Hydrodynamics of a rotating superfluid liquid

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Two systems of hydrodynamic equations exist for a rotating superfluid liquid. In one the expansion is carried out in terms of all the gradients, including the components of the velocity curl corresponding to uniform rotation. In the other the expansion is only in terms of nonequilibrium gradients. Exact nonlinear equations for both cases, with allowance for the deformation energy of the vortex lattice, are derived from the conservation laws. The specific modes of the hydrodynamics of fast rotation are investigated. Some features of the thermodynamics of the isolated-vortex-filament oscillations caused by the fact that the vibrational quanta have an intrinsic angular momentum are also considered.

A characteristic feature of a rotating superfluid liquid is that quantum vortex filaments are produced in it. The specific hydrodynamics of such a system is the subject of many papers, both earlier[1–3] and recent.[4–6] Nonetheless, a host of important questions, even in the very formulation of the problem, remain unanswered. The point is that two approaches are possible to a hydrodynamic description of rotating systems, i.e., to a description in which the quantities are assumed to vary slowly in space and in time and the expansions are in terms of gradients. In the first the initial state of the system is assumed to be at rest and the expansion is in all the gradients, including the components of the velocity curl connected with the uniform rotation. Since uniform rotation is always in thermodynamic equilibrium, another approach is possible, in which the velocity curl that corresponds to the uniform rotation is not assumed small, and the expansion is only in terms of nonequilibrium gradients, in the gradients that show up above the uniform rotation (hydrodynamics of fast rotations).

This raises the question of the meaning and the range of validity of the equations discussed in the papers cited above. These equations, just as in hydrodynamics of a nonrotating superfluid liquid, contain the two independent velocities of the normal and superfluid motions. The interactions of the excitations that make up the normal part of the thermal excitation with the vortex filaments are taken into account as a mutual friction force proportional to the difference between the normal and superfluid velocities. For such a description to be valid it is necessary in any case that the excitation free path time τ_N, which is connected with their scattering by one another, be considerably shorter than the analogous time τ_U due to the interaction of the excitation with the vortex filaments. Otherwise introduction of the velocity of the normal component as an independent thermodynamic variable is meaningless. But even if the condition $\tau_U \gg \tau_N$ is satisfied, the usual equations are valid, given the angular velocity of the rotation, only at frequencies that are not too low.

Let us clarify the situation, considering for the sake of argument, on the basis of the usual equations, the temperature oscillations and the concomitant oscillations of the relative velocity, perpendicular to the vortices, of the superfluid and normal components. In this case such oscillations are analogous in many respects to temperature oscillations in crystals under conditions of phonon hydrodynamics (see Ref. 7) with τ_N and τ_U playing the role of the times of phonon relaxation due to normal and umklapp processes, respectively. In both cases there are two oscillation modes, whose frequencies can be expressed as functions of the wave vectors in the form $\omega = -i\gamma \pm (c_2^2 k^2 - \gamma^2)^{1/2}$, where c_2 is the second-sound velocity and $\gamma = (1/\tau_U) + c_2^2 k^2 \tau_N$. In the case of a rotating superfluid liquid we have (see Ref. 1) $\tau_U \sim (B\Omega)^{-1}$, where Ω is the angular velocity of the rotation and B is one of the two parameters introduced by Hall and Vinen[2] to define the mutual friction force. From the viewpoint of hydrodynamics of slow rotations; both modes are hydrodynamic, since both frequencies ω tend to zero when k and tend simultaneously to zero. In fast-rotation hydrodynamics, however, i.e., as $k \to 0$ and at constant Ω, only one (heat-conduction) mode is hydrodynamic.

The usual equations are thus hydrodynamic in the sense of slow rotations. Given Ω, however, their validity is restricted by the condition $\tau_U \gg \tau_N$. If, however, this condition is satisfied and the motion frequency ω satisfies the inequality $\omega\tau_U \ll 1$, we can replace the ordinary equations by the much simpler equations, derived in the present paper, of fast rotations, in exactly the same way that at $\tau_U \gg \tau_N$ and $\omega\tau_U \ll 1$ we can replace the phonon-hydrodynamics equations by the usual equations of elasticity theory.

