in a plasma

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Two-stream instability is investigated in the long-wave approximation. Exact nonlinear solutions that describe the bunching and breakup of a beam into separate plasmoids and their motion in the plasma are obtained. An analogy is indicated between the nonlinear stage of the two-stream instability on the one hand and self-modulation and self-focusing of nonlinear waves on the other.

1. Two-stream instability in a plasma is one of the basic examples of a fundamental collective interaction and has extensive and important applications. There is at present an urgent need for developing a theory that takes consistent account of the nonlinear effects in the plasma and in the beam (see, e.g., the review by Dolginov and Toptygin,[1] devoted to the latest results of the theoretical, numerical, and experimental research into bunching and decay of beams into individual plasmoids). We present below a derivation and exact solutions of the equations for the interaction of a monoenergetic beam with a cold plasma in the approximation of long-wave excitations.

2. In the limit $v_0/l = kv_0 \ll \omega_{pe}$ (nonresonant two-stream instability), the quasineutrality condition is valid, i.e., the sum of the values of the beam density n_b and the plasma density n_p is equal to the ion density N:

$$n_p + n_b = N. \tag{1}$$

It is assumed next that the ions are immobile. It follows from (1) that the electron current density does not depend on the coordinates:

$$e(n_b v_b + n_p v_p) = -I(t). \tag{2}$$

Here v_b and v_p are the beam and plasma velocities. We consider below the case when I is independent of time.

Eliminating the electric field E from the plasma and beam equations of motion

$$\partial_t v_p + v_p \partial_z v_p = \partial_t v_b + v_b \partial_z v_b = -\frac{e}{m} E \tag{3}$$

and using the continuity equations for n_b, we obtain under conditions (1) and (2) a closed system of equations

$$\partial_\tau u + u \partial_x u = \partial_x \left(\frac{(1-u)^2 \nu}{1-\nu} \right), \tag{4}$$

$$\partial_\tau \nu + \partial_x (u\nu) = 0. \tag{5}$$

We have introduced here the dimensionless variables $x = z/l$, $\tau = tI/eNl$,

$$u = v_b eN/I, \quad \nu = n_b/N.$$

3. We apply to (4) and (5) the hodograph transformation, i.e., we change from the variables x to τ to u and ν. (More convenient variables are $w = \ln|1-u|$ and $\mu = 1/(1-\nu)$. It follows from (4) and (5) that

$$\partial_u x = [2(\mu-1)(1-u) + u]\partial_\mu \tau - (1-u)\partial_w \tau, \tag{6}$$

$$\partial_w x = u \partial_w \tau + \mu(\mu-1)(1-u)\partial_\mu \tau. \tag{7}$$

Eliminating x from (6) and (7) we get

$$\mu(\mu-1)\partial_{\mu\mu}^2 \tau - 2(\mu-1)\partial_{\mu w}^2 \tau + \partial_{ww}^2 \tau + \partial_w \tau + 2\partial_\mu \tau = 0. \tag{8}$$

The general solution of (8) is expressed in terms of integrals of hypergeometric functions. We confine ourselves here to a few particular solutions.

A relation of the form

$$\tau = \frac{2}{1-u} + 2\ln(1-u) - \frac{\nu}{1-\nu},$$
$$x = 2u + \frac{2}{1-u} + 4\ln(1-u) - \frac{\nu u}{(1-\nu)^2} \tag{9}$$

describes the breakup of the beam into individual parts (Fig. 1). After a finite time the beam density at the point $x = 0$ becomes zero. The velocity and density of the beam near the point $x = 0$ at $|\tau| \ll 1$ have the following coordinate and time dependences:

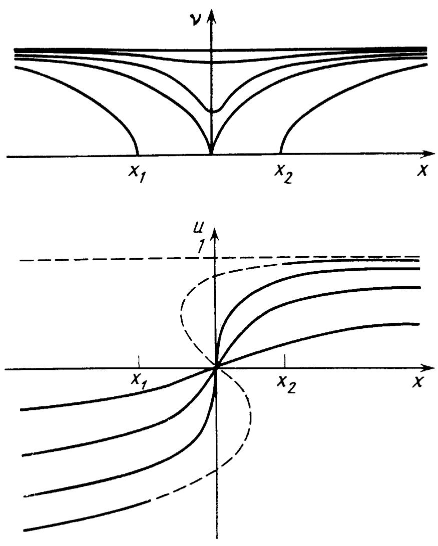

FIG. 1. Velocity and density of the beam vs the coordinates and the time corresponding to the onset of discontinuity.

