Neutron scattering in liquid He^3

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The spectrum of the short-wave sound excitations in liquid He^3 was determined. We observed purely diffusive transverse sound in addition to the weakly damped longitudinal sound.

1. RESULTS

1. In their experiments[1,2] on neutron scattering in liquid He^3, Sköld *et al.* measured the dynamic structure factor $S(k,\omega)$ for wave vectors in the wave-vector range $0.5p_F < k < 3p_F$ is the Fermi momentum) and at frequencies $\omega \lesssim 30$ K. The plot of S vs ω has two characteristic broad peaks. One is connected with the fluctuations of the total density of the liquid, i.e., with the longitudinal sound oscillations, and the other with the spin-density fluctuations, i.e., with the paramagnons. When the temperature T is changed from 0.015 to 1.2 K the sound peak remains clearly pronounced, while the paramagnon peak broadens greatly. A theoretical investigation by A. Akhiezer, I. Akhiezer, and Pomeranchuk[3] has shown that in the long-wave approximation $k \rightarrow 0$ the experiments on neutron scattering determine the spectrum of the zero sound whose existence was predicted by Landau.[4] Calculations by Aldrich, Pethick, and Pines[5,6] in the short-wave region $k \sim p_F$ gave no explanation why the sound peak of $S(\omega)$ is unusually broad. Although an attempt was made in Ref. 6 to generalize the Fermi liquid theory to include the region $k \sim p_F$, they used a typical gas approach connected with the assumption that a single-particle spectrum $\varepsilon = \varepsilon_p$ exists, with small damping, i.e., an approach based on the Landau theory in its traditional formulation.

2. We show in the present paper that a good single-particle $\varepsilon = \varepsilon_p$ spectrum is an unnecessary "luxury" for the theory. The Landau theory is based only on the Landau-Luttinger theorem that the numbers of the particles and quasiparticles are equal, on the Ward-Pitaevskiĭ identity, and on the results obtained by Galitskiĭ and Migdal[7] on the analytic properties of the Green's functions. The only important assumption is that there exists in the system some mechanism of single-particle motion. The single-particle density of states $n(\varepsilon)$ should be different from zero at $\varepsilon \approx 0$, i.e.,

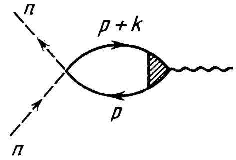
$$n(\varepsilon) = -2i \int \frac{d^3p}{(2\pi)^3} \operatorname{Im} G(p,\varepsilon) \neq 0. \qquad (1)$$

It turns out that this assumption alone is sufficient to develop a reasonable theory of liquid He^3. The fundamental difference between a Fermi gas and a Fermi liquid consists of the anomalously small jump $a \ll 1$ of the particle momentum distribution function n_p and of a strong difference between the physical scattering amplitude Γ^k from the unphysical one Γ^ω, i.e., $\Gamma^\omega \gg \Gamma^k$. For a gas, the particles practically coincide with the quasiparticles: $1 - a \ll 1$, $\Gamma^\omega \approx \Gamma^k$, and a theory can be constructed in terms of the spectrum $\varepsilon = \varepsilon_p$ and the function n_p. At $a = 0$, when the single-particle excitation branch vanishes, no catastrophe occurs in the theory. The condition $n(\varepsilon) \neq 0$ imposes a restriction only on the analytic properties of G as a function of p^2, but not of ε, and manifests itself in the existence of a pole of G in the p^2 plane:

$$G(p^2,\varepsilon) = b \Big/ \left[B(\varepsilon) - \frac{p^2 - p_F^2}{p_F^2} \right], \quad \frac{p^2 - p_F^2}{p_F^2} < 1. \qquad (2)$$

We have naturally arrived at the concept of a spectrum of single-particle excitations $p^2 = p^2(\varepsilon)$, and the ratio of Re B and Im B can be arbitrary.

3. The main result of this paper is observation of a pure diffusive transverse excitation branch with spectrum $\omega = -ick$ at $k \sim p_F$. Its existence is a concrete manifestation of the similarity between a liquid and a solid. A melting He^3 crystal loses transverse rigidity and long-range order, but two excitation branches cannot vanish without trace. The transverse sound in the liquid attenuates and cannot propagate in the usual sense. It is difficult to observe in experiment. The point is that sound excitation in a liquid in the course of neutron scattering proceeds in two stages. The neutron produces first a single-pair excitation, after which it goes over into an acoustic quantum and "runs" through the system:



$$= \sigma_{nHe^3}(G_p G_{p+k} g_{p,k}). \qquad (3)$$

For transverse oscillations the sound-production block $g \sim e^{i\varphi}$, where φ is the azimuthal angle of the vector **p** (the vector **k** is directed along the z axis). On the other hand the cross section σ_{nHe^3} for neutron scattering by an He^3 atom is independent, with high accuracy, of this angle. Therefore the averaging over the direction of p in (3) makes the amplitude of transverse-sound excitation in an isotropic medium equal to zero. In a system with long-range order, however, the function G in (3) depends on the angles of **p** and $\mathbf{k}_i$, where $\mathbf{k}_i$ are the periods of the crystal reciprocal-lattice structure, and this amplitude is finite after averaging over the **p** direction.