In fast-rotation hydrodynamics one introduces for a superfluid liquid with a vortex-filament lattice only one independent velocity of macroscopic motion in directions perpendicular to the vortices, i.e., the system behaves for these directions as an ordinary crystal. Since the longitudinal total momentum of the excitations is preserved by virtue of the homogeneity of the system in the filament direction, we introduce in this direction two velocities and the system behaves as a superfluid liquid. Fast-rotation hydrodynamics is valid, given Ω, at sufficiently low frequencies $\omega \ll \tau_U^{-1}$ and $\omega \ll \tau_N^{-1}$ no matter what the ratio of τ_U and τ_N. At temperatures on the order of 1 K in liquid He II the constant B of

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Hall and Vinen is of the order of unity[1] and the validity of fast-rotation hydrodynamics is restricted to frequencies lower than Ω. With decreasing temperature, however, the situation changes because flexural oscillations of the vortex filaments begin to make a substantial contribution to the density of the normal component. This contribution

$$\rho_n \sim (Tm^2/\hbar^2)(m\Omega/\hbar)^{1/2}\ln^{-2}(b/a),$$

where m is the mass of the liquid particles, $b \sim (\hbar/m\Omega)^{1/2}$ is the distance between the vortex filaments, and a is the interatomic distance, exceeds the phonon contribution if $\Omega \sim 1$ sec^{-1} at temperatures lower than 0.1 K, and at even higher temperatures at larger Ω. Since the flexural quasiparticles are localized on the vortex filaments and can move easily only along them, one can speak under these conditions only of fast-rotation hydrodynamics. A similar but even more clearly pronounced situation arises at low temperatures in solutions of ^{3}He in He II, owing to the absorption[8] of impurities on the vortex filaments. Finally, in superfluid ^{3}He-B, in which, as in He II, the orbital hydrodynamics is isotropic, the condition $\tau_U \sim \tau_N$ begins to be satisfied at temperatures $T \lesssim T_c$ for $\Omega \lesssim 1$ sec^{-1} because of the rather large τ_N. Higher angular velocities ^{3}He-B should be described by fast-rotation hydrodynamics.

The derivation of the hydrodynamic equations of a rotating superfluid is of interest in connection with the general problem of the hydrodynamic description of condensed media with different types of ordering. We regard as the most convenient and general method for deriving these equations the method based on the differential form of the conservation laws (see Ref. 9), which verifies, simultaneously with the derivation, also the uniqueness of the equations. As applied to the hydrodynamics of slow rotations of a superfluid liquid, this method was used by Bekarevich and Khalatnikov,[2] but they did not take into account the energy of the vortex-lattice deformation. Volovik and Dotsenko[5] used for this purpose a Poisson-bracket method developed by Dzyaloshinskiĭ and Volovik,[10] but their derivation is not purely phenomenological and calls for the use of "microscopic" equations of motion of an isolated vortex filament. Later Baym and Chandler[6] considered the slow-rotation equations only in a form linearized in the lattice deformation. Before proceeding to the fast-rotation equations, we present therefore a general derivation of the equations of slow-rotation dynamics on the basis of the conservation laws.

In the concluding section of the paper we consider very low angular velocities, such that in the equilibrium state there is only a single vortex filament along the center of the vessel. This case also lends itself readily to experimental investigation.[11] We shall show that owing to the presence of the finite rotation velocity the properties of a vortex filament differ substantially from those of an ordinary elastic filament. The amplitude of the thermal oscillations of an ordinary filament are known[12] to tend to infinity with increasing filament length. In the case of a vortex filament the angular velocity exerts a stabilizing action, so that when the length is increased the oscillation amplitude tends to a finite limit that depends on the angular velocity, and remains considerably smaller than the vessel radius. We shall calculate the contribution of the filament oscillations to the thermodynamic functions of the system. This contribution is also found to be substantially dependent on the angular velocity. We shall show, finally, that a unique effect should take place, wherein the angular momentum is transferred by the motion of the normal component along the vortex filaments.

1. HYDRODYNAMICS OF SLOW ROTATIONS

The state of a vortex lattice whose parameters vary slowly in space and in time can be described by specifying the vectors $\mathbf{e}_a(\mathbf{r},t)$, $a = 1,2$, which are equal to the local values of the elementary vectors of lattice-translation in a plane perpendicular to the vortex direction. From these we can determine the corresponding reciprocal-lattice vectors

$$\mathbf{e}^a = -(1/s)\varepsilon^{ab}[\boldsymbol{\nu}\times\mathbf{e}_b],$$

where s is the area of the unit cell of the direct lattice, $\varepsilon^{12} = -\varepsilon^{21} = 1$, $\varepsilon^{11} = \varepsilon^{22} = 0$, $\boldsymbol{\nu} = (1/s)\mathbf{e}_1\times\mathbf{e}_2$ is a unit vector along the axis of the vortices.