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$$\nu\approx(x/\tau)^2-\tau,\qquad u=-x/\tau.$$

The density distribution acquires near the singularity as $\tau\to 0$ a characteristic sharply peaked form:

$$\nu\approx(3x/5)^{2/3}-\tau,\qquad u\approx(3x/5)^{1/3}.$$

After formation of the break at $\tau\ll 1$ the individual parts of the beam move in accord with the law $x_{1,2}=\pm 2\tau^{3/2}/3$ at a velocity $u_{1,2}=\pm\tau^{1/2}$. Next, at $\tau\gg 1$ the break expands in such a way that one edge moves with constant velocity $u_2=1$, and the other exponentially $u_1\approx-\exp(\tau)$.

From (6) and (7) we can find the rate of expansion of the break in the general case. If the plasma velocity in the beam has a power-law dependence on the x coordinate, $u\propto x^{\beta}$, the rate of expansion of the break is a powerlaw function of the time: $u_{1,2}\propto\pm\tau^{\beta/(\beta-1)}$ at $\beta\neq 1$. If $\beta=1$, then $u_{1,2}\propto\pm\exp(\tau)$.

Another solution of the system (6) and (7)

$$\tau=\frac{\mu-1+c}{\mu(\mu-1)(1-u)^2},\tag{10}$$

$$x=\tau\left[1+\frac{\mu^2(1+c)-2\mu(2+c)+3}{\mu-1+c}(1-u)\right],$$

where c is a constant, describes at $-1<c<0$ interpenetrating plasma streams. As $x\to+\infty$ the beam density vanishes and the velocity tends to infinity: $u\approx x-\tau+1$. As $x\to-\infty$ the beam density $\nu\to|c|$ and the velocity $u\to 1$.

4. If $\nu\ll 1$, a frequently realized experimental case of low beam densities, the system (4) and (5) can be simplified. In the limit $\nu\ll 1$, $|u|\ll 1$ we obtain from (4) the equation

$$\partial_\tau u+u\partial_x u=\partial_x\nu,\tag{11}$$

which together with (5) is completely identical with the system of equations considered earlier for self-modulation and self-focusing of nonlinear waves (see Refs. 2–4 and the citations there). The solutions of this system are well known.

For example, we consider the evolution of a beam of finite size, in which the density depends parabolically on the coordinates, and the velocity is a linear function of x:

$$\nu=\nu_0(\tau)+\nu_2(\tau)x^2,\qquad u=u_1(\tau)x.$$

As applied to self-focusing this case was investigated in Ref. 5. At $u_1^2(0)<4|\nu_2(0)|$ bunching of the beam takes place, i.e., its density becomes infinite and the beam contracts to a point. Within the framework of these solutions there exists a regime of periodic contractions and expansions, with a period

$$T=\pi/(2(1-u_1^2(0)/4|\nu_2(0)|)^{1/2}).$$

The system (5), (11) has self-similar solutions

$$\nu=\tau^{-\alpha}\Phi(x/\tau^{1-\alpha/2}),\qquad u=\tau^{-\alpha/2}K(x/\tau^{1-\alpha/2}).$$

Expanding these relations in powers of x we find that near the singularity as $x\to 0$ we have

$$\nu\approx|\tau|^{-\alpha}-2(\alpha-\alpha^2)x^2/\tau^2,\qquad u\approx\alpha x/\tau.\tag{12}$$

These relations describe flows in which at $0<\alpha\leqslant\frac{2}{3}$ the density becomes infinite at $\tau\to 0$ and turns to zero at $\alpha<0$. Far from the origin we have

$$\nu\infty|x|^{2\alpha/(\alpha-2)},\qquad u\infty|x|^{\alpha/(\alpha-2)}\operatorname{sign}(-\alpha x).$$

The solutions of the system (5), (11) can also correspond to singularities at which local minima or maxima occur with finite values of the density and with sharply peaked shape (see Ref. 4). The equations employed are then invalid. Multistream flow is formed in the plasma and discontinuities can set in, at which the beam density is zero.

5. The equations obtained and their solutions allows us to investigate, with allowance for the nonlinearity of both the plasma and of the beam, the bunching and the decay of the beam. There exist solutions in which even a beam density that is low at the initial instant of time becomes infinite at individual points [see (12)], i.e., it becomes important to take the plasma nonlinearity into account. This process differs from the bunching that occurs for self-intersecting trajectories in particle beams in vacuum: for example, characteristic acute-angle profiles are produced. Beam decay after a finite time has no analog whatever in the solutions for beams of noninteracting particles and corresponds to toppling of rarefaction waves (see Fig. 1).

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