# Stochastic motion of magnetization vector of superfluid $^3$He

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The superfluid-He$^3$ magnetization motion induced by a periodic sequence of rf pulses is investigated. It is shown that under certain conditions the magnetization motion is stochastic. To describe the stochastic motion of the magnetization, a kinetic equation is derived for the distribution function of points in phase space, and is solved for a stationary case. The obtained distribution function is used to calculate the stationary magnetization produced by the action of a periodic sequence of rf pulses.

**1.** The spin-system dynamics of superfluid $^3$He is described by the system of Leggett's equations.[1,2] Fomin[3] used them to show that the NMR frequency of superfluid $^3$He is subject to a shift that depends on the angle of inclination of the magnetization field. If the alternating field is at resonance at the initial instant with the NMR frequency, the magnetization starts to deviate from the equilibrium position. Deviation of the magnetization, however, is accompanied by a change of the resonance frequency. This in turn upsets the resonance condition and as a result, in sufficiently weak alternating fields in which the condition $\gamma h \ll \omega_p$ is satisfied (here $2h$ is the alternating-field amplitude, $\gamma$ is the gyromagnetic ratio, and $\omega_p$ characterizes the frequency shift), the magnetization inclination angle can be only small, of the order of $\gamma h/\omega_p$. Large inclinations can be obtained, naturally, with high-power pulses with amplitudes $h \gg \omega_p/\gamma$, at which the dynamic character of the shift becomes of little significance. It is difficult, however, to produce such pulses in practice. Osheroff and Corrucini[5] achieved a considerable inclination of the magnetization with pulses satisfying the conditions $\gamma h \gtrsim \omega_p$ and $\gamma h \approx 1/\tau$ ($\tau$ is the pulse duration). The maximum inclination obtained in these experiments agrees with the result of Ref. 6, based on Leggett's equations. Leggett's equations can be solved for arbitrary ratio of $\gamma h$ and $\omega_p$ only by numerical methods.[4]

In this paper is proposed a method of obtaining large inclinations of the magnetization of superfluid $^3$He by a periodic sequence of relatively weak ($\gamma h \lesssim \omega_p$, $\gamma h \ll 1/\tau$) rf pulses.

**2.** Fomin[3] obtained on the basis of Leggett's equations, in the limit of strong magnetic fields $H_0 \gg \Omega_{A,B}/\gamma$, ($\Omega_{A,B}$ are the longitudinal-oscillation frequencies of the $A$ and $B$ phases, respectively), for the nonresonant case realized when the dc field is changed jumpwise by an amount $\Delta H$ of the order of $H$ itself, the following equations of motion for the magnetization[1)]

$$\dot{\theta}=V(\varphi,t), \quad \dot{\varphi}=\omega(\theta)+\frac{\partial V(\varphi,t)}{\partial\varphi}\operatorname{ctg}\theta, \tag{1}$$

where

$$\omega(\theta)=\Delta\omega+\omega_p^{A,B}(\theta), \quad V(\varphi,t)=\gamma h(t)\sin\varphi,$$

$$\Delta\omega=\omega_0-\omega, \quad \omega_0=\gamma H_0, \quad \omega_p^{A,B}(\theta)=\omega_p^{A,B}\cos\theta,$$

$$\omega_p^A=\frac{1}{4}\frac{H_0}{H}\frac{\Omega_A^2}{\omega_0}, \quad \omega_p^B=\frac{2}{5}\frac{H_0}{H}\frac{\Omega_B^2}{\omega_0}, \quad H=H_0-\Delta H,$$

$\theta$ and $\varphi$ are the polar and azimuthal angles of the magnetization vector, the $z$ axis is directed along the constant magnetic field, while $\omega$ and $2h(t)$ are the frequency and slowly varying amplitude of the alternating field applied along the $x$ axis. The system (1) was written in a coordinate frame rotating with frequency $\omega$ around the $z$ axis.

Assume that the alternating field is a sequence of $\delta$-like pulses:

$$h(t)=h_1\sum_{n=-\infty}^{\infty}\delta\left(\frac{t}{T}-n\right), \tag{2}$$

where $T$ is the interval between the pulses. We note that such a representation of the alternating field is valid if the pulse duration $\tau$ satisfies the conditions

$$\omega,\ \omega_0 \gg 1/\tau \gg \Delta\omega, \quad \gamma h, \quad \omega_p^{A,B}, \tag{3}$$

and $h_1$ is connected with the ac field amplitude at the pulse peak by the relation $h \approx h_1(T/\tau)$.

It follows from Eqs. (1), in the weak-pulse approximation $\gamma h\tau \ll 1$, that the jumplike changes of the angles $\theta$ and $\varphi$ assume after the action of the $n$th pulse the form

$$\Delta\theta_n=\omega_1 T\sin\theta_n, \quad \Delta\varphi_n\approx\Delta\alpha_n+K'\cos\varphi_n, \tag{4}$$

where

$$\Delta f_n=f_{n+1}-f_n, \quad f_n=\theta_n,\ \varphi_n,\ \alpha_n,$$
$$\Delta\alpha_n\approx\omega(\theta_n)T-K_n\sin\varphi_n, \quad \omega_1=\gamma h_1, \tag{5}$$
$$K_n=\omega_1\omega_p^{A,B}T^2\sin\theta_n, \quad K_n'=\omega_1 T\operatorname{ctg}\theta_n,$$

$\theta_n$ and $\varphi_n$ are the values of the polar and azimuthal angles at the instant of application of the $n$th pulse.