The following obvious equalities hold:

$$\mathbf{e}^a\mathbf{e}_b = \delta_b{}^a, \qquad e_{ai}e_k{}^a = \delta_{ik} - \nu_i\nu_k. \tag{1}$$

We introduce the metric tensor g_{ab} and its inverse g^{ab}; they satisfy the relations

$$g_{ab} = \mathbf{e}_a\mathbf{e}_b, \qquad g^{ab} = \mathbf{e}^a\mathbf{e}^b, \qquad g_{ac}g^{cb} = \delta_a{}^b. \tag{2}$$

Let $\mathbf{v}_L(\mathbf{r},t)$ be the local vortex-filament velocity perpendicular to the vortex axis: $\mathbf{v}_L\cdot\boldsymbol{\nu} = 0$. If $\mathbf{v}_L$ is known and the functions $\mathbf{e}^a(\mathbf{r})$ at the initial instant of time are specified, we can determine $\mathbf{e}^a$ at a nearby instant of time, i.e., the derivatives $\partial\mathbf{e}^a/\partial t$. To establish this connection we note that the unit vector $\mathbf{n} = \mathbf{e}^1/e^1$ is the normal to the corresponding crystallographic plane, and $d = 1/e^1$ is proportional to the local value of the interplanar distance. From simple geometric considerations we get

$$\dot{\mathbf{n}} + V(\mathbf{n}\nabla)\mathbf{n} = -\nabla V + \mathbf{n}(\mathbf{n}\nabla V), \quad \dot{d} + V(\mathbf{n}\nabla)d = d(\mathbf{n}\nabla V), \tag{3}$$

where $V = \mathbf{v}_L\cdot\mathbf{n}$ is the normal velocity of the considered crystallographic plane. From (3) we obtain

$$\dot{\mathbf{e}}^1 + \nabla(\mathbf{e}^1\mathbf{v}_L) = V[\mathbf{n}\times\operatorname{rot}\mathbf{e}^1]. \tag{4}$$

A similar relation holds also for $\mathbf{e}^2$.

We express a physically infinitely small (i.e., large compared with the lattice period but small compared with the distance over which the vortex configuration varies) differential of the coordinates in the form

$$d\mathbf{r} = \mathbf{e}_a dN^a. \tag{5}$$

The quantities dN^a for two lattice points separated by a distance $d\mathbf{r}$ are the two projections of $d\mathbf{r}$ measured in units of the corresponding lattices. These quantities, obviously, are not altered by an arbitrary elastic deformation (in the absence of dislocations). From (5) we get

$$dN^a = \mathbf{e}^a d\mathbf{r} = \mathbf{e}_0{}^a d\mathbf{r}_0, \tag{6}$$

where $\mathbf{e}_0{}^a$ and $d\mathbf{r}_0$ are respectively the periods of the reciprocal lattice and the difference between the coordinates of the considered points in the undeformed state; we see therefore that

$$\mathbf{e}^a=\nabla N^a, \tag{7}$$

and therefore the specified functions $N^a(\mathbf{r},t)$ determine completely the configuration of the vortex filaments. Moreover, with the aid of (4) we can express in terms of N^a also the velocity $\mathbf{v}_L$. Since curl $\mathbf{e}^a=0$ by virtue of (7), we have

$$\dot{\mathbf{e}}^a+\nabla(\mathbf{v}_L\mathbf{e}^a)=0, \tag{8}$$

so that

$$\mathbf{v}_L=-\mathbf{e}_a\dot{N}^a. \tag{9}$$

Variables analogous to N^a were considered for the description of a vortex lattice by Volovik and Dotsenko[5].