4. The crucial point of the theory is the question of the fluctuations of the spin density ω_k. It is purely diffusive and corresponding to it are a pole of the particle-scattering amplitude $\Gamma(\omega)$ at $\omega = -i\omega_k$ and a peak of the function S with an ω-dependence in the form

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$$S(\omega)\propto\frac{\omega}{\omega^2+\omega_k^2}. \qquad (4)$$

The experimental accuracy is such[1,2] that it is impossible to determine the dependence of ω_k on k. Two types of spectrum are admissible: phonon, when ω_k is an increasing function of k, viz.,

$$\omega_k=ck+c_1k^3, \quad c_1>0,$$

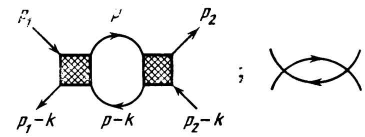
and roton,

$$\omega_k=ck(1+\gamma^2(k^2-k_0^2)^2),$$

when the frequency ω_k has a minimum $k=k_0$. The results of the present paper are not sensitive to the form of the spectrum, and in the actual calculations we shall use the averaged spectrum $\omega_k=\omega_D(k/k_D)$, i.e., we introduce in the usual manner the Debye frequency ω_D and momentum k_D, of the order of $\omega_D\sim 1$ K and $k_D\sim p_F$. As shown in Ref. 8, the role of the spin-density fluctuations is analogous to the role of the phonons in a solid. Exerting a strong influence on the single-particle spectrum near the Fermi surface, they are rapidly turned off slightly away from this surface. The Landau quasiparticles exist both near ($\varepsilon\ll\omega_D$) and far ($\varepsilon\gg\omega_D$) from the Fermi surface. At $\varepsilon<\omega_D$ these are heavy quasiparticles, dressed by a paramagnetic "jacket," with $m^*\approx 3m$, while at $\varepsilon>\varepsilon_D$ these are light bare quasiparticles with $m_0^*\approx 0.3m$. It was found that the paramagnons are unimportant for the determination of the spectrum of the short-wave excitations with $k\sim p_F$. We began to understand the nature of these excitations only after we had learned to neglect in correct fashion the spin-density fluctuations and had seen that the longitudinal and transverse sounds are excitations of a gas of light bare quasiparticles. In the Landau limit $k\to 0$ and $\omega\to 0$, on the contrary, correct account of the paramagnons is necessary, in which case the sounds are excitations in a system of heavy quasiparticles. It is typical that the velocities of both the Landau zero sound and of the short-wave longitudinal sound are almost equal to the velocity u_0 of the usual thermodynamic sound: $u_0^2=(n/m)\partial\mu/\partial n$, where μ is the chemical potential and n is the density. As $m^*\to\infty$ and $m_0^*\to 0$ this equality is exact. What is customarily called zero sound at $k\sim p_F$, is simply ordinary sound in a liquid which for known reasons has not solidified, while the statistics of the particles does not matter here. The short-wave sound branch is located in the region $\omega<kv_0$, where $v_0=p_F/m_0^*$, i.e., where it should attenuate strongly in accord with the Fermi-gas theory. Nevertheless it attenuates weakly, and moreover its attenuation vanishes as $m_0^*\to 0$. There is no paradox here: we have found the sound-excitation spectrum typical of any cold liquid. The parameters of the Landau theory were themselves chosen such that in the short-wave region, $k\sim p_F$, the effects connected with the statistics are inconsequential.

2. SPIN-DENSITY FLUCTUATION SPECTRUM

In the long-wave limit $\omega\to 0$ and $k\to 0$ the dependence of the scattering amplitude Γ on ω and k is determined in the Landau theory and is connected with the "dangerous" cross section of this amplitude in the particle—hole channel, shown in the diagram



$$=\Pi(\mathbf{k},\omega)=-2i\int G_pG_{p-k}\frac{d^3p\,d\varepsilon}{(2\pi)^4}. \qquad (5)$$

On the basis of only the analytic properties of the Green's functions[7] we shall show that the principal dependence of Γ on ω is connected with the cross section not only for small k but for all $k<2p_F$ and small ω. To this end, using the connection between G and the function G^R and G^A (Ref. 7), we shall turn the contour of the integration with respect to ε in (5) from the right-hand semiaxis into the upper ε half-plane and write

$$\operatorname{Im}\Pi(\mathbf{k},\omega)=4\int_0^\omega \operatorname{Im}G_p\operatorname{Im}G_{p-k}\frac{d^3p\,d\varepsilon}{(2\pi)^4}, \qquad \omega>0,$$

$$\operatorname{Re}\Pi(\mathbf{k},\omega)=4\int_{-\infty}^0 \operatorname{Im}G_p\{\operatorname{Re}G_{p+k}+\operatorname{Re}G_{p-k}\}\frac{d^3p\,d\varepsilon}{(2\pi)^4}. \qquad (6)$$

The imaginary part of Π is determined by an integral over a narrow region $\varepsilon<\omega$, and the real over a wide one. Therefore $\operatorname{Im}\Pi$ depends more strongly on ω than $\operatorname{Re}\Pi$, namely:

$$\operatorname{Im}\Pi\propto\omega, \quad \operatorname{Re}\{\Pi(\omega)-\Pi(0)\}\propto\omega^2, \quad \omega\to 0.$$

We introduce now a parametrization of $G(p,\varepsilon)$ (Ref. 8), We expand the smooth functions in powers of ε and $p^2-p_F^2$, separate the contribution of the spin excitations, after which we obtain in self-consistent manner:

$$a_0G^{-1}(\mathbf{p},\varepsilon)=\varepsilon-\frac{p^2-p_F^2}{2m_0^*}-a_0\Sigma^\sigma(\varepsilon), \quad \Sigma=\Sigma^R+\Sigma^\sigma. \qquad (7)$$

The quantities a_0 and m_0^* are connected with the expansion of the smooth part of Σ:

$$\frac{\partial\Sigma^R}{\partial\varepsilon}=1-\frac{1}{a_0}, \quad \frac{m}{m_0^*}=\left(1+2m\frac{\partial\Sigma^R}{\partial p^2}\right)a_0.$$