A variation of the angle similar to that given by (5) for $\alpha_n$ was considered in Refs. 7 and 8 in connection with an investigation of the motion of a nonlinear oscillator acted upon by repeated $\delta$-like jolts. It was shown in these references, in particular, that the values of $\alpha_0, \alpha_1, \ldots$ at $K \gg 1$, where $K$ is the mean value of $K_0, K_1, \ldots$, are statistically independent. The correlation becomes uncoupled after a time $t_0 \approx 2T/\ln K$ shorter than the interval between the pulses. Since the rate of change of the polar angle depends, according to (1) and (4), on the angle $\alpha$, the change of $\theta$ will also be random. The system considered here is therefore, at $K \gg 1$, one example of a manifestation of stochasticity in nonlinear systems with a small number of freedom, which are intensively investigated of late.[7–11]

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3. Following Refs. 7 and 8, we set up a statistical ensemble of the quantities $\{\theta_0,\varphi_0;\theta_1,\varphi_1;...\}$ and write down the Liouville equation for the distribution function $\rho(\theta,\varphi;t)$ of these points in phase space:

$$\frac{\partial}{\partial t}\rho(\theta,\varphi;t)+\frac{\partial}{\partial\theta}[\dot\theta\rho(\theta,\varphi;t)]+\frac{\partial}{\partial\varphi}[\dot\varphi\rho(\theta,\varphi;t)]=0. \quad (6)$$

We expand $\rho$ and $V$ in Fourier series

$$\rho(\theta,\varphi;t)=\sum_{n=-\infty}^{\infty}\rho_n(\theta,t)e^{in\varphi},\quad V(\varphi,t)=\sum_{n=-\infty}^{\infty}V_n(t)e^{in\varphi}. \quad (7)$$

Substituting (7) in (6) and taking the equations of motion (1) into account we obtain for $\rho_n$ the equation

$$\frac{\partial}{\partial t}\rho_n(\theta,t)+in\omega(\theta)\rho_n(\theta,t) + \sum_{n=-\infty}^{\infty}B_{nn'}V_{n'}(t)\rho_{n-n'}(\theta,t)=0, \quad (8)$$

where

$$B_{nn'}=\partial/\partial\theta-[1+n'(n-n')]\operatorname{ctg}\theta. \quad (9)$$

Changing in (8) to the interaction representation with the aid of the transformation

$$\rho_n(\theta,t)\to\rho_n(\theta,t)\exp\left\{-in\int_0^t\omega(\theta)\,dt\right\}, \quad (10)$$

we obtain

$$\frac{\partial}{\partial t}\rho_n(t) =-\sum_{n'=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}B_{nn'}V_{n'k}\rho_{n-n'}(t)\exp i[\Delta\alpha(t)+k\Omega t], \quad (11)$$

where

$$\Delta\alpha(t)=\alpha(t)-\alpha(0)=\int_0^t\omega(\theta)\,dt,\quad \rho_n(\theta,t)\equiv\rho_n(t), \quad (12)$$

and $V_{nk}$ a coefficient of the expansion

$$V_n(t)=\sum_{k=-\infty}V_{nk}e^{ik\Omega t},\quad \Omega=\frac{2\pi}{T}. \quad (13)$$

Since $\theta$ varies slowly with time at $\omega_1 T\ll 1$, we can put

$$\Delta\alpha(t)\approx\omega(\theta)t. \quad (14)$$

It is now convenient to change to the Laplace transformation for $\rho_n(t)$, using

$$g_n(s)=\int_0^{\infty}e^{-st}\rho_n(t)\,dt \quad (15)$$

and to reduce (11), with allowance for (14), to the form

$$g_n(s)=\frac{\rho_n(0)}{s}-\frac{1}{s}\sum_{n'=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}B_{nn'}V_{n'k}g_{n-n'} \times[s-i(n'\omega(\theta)+k\Omega)]. \quad (16)$$

We shall be interested hereafter to the asymptotic behavior as $t\to\infty$, which in turn is equivalent to the limit $s\to 0$. Taking this into account, we iterate Eq. (1) up to second order in the interaction and retain only the resonant terms. This yields

$$g_0(s)=\frac{\rho_0(0)}{s}-\frac{1}{s}\sum_{n>0}\sum_{k>0}B_{0n}\left\{\frac{V_{n-k}\rho_{-n}(0)}{s-i[n\omega(\theta)-k\Omega]}+\text{c.c.}\right\} +\frac{2}{s}\sum_{n>0}\sum_{k>0}B_{0n}\frac{|V_{n,-k}|B_{nn}|V_{n,-k}|}{s^2+[n\omega(\theta)-k\Omega]^2}\rho_0(0), \quad (17)$$

or, returning to the $t$-representation as $s\to 0$,

$$\frac{\partial}{\partial t}\rho_0(t) =-\sum_{n>0}\sum_{k>0}B_{0n}\{\exp[i(n\omega(\theta)-k\Omega)t]V_{n,-k}\rho_{-n}(0)+\text{c.c.}\} +2\pi\sum_{n>0}\sum_{k>0}B_{0n}|V_{n,-k}|\delta(n\omega(\theta)-k\Omega)B_{nn}|V_{n,-k}|\rho_0(0). \quad (18)$$