In macroscopic hydrodynamics of a rotating superfluid liquid one introduces the averaged velocity $\mathbf{v}_s$ of the superfluid component, whose curl is determined by the direction and density of the vortex filaments, as well as the circulation quantum $2\pi\hbar/m$. Since the unit-cell area s is equal to $g^{1/2}$, where g is the determinant of the metric tensor g_{ab}, we have

$$\operatorname{rot}\mathbf{v}_s=\frac{2\pi\hbar}{ms}\mathbf{v}=\frac{2\pi\hbar}{mg}[\mathbf{e}_1\times\mathbf{e}_2]=\frac{2\pi\hbar}{m}[\mathbf{e}^1\times\mathbf{e}^2]. \tag{10}$$

By differentiating (10) with respect to time and using (8) we obtain the conservation condition for the vortex filaments

$$\operatorname{rot}\dot{\mathbf{v}}_s=\operatorname{rot}[\mathbf{v}_L\times\operatorname{rot}\mathbf{v}_s]. \tag{11}$$

In accord with the general method of deriving the hydrodynamics equations from the conservation laws,[9] we introduce in the case of slow rotations, besides the velocity $\mathbf{v}_s$, the velocity $\mathbf{v}_n$ of the normal component and seek equations in the form of conservation laws

$$\dot\rho+\operatorname{div}\mathbf{j}=0,\quad \partial j_i/\partial t+\partial\Pi_{ik}/\partial x_k=0,$$
$$\dot S+\operatorname{div}\{S\mathbf{v}_n+\mathbf{q}/T\}=R/T,\quad \dot{\mathbf{v}}_s=[\mathbf{v}_L\times\operatorname{rot}\mathbf{v}_s]+\nabla\varphi. \tag{12}$$

Here ρ, S, and $\mathbf{j}$ are the mass, entropy and momentum per unit volume, while Π_{ik}, $\mathbf{q}$, $R>0$, and φ are quantities to be determined. We must also find the connection between the velocities $\mathbf{v}_s$, $\mathbf{v}_n$, and $\mathbf{v}_L$. The criterion is the requirement that the energy-conservation equation be automatically obtained from Eqs. (12).

The Galileo transformation formulas

$$E=\rho v_s^2/2+\mathbf{j}_0\mathbf{v}_s+E_0,\quad \mathbf{j}=\rho\mathbf{v}_s+\mathbf{j}_0 \tag{13}$$

connect the energy E per unit volume and the momentum $\mathbf{j}$ with their values E_0 and $\mathbf{j}_0$ in a system where $\mathbf{v}_s=0$. The energy E_0 can be regarded as a function of S, ρ, $\mathbf{j}_0$, and the metric tensor g^{ab}, so that

$$dE_0=TdS+\mu d\rho+(\mathbf{v}_n-\mathbf{v}_s,\ d\mathbf{j}_0)+{}^1/_2h_{ab}dg^{ab}. \tag{14}$$

Equations (12)–(14) differ from the corresponding equations of Bekarevich and Khalatnikov[2] in that the vortex-conservation condition (11) is taken into account in (12) in the equation of the superfluid motion, and also in that the dependence of the energy on the deformation of the vortex lattice is fully taken into account in the identity (14).

Differentiating with respect to time the first equation of (13), we obtain by using (13) and (14), after some transformations,

$$\dot E+\operatorname{div}\{\mathbf{Q}_0+\mathbf{q}+v_{nk}\pi_{ki}+\psi(\mathbf{j}-\rho\mathbf{v}_n)+v_{Lk}h_{ab}e_i^{a}e_k^{b}\}$$
$$=R+\frac{\mathbf{q}\nabla T}{T}+\pi_{ik}\frac{\partial v_{ni}}{\partial x_k}$$
$$+\psi\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n)+\{\mathbf{v}_L-\mathbf{v}_n,\ [\operatorname{rot}\mathbf{v}_s\times\mathbf{j}-\rho\mathbf{v}_n]+\mathbf{e}^a\operatorname{div}(h_{ab}\mathbf{e}^b)\}, \tag{15}$$

where

$$\mathbf{Q}_0=(\mu+v_s^2/2)\mathbf{j}+TS\mathbf{v}_n+\mathbf{v}_n(\mathbf{j}_0\mathbf{v}_n),$$
$$\pi_{ik}=\Pi_{ik}-P\delta_{ik}-v_{si}j_k-v_{nk}j_{0i}-h_{ab}e_i^{a}e_k^{b},$$
$$\psi=-(\mu+v_s^2/2+\varphi),\quad P=-E_0+TS+\mu\rho+(\mathbf{v}_n-\mathbf{v}_s,\ \mathbf{j}_0).$$