It is essential that Σ^σ is a strong function of ε, but a weak one of p^2. We can threfore integrate with respect to p^2 without expanding the dependence of Σ^σ on ε. Since $\operatorname{Im}\Sigma$ vanishes at $\varepsilon=0$ (Ref. 7), there exists an interval where $\operatorname{Im}\Sigma$ is small but $\operatorname{Im}G$ is a δ function of $p^2-p_F^2$:

$$\operatorname{Im}G(p^2,\varepsilon)$$
$$=\operatorname{Im}\Sigma(\varepsilon)\Big/\left[\left(\frac{p^2-p_F^2}{2m_0^*}\right)^2+(\operatorname{Im}\Sigma(\varepsilon))^2\right]\propto\delta(p^2-p_F^2).$$

The real part of Σ is inessential here, and is therefore omitted. We can now dispense in (5) with integration with respect to p^2 and with respect to the angle between p and k:

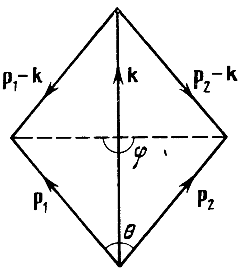
$$\operatorname{Im}\Pi(\mathbf{k},\omega)$$
$$\approx-\frac{|\omega|}{kv_0}v_0\operatorname{arctg}\frac{kv_0}{\operatorname{Im}\Sigma(\omega)}\delta(p^2-p_F^2)\delta((\mathbf{p}-\mathbf{k})^2-p_F^2)\delta(\varepsilon). \qquad (8)$$

Here $\nu_0 = p_F^2 a_0^2/\pi^2 v_0$ is the unrenormalized density of states and $v_0 = p_F/m_0^*$ velocity of the bare quasiparticles. For all the k of significance here, the arctangent in (8) can be replaced by $\pi/2$. In the derivation of (8) we made only one assumption: Σ is expanded in terms of $p^2 - p_F^2$; the dependence of Σ on ε turns out to be inessential, since the unrenormalized jump a_0 of the particle momentum distribution function n_p is cancelled out in the combination ν_0/v_0 in (8), so that this expression valid also at $a = 0$.

Since both momenta p and $p - k$ in (5) turned out to be clamped to the Fermi surface at all $k < 2p_F$, a closed equation for the amplitude $\Gamma(p_1, p_2, k)$ can be obtained when all its external momenta lie on this surface. At $\omega = 0$ the amplitude in questions the one contained in the collision integral in the kinetic equation:

$$\Gamma(\omega)=\Gamma(0)+\Gamma(0)\delta\Pi(\omega)\Gamma(\omega), \quad \delta(\Pi)=\Pi(\omega)-\Pi(0). \quad (9)$$

In (9) there is only one integration with respect to the azimuthal angle of the momentum **p**; see Eq. (5) and



$$p_1 = p_2 = |\mathbf{p}_1 - \mathbf{k}| = |\mathbf{p}_2 - \mathbf{k}| = p_F \quad (5')$$

If Γ is expanded in a Fourier series in the angle $\varphi = \varphi_1 - \varphi_2$:

$$\Gamma(k, 0, \varphi) = \sum_m D_m(k, 0)\, e^{im\varphi},$$
$$\Gamma(k, \omega, \varphi) = \sum_m D_m(k, \omega)\, e^{im\varphi}, \quad (10)$$

Eqs. (9) for the different terms of the series are separated. From (9) and (10) we have

$$D_m(k, \omega) = -\frac{g^2(k)}{\omega_m(k) - i|\omega|}. \quad (11)$$

We have introduced the notation

$$g^2 = \frac{2kv_0}{\pi\nu_0}, \quad \omega_m(k) = -\frac{2}{\pi} k v_0 \nu_0^{-1} D_m^{-2}(k, 0).$$

Expression (11) has the standard "sound" form. Strong attraction for one of the quantum numbers m, i.e., $-D_m^{-1}\nu_0^{-1} \ll 1$, leads to the appearance of a soft diffusion branch of excitations with spectrum $\omega = -i\omega_k, \omega_k \ll kv_0$. The onset of excitations with imaginary frequencies does not contradict the stability conditions, as explained in detail in Refs. 8 and 9. We note that in the limit as $k \to 0$ the quantities D_m do not go over into the functions Γ_l^k of the Landau theory. Thus, the zeroth harmonic of D_0 in terms of the angle φ is connected with $\Gamma(k, \cos\theta)$ by the relation

$$D_0(k, 0) = \frac{1}{\pi} \int_{\cos\theta_0}^{1} \Gamma(k, \cos\theta) \frac{d\cos\theta}{(1-\cos\theta)^{1/2}(\cos\theta - \cos\theta_0)^{1/2}},$$

$$\cos\theta_0 = \frac{k^2}{2p_F^2} - 1. \quad (12)$$

Here θ is the angle between the vectors p_1 and p_2 [see (5')]. At $k = 0$ we have

$$D_0(0, 0) = \frac{1}{\pi} \int_{-1}^{1} \Gamma^k(\cos\theta) \frac{d\cos\theta}{\sin\theta}.$$

The generally accepted representation for $\Gamma_0(\omega)$ at small k and ω turned out to be inexact:

$$\Gamma_0(\omega) \neq \frac{\Gamma_0^k}{1 - \Gamma_0^k \delta\Pi(k, \omega)}.$$

The expansions of Γ in terms of φ and θ are not repetitions of each other.