The terms of first order in the interaction in (18) contain products of oscillating functions and the initial conditions $\rho_n(0)$. It will be shown below that owing to the nonlinear character of the motion of the magnetization, the contribution from the first-order terms, and hence from the initial conditions, becomes negligible in the change of the distribution function. We note that if the magnetization motion were linear, the difference $n\omega(\theta)-k\Omega$ could be made small enough. It would then be impossible to get rid of the first-order terms of (18) by averaging over a specified finite time interval.

If (18) is averaged over the initial value of the azimuthal angle $\varphi_0$, the first-order terms of (18) become proportional to the correlation function

$$R(t)\equiv R_N=\frac{1}{2\pi}\int_0^{2\pi}d\varphi_0\exp i(\alpha_N-\alpha_0). \quad (19)$$

For long times $t\approx NT$, where $N\gg 1$, we have introduced here a discrete time scale (with time interval $T$): $\alpha(t)\equiv\alpha_N$, $\alpha(0)\equiv\alpha_0$. The dependence of $\alpha_N$ on $\varphi_0$ is obtained from (5). The correlation function (19) was considered in Refs. 7 and 8, where it is shown that

$$R_N\infty\exp\{-(N/2)\ln K\},\quad K\gg 1.$$

For times $t\gg t_0$, neglecting the first-order terms, we can therefore obtain from (18) for the distribution function

$$\langle\rho_0(\theta,t)\rangle=\frac{1}{2\pi}\int_0^{2\pi}d\varphi_0\,\rho_0(\theta,t) \quad (20)$$

at $K\gg 1$ the kinetic equation

$$\frac{\partial}{\partial t}\langle\rho_0(\theta,t)\rangle =2\pi\sum_{n>0}\sum_{k>0}B_{0n}|V_{n,-k}|\delta(n\Delta\omega-k\Omega)B_{nn}|V_{n,-k}|\langle\rho_0(\theta,t)\rangle. \quad (21)$$

Taking (9) and (13) into account, assuming that resonance obtains at the overtone $\Delta\omega=k\Omega$, where $k$ is an integer, we get from (21)

$$\frac{\partial}{\partial t}\langle\rho_0(\theta,t)\rangle=\frac{1}{4}\omega_1^2T\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}-\operatorname{ctg}\theta\right)\langle\rho_0(\theta,t)\rangle. \quad (22)$$

A general analysis of (22) is difficult. We analyze therefore a stationary case. Equating the right-hand side of (22) to zero, we get the equation

$$\frac{\partial}{\partial\theta}\langle\rho_0\rangle_{st}=\langle\rho_0\rangle_{st}\operatorname{ctg}\theta, \tag{23}$$

whose solution normalized to unity is

$$\langle\rho_0\rangle_{st}={}^1/_2\sin\theta. \tag{24}$$

With the aid of the distribution function (24) we can calculate the average stationary value of the transverse magnetization produced by a periodic sequence of rf pulses:

$$\bar{M}_\perp=M_0\,\overline{\sin\theta}={}^1/_4\pi M_0\approx 0.78M_0, \tag{25}$$

where

$$\overline{\sin\theta}=\int_0^\pi d\theta\sin\theta\langle\rho_0\rangle_{st},$$

$M_0$ is the value of the equilibrium magnetization.

It is known[5] that the magnetization relaxation time of superfluid $^3$He is of the order of 0.1 sec. This permits $N\approx 50$ pulses at intervals $T\sim 10^{-3}$ sec and duration $\tau\sim 10^{-5}$ sec to be applied before a substantial manifestation of relaxation processes. At a Larmor frequency $\omega_0\approx 10^6$ sec$^{-1}$, a longitudinal-oscillation frequency in the $B$ phase (Ref. 12) $\Omega_B\approx 2\times 10^5$ sec$^{-1}$, and $\Delta H/H_0=3/2$ we obtain for the frequency shift $\omega_p^B\approx 4.10^4$ sec$^{-1}$. The stochasticity condition a $K\approx\omega_1\omega_p^B T^2\gg 1$ and the weak-interaction condition $\omega_1 T\ll 1$ are then satisfied at $\omega_1\approx 200$ sec$^{-1}$, corresponding to a pulse-peak amplitude $h\approx 1$ Oe.

Thus, a periodic sequence of rf pulses can induce stochastic motion of the magnetization of superfluid $^3$He, and the transverse magnetization attainable thereby is given by expression (25).

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[1)]We neglect here the $\theta$-independent part of the resonance-frequency shift, which is immaterial in the present analysis.

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