In the derivation of (15) we used also the equality

$$\frac{1}{2}h_{ab}\dot g^{ab}+\frac{1}{2}v_{ni}h_{ab}\frac{\partial g^{ab}}{\partial x_i}=-\operatorname{div}\{h_{ab}\mathbf{e}^a(\mathbf{v}_L-\mathbf{v}_n,\mathbf{e}^b)\}$$
$$-\frac{\partial v_{ni}}{\partial x_k}(h_{ab}e_i^{a}e_k^{b})+\{\mathbf{v}_L-\mathbf{v}_n,\ \mathbf{e}^a\operatorname{div}(h_{ab}\mathbf{e}^b)\},$$

which is easily obtained from the second equation of (2) and from (8).

The form of (15) enables us to determine the energy-flux vector

$$\mathbf{Q}=\mathbf{Q}_0+\mathbf{q}+v_{nk}\pi_{ki}+\psi(\mathbf{j}-\rho\mathbf{v}_n)+v_{Lk}h_{ab}e_i^{a}e_k^{b}$$

and the dissipation function

$$R=-\frac{\mathbf{q}\nabla T}{T}-\pi_{ik}\frac{\partial v_{ni}}{\partial x_k}-\psi\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n)$$
$$-\{\mathbf{v}_L-\mathbf{v}_n,\ [\operatorname{rot}\mathbf{v}_s\times\mathbf{j}-\rho\mathbf{v}_n]+\mathbf{e}^a\operatorname{div}(h_{ab}\mathbf{e}^b)\}. \tag{16}$$

From the condition that R be positive it follows that the unknown quantities $\mathbf{q}$, π_{ik}, ψ, and $\mathbf{v}_L-\mathbf{v}_{n\perp}$ (the symbol $\perp$ means that we are dealing with the projection of the corresponding vector on a plan perpendicular to $\mathbf{v}$) can be written in the general case as linear combinations of all the conjuctated variables ∇T, $\partial v_{ni}/\partial x_k$, etc., contained in (16). We shall not write out the unwieldy general formulas and confine ourselves, as usual, to the mutual friction effects described by the last term of (16). We have

$$\mathbf{v}_L-\mathbf{v}_{n\perp}=-\alpha\left\{\mathbf{j}_\perp-\rho\mathbf{v}_{n\perp}+\frac{mg^{1/2}}{2\pi\hbar}[\mathbf{e}^a\times\mathbf{v}]\operatorname{div}(h_{ab}\mathbf{e}^b)\right\}$$
$$-\beta\left\{[\mathbf{v}\times\mathbf{j}-\rho\mathbf{v}_n]+\frac{mg^{1/2}}{2\pi\hbar}\mathbf{e}^a\operatorname{div}(h_{ab}\mathbf{e}^b)\right\}, \tag{17}$$

where α and β are certain coefficients, with $\beta>0$. Since at $T=0$ there is no normal part, so that $\mathbf{v}_L$ should be independent of $\mathbf{v}_n$, it follows that the constant α should equal $-1/\rho$. If we put $\alpha=-(1/\rho_s)+\beta'$, the constants β and β' will vanish as $T\to0$. They coincide with the mutual friction coefficients introduced by Bekarevich and Khalatnikov[2] and differ from the constants of Hall and Vinen[1] by a factor $\rho_n/2\rho\rho_s$.

At zero temperature we obtain from (17)

$$\mathbf{v}_L-\mathbf{v}_{s\perp}=\frac{mg^{1/2}}{2\pi\hbar\rho}[\mathbf{e}^a\times\mathbf{v}]\operatorname{div}(h_{ab}\mathbf{e}^b), \tag{18}$$

which is a generalization of a known relation[13,14] to the case of arbitrary and not small deformations of the vortex lattice.