3. SINGLE-PARTICLE SPECTRUM

1. We separate now the contribution made to Σ (8) by the spin excitations with $m = 0$:

$\Sigma^\sigma(\varepsilon) =$ 

$$= 3i \int \frac{d^3k\, d\omega}{(2\pi)^4} D_0^\sigma(k, \omega) \{G(\mathbf{p}+\mathbf{k}, \varepsilon+\omega) - G(\mathbf{p}+\mathbf{k}, \omega)\}. \quad (13)$$

The wavy line in (13) corresponds to $D_0^\sigma(k,\omega)$. The Lehmann expansion for Σ (Ref. 10) guarantees that the contribution of D to Σ enters with the vertices $g(k)$ defined in (11), the question of renormalizing the vertices does not arise. The principal part of $\Sigma^\sigma(\varepsilon)$ was calculated long ago by the known Migdal procedure[11]: integrate first with respect to the angle between p and k and then with respect to ω. Just as for $\delta\Pi(\omega)$, the strongest dependence of Σ^σ on ε is associated with the integration over the narrow interval of ω in (13):

$$a_0 \operatorname{Re}\Sigma^\sigma(\varepsilon) = -\frac{3v_0}{2\pi p_F^2} \int_0^{2p_F} k^2 dk \operatorname{arctg}\frac{\varepsilon}{\omega_k},$$
$$a_0 \operatorname{Im}\Sigma^\sigma(\varepsilon) = -\frac{3v_0}{4\pi p_F^2} \operatorname{sign}\varepsilon \int_0^{2p_F} k^2\, dk \ln\left(1 + \frac{\varepsilon^2}{\omega_k^2}\right). \quad (14)$$

To track the transition to the Landau theory, we consider the limit as $\varepsilon \to 0$:

$$aG^{-1}(\mathbf{p}, \varepsilon) = \varepsilon - v(p - p_F) + i\delta \operatorname{sign}\varepsilon,$$
$$v = v_0 \frac{a}{a_0}, \quad \nu = \frac{a_0^2 p_F^2}{\pi^2 v} = \nu_0 \frac{a}{a_0}, \quad (15)$$
$$\frac{m^*}{m_0^*} = \frac{a_0}{a} = 1 + \frac{3v_0}{2\pi p_F^2} \int_0^{2p_F} \frac{k^2\, dk}{\omega_k}.$$

We express a_0/a also in terms of $D_0^\sigma(k,0)$:

$$\frac{a_0}{a} = 1 - \frac{3}{2} \int_0^{4p_F^2} D_0^\sigma(k, 0) \frac{dk^2}{4p_F^2} \nu_0, \quad (16)$$

$$\frac{a}{a_0} = 1 + \frac{3}{2} \int_0^{4p_F^2} D_0^\sigma(k, 0) \frac{dk^2}{4p_F^2} \nu. \quad (17)$$

Allowance for the spin-density fluctuations changed the density of states ν_0, the residue a_0, and the velocity v_0 by the same factor a_0/a. The softer the ω_k spectrum, the heavier the quasiparticles and the lower the density of states ν. In the limit as $\omega_k \to 0$, according to (15), the bare mass m_0^* does not enter in the connection of m^* with ω_k:

$$m^* = m_0^* + \frac{3}{2\pi p_F} \int_0^{2p_F} \frac{k^2 dk}{\omega_k}.$$

It can be seen from (16) that since $D_0^\sigma(k,0) < 0$, the quantity D_0^σ should be made nondimensional not by the true density of states ν, but by the unrenormalized ν_0. The slight error incurred when a/a_0 is calculated by (17) can lead to a negative value of a, since $a \ll a_0$.

To get a feel of the dependence of Σ on ε, we carry out the calculations for the simplest spectrum $\omega_k = \omega_D (k/k_D)$; in this case

$$a_0 \operatorname{Re} \Sigma^\sigma = -\frac{\varepsilon_F{}^0 k_D{}^3}{\pi p_F{}^3} \left\{ \operatorname{arctg} y + \frac{1}{2} y - \frac{1}{2} y^3 \ln\left(1 + \frac{1}{y^2}\right) \right\},$$

$$a_0 \operatorname{Im} \Sigma^\sigma = -\frac{\varepsilon_F{}^0 k_D{}^3}{2\pi p_F{}^3} \left\{ \ln(1+y^2) + 2y^2 - 2y^3 \operatorname{arctg} \frac{1}{y} \right\} \operatorname{sign} \varepsilon, \tag{18}$$

where

$$y = \frac{\varepsilon}{\omega_D}, \quad \varepsilon_F{}^0 = \frac{p_F{}^2}{2m_0^*}, \quad \frac{a_0}{a} = 1 + \frac{\varepsilon_F{}^0}{\omega_D} \frac{3k_D{}^3}{2\pi p_F{}^3}.$$

We present also the expressions for the limiting values of G:

$$aG^{-1}(\mathbf{p}, \varepsilon) = \varepsilon - (p - p_F) v + i \frac{\varepsilon |\varepsilon|}{\omega_D}, \quad \varepsilon < \omega_D,$$

$$a_0 G^{-1}(\mathbf{p}, \varepsilon) = \varepsilon - (p - p_F) v_0 + \omega_0 \operatorname{sign} \varepsilon \left\{ 1 + \frac{i}{\pi} \ln \frac{\varepsilon^2}{\omega_D{}^2} \right\}, \tag{19}$$

$$\omega_0 = k_D v_0 \left(\frac{k_D}{2p_F} \right)^2, \quad \omega_D < \varepsilon.$$

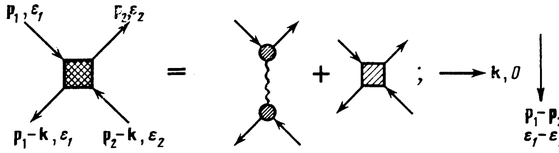
At $\varepsilon > \omega_D$ the contribution of the spin excitations to $\operatorname{Im} \Sigma$ diverges logarithmically, and at $\varepsilon \sim k_D v_0$ this growth stops. There is a fundamental difference between diffusion excitations and ordinary phonons. The propagation function D of a phonon decreases rapidly like ω^{-2} with increasing ω, while the diffusion excitations are turned off slowly: $D \propto \omega^{-1}$ at $\omega > \omega_D$. The criterion for the applicability of pertubation theory for diffusion is therefore also more stringent: $(k_D/2p_F)^3 < 1$. This condition limits the phase space of the fluctuations. If $(\varepsilon_F^0/\omega_D)(k_D/2p_F)^3 < 1$, the spin fluctuations make a small contribution and $m_0^* \approx m^*$; $a_0 \approx a$; $v_0 \approx v$. If, however, the inequality is reversed, the properties of the system are greatly altered, but only in a narrow region near the Fermi surface. Besides the regions $\varepsilon < \omega_D$ and $\varepsilon > \omega_D$ there is an intermediate region $\omega_D < \varepsilon < \omega_0$ where G is practically independent of ε:

$$a_0 G^{-2}(\mathbf{p}, \varepsilon) = -v_0 (p - p_F) + i \operatorname{sign} \varepsilon \frac{\omega_0}{\pi} \ln \frac{\varepsilon^2}{\omega_D{}^2}.$$

The appearance of one more energy scale ω_0 on top of ω_D complicates the problem greatly, since it is unknown whether there exists an interval $\omega_0 < \varepsilon < \varepsilon_F^0$, in which bare particles with effective mass m_0^* can be correctly defined. We shall assume that such an interval exists. From a comparison of the experimental and calculated dependences of the heat capacity $C(T)$ and of the entropy on T it follows that for He^3 we have $\omega_D \sim 0.5$–1 K, $\varepsilon_F^0 \sim 12$–20 K, $k_D \sim p_F$; $\omega_0 \sim 2$–5 K; $m_0^* \sim 0.25$–$0.4m$.

4. SPECTRUM OF SOUND EXCITATIONS

1. We obtain the sound spectrum as the poles of the spin-independent amplitude $\Gamma(\mathbf{k},\omega)$. Just as for $\Sigma(\mathbf{p},\varepsilon)$, we separate in Γ the abrupt part connected with the paramagnons, and regularize the regular part[8]:



(20)

In analytic form this means:

$$\Gamma(\mathbf{p}_1, \varepsilon_1, \mathbf{p}_2, \varepsilon_2, \mathbf{k}, 0) = \frac{3}{2} \frac{g^2(\mathbf{p}_-)}{\omega_{\mathbf{p}_-} - i|\varepsilon_-|} + \Gamma^R(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}). \tag{20'}$$

We have parametrized the amplitude $\Gamma(k,0)$ which is static in the ω channel, i.e., we are transfering from left to right in (20) a momentum k, a zero frequency $\omega = 0$, and a zero spin. Along the second particle—hole channel (downward) are transferred a momentum $\mathbf{p}_- = \mathbf{p}_1 - \mathbf{p}_2$, a frequency $\varepsilon_- = \varepsilon_1 - \varepsilon_2$, and a zero spin. The retardation with respect to the variable ε_- is taken into account only in the D-function of the spin excitations, and Γ^R is a static amplitude also in ε_-. The equation for $\Gamma(\omega)$ is the same as for the spin excitations (9). Here, however, we are interested in small $\omega \ll \varepsilon_F^0$ and small $k \ll 2p_F$, and the important relation is between ω, kv, and kv_0. It is therefore not enough to retain in $\delta\Pi$ only $\operatorname{Im} \delta\Pi$, but account must be taken also of $\operatorname{Re} \delta\Pi$. At small ω and k, however, we can use in the calculation of $\delta\Pi$ also the representation of G in the form (7) and integrate first with respect to p in (9). In view of the rapid convergence of this integral we can close the contour of the integration with respect to p in the upper half-plane:

$$\Gamma(\mathbf{n}_1, \varepsilon_1, \mathbf{n}_2, \varepsilon_2, \mathbf{k}, \omega) = \Gamma(\mathbf{n}_1, \varepsilon_1, \mathbf{n}_2, \varepsilon_2, \mathbf{k}, 0) - \nu_0 \int_{-\omega/2}^{\omega/2} d\varepsilon \int \frac{d\Omega_\mathbf{n}}{4\pi}$$

$$\times \frac{\Gamma(\mathbf{n}_1, \varepsilon_1, \mathbf{n}, \varepsilon, \mathbf{k}, 0)\,\Gamma(\mathbf{n}, \varepsilon, \mathbf{n}_2, \varepsilon_2, \mathbf{k}, \omega)}{\mathbf{k}\mathbf{n}v_0 - \omega + a_0\Sigma^\sigma(\varepsilon + \omega/2) - a_0\Sigma^\sigma(\varepsilon - \omega/2)}. \tag{21}$$

Equation (21) for $\Gamma(\omega)$ is closed with respect to the momenta: they all lie on the Fermi surface: $\mathbf{n}_1 = \mathbf{p}_1/p_F$, $\mathbf{n}_2 = \mathbf{p}_2/p_F$, $\mathbf{n} = \mathbf{p}/p_F$. The equation is closed also with respect to the frequencies ε_1, ε_2, and ε, since the integration interval in (21) is finite: $-\omega/2 < \varepsilon < \omega/2$.

According to (14) and (20), $\Gamma(\omega = 0)$ and $\Sigma^\sigma(\varepsilon)$ are functionals of the spectrum of the spin fluctuations ω_k, therefore the solution (21) also depends on the form of ω_k. There are, however, two limiting regions $\omega < \omega_D$ and $\omega > \omega_0 \ln(\omega_0/\omega_D)$ with $\omega_0 = k_D^3 v_0/4p_F^2$, where the detailed dependence of ω_k on k is unimportant. The first region is the Landau limit; the retardation with respect to ε_- is inessential here since a frequency $\varepsilon_- < \omega_D$ is transferred in the transverse channel. In this region

$$\Gamma(\mathbf{n}_1, \mathbf{n}_2) = \Gamma^k(\mathbf{n}_1, \mathbf{n}_2) + \nu\omega \int \frac{d\Omega_v}{4\pi} \frac{\Gamma^k(\mathbf{n}_1, \mathbf{n}) \Gamma(\mathbf{n}, \mathbf{n}_2)}{\omega - \mathbf{kv}}. \quad (22)$$

In the second region the retardation is so much stronger, that the contributions of the spin excitations to Γ and Σ can be neglected, since the characteristic value $\varepsilon_- \sim \omega > \omega_0 \ln(\omega_0/\omega_D)$ and therefore

$$\Gamma(\mathbf{n}_1, \mathbf{n}_2) = \Gamma^R(\mathbf{n}_1, \mathbf{n}_2) + \nu_0\omega \int \frac{d\Omega_{v_0}}{4\pi} \frac{\Gamma^R(\mathbf{n}_1, \mathbf{n}) \Gamma(\mathbf{n}, \mathbf{n}_2)}{\omega - \mathbf{kv}_0}. \quad (23)$$

The difference between (22) and (23) is that $\Gamma^k, \nu, v \to \Gamma^R, \nu_0, v_0$. We now represent Γ^R in the form

$$\Gamma^R(\mathbf{n}_1, \mathbf{n}_2) = \Gamma_0^R + \Gamma_1^R \mathbf{n}_1 \mathbf{n}_2.$$

If we confine ourselves to this representation, we can find Γ_0^R and Γ_1^R from experiment: according to (20), the vertex Γ coincides at $\varepsilon_1 = \varepsilon_2 = 0$ with Γ^k of the Landau theory, and the contribution $\Gamma - \Gamma^R$ of the spin excitations is connected with the quantity m^*/m_0^*. The first two harmonics Γ_0^k and Γ_1^k, however, are known from data on the heat capacity and the compressibility of the liquid, so that

$$\Gamma_0^k \nu_0 = \Gamma_0^R \nu_0 + \frac{m^*}{m_0^*} - 1, \quad \Gamma_1^k \nu_0 = \Gamma_1^R \nu_0 + 3\left(\frac{m^*}{m_0^*} - 1\right). \quad (24)$$

To obtain (24) we used the connection between the momentum and the angle: $p_-^2 = 2p_F^2(1 - \mathbf{n}_1 \cdot \mathbf{n}_2)$, and neglected the corrections $\sim k_D^2/4p_F^2$. According to the Landau theory the quantity $\Gamma_1^k \nu$ is connected with m^*/m, so that (24) leads to a connection of m/m_0^* with Γ_0^R and Γ_1^R:

$$m/m_0^* = 1 - \Gamma_1^R \nu_0/3, \quad (1 - \Gamma_0^k \nu) m^*/m_0^* = 1 - \Gamma_0^R \nu_0,$$

$$m/m^* = 1 - \Gamma_1^k \nu/3. \quad (25)$$

In the Landau notation[4] we have

$$\Gamma_0^k \nu = A_0, \quad F_0 = \frac{A_0}{1 - A_0}, \quad \Gamma_1^k \nu = A_1, \quad F_1 = \frac{A_1}{1 - A_1/3}.$$

It is natural to introduce a corresponding notation also for Γ^R:

$$\Gamma_0^R \nu_0 = A_0^0, \quad F_0^0 = \frac{A_0^0}{1 - A_0^0} \quad \Gamma_1^R \nu_0 = A_1^0, \quad F_1^0 = \frac{A_1^0}{1 - A_1^0/3}.$$

It can be seen from (25) that m/m_0^* is also connected with A_1^0 as m/m_0^* is connected with A_1. This analogy can be continued even farther: the speed of sound u_0, according to the second relation in (25), can be expressed in terms of m^* and F_0 as well as in terms of m_0^* and F_0^0:

$$u_0^2 = \frac{p_F^2}{3mm^*}(1 + F_0) = \frac{p_F^2}{3mm_0^*}(1 + F_0^0). \quad (26)$$

This result was obtained earlier in Refs. 8 and 9: in the self-consistent-field approximation the spin excitations make no contribution to the compressibility of the liquid. We can now estimate the first two harmonics A_0^0 and F_0^0: $0 < F_0^0 < 0.6$; $-9 < A_1^0 < -4.5$. We recall that $F_0 = 11$ and $A_1 = 2$. The quantities F_0^0 and A_1^0 are not defined because m_0^* is not uniquely determined from the data for the heat capacity at high temperatures: $0.25m < m_0^* < 0.4m$. Since $-A_1^0 \gg F_0^0$, we introduce a simplification of no further importance and put $F_0^0 = 0$. It will be seen below that the large number A_1^0 yields an unusual amount of physical information. We shall not need the exact value of A_1^0, all that matters is that $-A_1^0 \gg 1$.

2. We solve now Eq. (23) for $\Gamma(\omega)$:

$$\nu_0 \Gamma(k, \omega) = \cos\theta_1 \cos\theta_2 A_\parallel(k, \omega) + \exp\{i(\varphi_1 - \varphi_2)\} \sin\theta_1 \sin\theta_2 A_\perp(k, \omega),$$

where $\theta_1, \varphi_1, \theta_2, \varphi_2$ are the angles between the vectors $\mathbf{k}, \mathbf{p}_1$ and $\mathbf{k}, \mathbf{p}_2$ [see (5')]

$$A_\parallel(k, \omega) = \frac{A_1^0}{1 - A_1^0 W(s) s^2}, \quad (27)$$

$$A_\perp(k, \omega) = \frac{A_1^0/2}{1 - A_1^0/2 - (A_1^0/2) W(s)(1 - s^2)}.$$

Here $s = \omega/kv_0$, and W is the Landau function

$$-W(s) = 1 + i\frac{\pi}{2}|s| - \frac{s}{2}\ln\frac{1+s}{1-s}; \quad |s| < 1.$$

The poles $A_\parallel$ and $A_\perp$ yield respectively the spectra of the longitudinal and transverse oscillations. Both excitation branches lie in the region $\omega < kv_0$, with

$$-W \approx 1 + i\frac{\pi}{2}s, \quad s > 0,$$

and from (27) we get

$$A_\parallel(k, \omega) = \frac{k^2 v_0^2}{\omega^2 - k^2 u_1^2 + i\gamma_0(k, \omega)},$$

$$u_1^2 = u_0^2 \frac{1}{1 - m_0^*/m}, \quad \gamma_0(k, \omega) = \frac{\pi}{2}\frac{\omega^2|\omega|}{kv_0}. \quad (28)$$

The amplitude $A_\parallel$ has a pole at

$$\omega = k(u_1 - iu_1^0), \quad u_1^0 = \frac{\pi}{4}\frac{u_1^2}{v_0}. \quad (28)$$

Since $u_1 < v_0$, the damping of the longitudinal excitations is moderately small:

$$\frac{u_1^0}{u_1} = \frac{\pi}{4}\left(\frac{m_0^*}{3(m - m_0^*)}\right)^{1/2} \approx 0.25 - 0.35.$$

The spectrum of the longitudinal excitations runs counter to all rules: it lies under the line $\omega = kv_0$ and attenuates weakly. As $m_0^* \to 0$ its velocity coincides with the velocity of the thermodynamic sound u_0 (26), and the damping of u_1^0 vanishes. There is nothing surprising here, we have obtained ordinary sound, but its characteristics are expressed in terms of Fermi-liquid constants. The same picture is obtained also in the Landau limit $\omega < \omega_D$. The zero sound velocity expressed by Abrikosov and Khalatnikov[12] in terms of the constant F_0 and F_1 was practically equal to the velocity u_0 of ordinary sound. This agreement appears also in experiment. So long as dispersion equation for zero sound admits of a solution with $\omega > kv$ or $\omega < kv_0$, this agreemet must take place. It is no accident that u_0 has landed in the interval $v^2 < u_0^2 < v_0^2$.

It is interesting that the sound damping $\omega > \omega_0 \ln(\omega_0/\omega_D)$ has no Fermi-liquid character, and there was no need for us to take it into account that the quasiparticles, have a damping $\sim\omega_0 \ln(\varepsilon^2/\omega_D^2)$ (19). In contrast to zero sound, Landau damping of short-wave sound is $\propto k$ and not $\propto k^2$. The imaginary part of $A_\parallel$ is a quantity that can be measured in experiments, since $\operatorname{Im} A_\parallel(\omega)$ and $S(\omega)$ have the same ω dependence:

$$S_\parallel(\omega) \propto \operatorname{Im} A_\parallel(\omega) \propto \frac{\gamma_0(k,\omega)}{(\omega^2-k^2u_1^2)^2+\gamma_0^2(k,\omega)}. \tag{29}$$

Corresponding to longitudinal sound is a peak in the dependence of S on ω, and the relation between its width and position does not depend on k:

$$S_\parallel(\omega) \propto \Lambda_0 \Big/ \left[\left(\frac{\omega^2}{\omega_1^2}-1\right)^2+\Lambda_0^2\right]; \quad \omega_1 = ku_1,$$

$$\Lambda_0 = \frac{\pi}{2}\frac{u_1}{v_0} < 1. \tag{29'}$$

3. The transverse branch of the excitation is purely diffusive:

$$A_\perp(k,\omega) = -\frac{2}{\pi}\frac{kv_0}{u_2k-i|\omega|}, \quad u_2 = \frac{4p_F}{3\pi(m-m_0^*)}. \tag{30}$$

Here is a new unexpected fact: at $m_0^* \ll m$ the quantity u_2 is expressed only in terms of the density $n = p_F^3/3\pi^2$ and of the bare mass m. This again does not mean that have find a "foreign" branch of excitations for the Fermi liquid. Such excitations should be possessed also by liquid He^4. The "velocity" of the transverse sound is lower than that of longitudinal sound: $u_2 < u_1$. With increasing density of the liquid this inequality becomes stronger, since the longitudinal branch becomes more rigid. Therefore the thermodynamics of He^3 and He^4 at high densities and temperatures is determined completely by the contribution of the transverse sound. The linear contribution of transverse sound to the heat capacity (as that of any diffusion excitation) exceeds the contribution $\propto T^3$ from longitudinal sound. For both He^3 and He^4 there is a region of high temperatures where the heat capacity is linear in T. Another explanation of this dependence was proposed by Andreev.[13]

4. Let us make clear now what happens to the longitudinal and transverse branches of the excitation at large $k \sim 2p_F$, but at small ω as before. We have learned how to do this in the beginning of the paper: at such k and ω we can neglect $\operatorname{Re}\delta\Pi$ in (9) and we can calculate $\operatorname{Im}\Pi$ at all $k < 2p_F$ and $\omega < \varepsilon_F^0$. We consider first the limit $\omega \to 0$, at which

$$A_\parallel(k,\omega) = -\frac{g_1^2(k)}{\omega_3(k)-i|\omega|}, \quad g_1^2(k) = \frac{2}{\pi}kv_0\frac{4p_F^2}{k^2} \tag{31}$$

$$\omega_3(k) = \frac{8p_F^3A_1^0(0)}{3\pi kA_1^0(k)(m-m_0^*)}.$$

We do not know the $A_1^0(k)$ dependence. It is reasonable to represent it in the form

$$A_1^0(k) = A_1^0(0)(1+k^2/k_0^2)^{-1},$$

where k_0 is the cutoff momentum. We recall that allowance for the dependence A_1^0 on k is not an exaggeration of the accuracy: $A_1^0(k)$ is the first harmonic of the amplitude as a function of the angle θ [see (5')] and all the external momenta of this amplitude lie on the Fermi surface:

$$\omega_3 \propto \frac{1}{k}+\frac{k}{k_0^2}.$$

A diffusive branch of excitation appeared, with a frequency ω_k that increases as $k \to 0$. At small k this growth, of course, ceases since the equation valid there is (28) and not (31). On the basis of (6), (28), and (31) we can obtain for $A_\parallel$ dependences on ω and k that are valid at all $k < 2p_F$ and $\omega_0 \ln(\omega_0/\omega_D) < \omega < \varepsilon_F^0$:

$$A_\parallel(k,\omega) = \frac{k^2v_0^2}{\omega^2-k^2u_1^2(k)+i\gamma(k,\omega)},$$

$$u_1^2(k) = u_1^2(0)\left(1+\frac{k^2}{k_0^2}\right), \quad u_1^2(0) = u_0^2\left(1-\frac{m_0^*}{m}\right)^{-1}, \tag{32}$$

$$\gamma = \frac{\pi}{2}\omega kv_0\left(\frac{\omega}{kv_0}+\frac{k}{2p_F}\right)^2, \quad u_0^2 = \frac{p_F^2}{3mm_0^*}(1+F_0^0).$$

Damping with good dispersion gives way to pure diffusion at $u_1k \approx k^2/2m_0^*$, i.e., at $k \sim 2p_F(m_0^*/m)^{1/2}$. This agrees with the experimentally observed situation: at $k > p_F$ the peak of $S(\omega)$ broadens greatly. We present also for $S(\omega)$ an expression valid at $k < 2p_F$ and $\omega_0 \ln(\omega_0/\omega_D) < \omega < \varepsilon_F^0$:

$$S(\omega) \sim \frac{\Lambda(k,\omega)}{(\omega^2/\omega_1^2(k)-1)^2+\Lambda^2(k,\omega)},$$

$$\omega_1 = ku_1(k),$$

$$\Lambda = \frac{\pi}{2}\frac{\omega^3u_1(0)}{\omega_1^3(k)v_0}\left(1+\frac{k^2}{2m_0^*\omega}\right)^2 = \frac{\omega}{\omega_3(k)}\left(1+\frac{2m_0^*\omega}{k^2}\right)^2. \tag{33}$$

The quantities $u_1(k)$ and $\omega_3(k)$ are defined by (32) and (31). At small k, i.e., at $\omega > k^2/2m_0^*$, we have $\Lambda \propto \omega^3$ and the peak of $S(\omega)$ is narrow, while at large k, i.e., at $\omega < k^2/2m_0^*$, we have $\Lambda \propto \omega$ and a broad peak.

5. The transverse excitation branch becomes more rigid at large k:

$$A_\perp(k,\omega) = -\frac{g_2^2(k)}{\omega_2(k)-i|\omega|},$$

$$g_2^2(k) = \frac{2}{\pi}kv_0\left(1-\frac{k^2}{4p_F^2}\right)^{-1}, \tag{34}$$

$$\omega_2(k) = \frac{4p_Fk(1+k^2/k_0^2)}{3\pi(m-m_0^*)(1-k^2/4p_F^2)}.$$

The singularities of g^2 and ω_2 at $k = 2p_F$ are, of course fictitious, and Eq. (34) is valid at $\omega_2 < kv_0$. At $k \ll 2p_F$ Eq. (34) goes over into (30).

5. CONCLUSION

1. Let us indicate the weak spot of the paper. What was considered above was in fact the self-consistent-field approximation. We have assumed the parameter $(k_D/2p_F)^3$ to be small and took the spin-density fluctuation into account to first order in this parameter, after which we neglected it again only in first order. This yielded the principal equation (26), which enabled us to transform from the Fermi-liquid amplitude of the Landau scattering, $A = \mathbf{A}_0 + A_1\mathbf{n}_1\cdot\mathbf{n}_2$ to the unrenormalized $A^0 = A_0^0 + A_1^0\mathbf{n}_1\cdot\mathbf{n}_2$. We cannot offer an estimate of this approximation, but it is known from experi-

ment that the temperature dependence of the speed of sound in He^3 has no anomalies whatever. On the other hand the heat capacity $C(T)$ and the magnetic susceptibility $\chi(T)$ depend very strongly on T. If the paramagnons were to make a noticeable contribution not only to $C(T)$ but also to the compressibility of the heat capacity would be similar. The next higher orders of perturbation theory in $(k_D/2p_F)^3$ were taken into account in Ref. 9 in a model that admits of an exact solution.

2. Our interpretation of the experimental data on neutron scattering differs in principle from that proposed in Refs. 5 and 6. We improve the Landau quasiparticles by taking into account the dependences of Γ and Σ on the frequency, whereas in Refs. 5 and 6 this is done by taking into account the momentum dependences of these quantities. In Refs. 5 and 6 in the self-consistent-field approximation there is therefore no zero-sound damping, and to find it we must introduce the concept of multipair excitation. In our approach, however, the sound is damped even in the mean-field approximation.

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