When expanding the equations in powers of the deformation it is convenient to put $N^a = N_0^a - u^a$, where $N_0^a = \mathbf{e}_0^a\cdot\mathbf{r}$, and introduce the variable $\mathbf{u} = \mathbf{e}_{0a}u^a$, which has the meaning of a two-dimensional displacement vector perpendicular to the axes of the undeformed vortices. We represent the elastic energy E_{el} per unit lattice volume in the form $E_{el} = E_1 + E_2$, where

$$E_1=(\pi\rho_s/2)(\hbar/m)^2 n\ln(1/na^2)$$

is the elastic energy with the dependence on the shape of the unit cell neglected ($n = 1/s = g^{-1/2}$) and E_2 is the shear energy. Accordingly,

$$h_{ab}=2\partial E_{el}/\partial g^{ab}=h_{ab}^{(1)}+h_{ab}^{(2)}.$$

Differentiating E_1 with respect to g^{ab} with allowance for the identity $dg = -gg_{ab}dg^{ab}$ and linearizing the result in the deviation δg^{ab} of the metric tensor from its value $g^{ab(0)}$ in an undeformed triangular lattice, we obtain

$$h_{ab}^{(1)}=\rho_s\frac{\hbar\Omega}{m}\left\{\ln\frac{b}{a}\left(g_{ab}^{(0)}-\delta g_{ab}+\frac{1}{2}g_{ab}^{(0)}\delta g_c{}^c\right)-\frac{1}{4}g_{ab}^{(0)}\delta g_c{}^c\right\}, \tag{19}$$

where $\Omega = (\pi\hbar/m)g_0^{-1/2}$ is the angular velocity the rotation and

$$\delta g_{ab}=g_{ac}^{(0)}g_{bd}^{(0)}\delta g^{cd},\qquad \delta g_c{}^c=g_{cd}^{(0)}\delta g^{cd}.$$

The constant term appears in (19) because at equilibrium the energy that has a minimum is the one in the rotating coordinate frame. We express the shear part $h_{ab}^{(2)}$ of h_{ab} in the form

$$h_{ab}^{(2)}=\mu_s(\delta g_{ab}-{}^1/_2 g_{ab}^{(0)}\delta g_c{}^c), \tag{20}$$

where $\mu_s = \rho_s\hbar\Omega/4m$ is the shear modulus calculated by Tkachenko[15] for a triangular lattice.

The quantities δg^{ab} and $\mathbf{e}^a$ can be easily expanded in the displacement $\mathbf{u}$ by using Eqs. (2) and (7). As a result we get the following expressions for the elastic terms that enter in the equations:

$$\begin{aligned}\mathbf{F}&\equiv\mathbf{e}^a\operatorname{div}(h_{ab}\mathbf{e}^b)\\&=2\Omega\lambda\nabla_\perp\rho_s+\rho_s\frac{\hbar\Omega}{4m}(2\nabla_\perp\operatorname{div}\mathbf{u}-\Delta_\perp\mathbf{u})-\rho_s 2\Omega\lambda\frac{\partial^2\mathbf{u}}{\partial z^2},\end{aligned} \tag{21}$$

$$\frac{\partial}{\partial x_k}(h_{ab}e_i{}^a e_k{}^b)=-\rho_s 2\Omega\lambda\nabla\operatorname{div}\mathbf{u}+\mathbf{F},$$

where $\lambda = (\hbar/2m)\ln(b/a)$. The first term in the right-hand side of the second equation of (21) can be left out upon normalization of the pressure $P\to P-\rho_s 2\Omega\lambda\operatorname{div}\mathbf{u}$ and simultaneous replacement of μ in the equation for the superfluid motion by the chemical potential of the liquid without allowance for elasticity. Indeed, in the linear approximation we have

$$\begin{aligned}d\mu&=-\frac{S}{\rho}dT+\frac{1}{\rho}dP-\frac{1}{\rho}\mathbf{j}_0 d(\mathbf{v}_n-\mathbf{v}_s)+\frac{1}{2\rho}h_{ab}dg^{ab}\\&=-\frac{S}{\rho}dT-\frac{1}{\rho}\mathbf{j}_0 d(\mathbf{v}_n-\mathbf{v}_s)+\frac{1}{\rho}d(P-\rho_s 2\Omega\lambda\operatorname{div}\mathbf{u}).\end{aligned}$$

When the first equation of (21) is substituted in it, Eq. (18) becomes the customarily employed expression.[13,14]

We express also $\mathbf{v}_L$ in terms of $\mathbf{u}$, using relation (9). For this purpose it suffices to note that $\mathbf{e}_0^a = \nabla N_0^a$ are the vectors of an undeformed but uniformly rotating lattice. Therefore $\dot{\mathbf{e}}_0^a = \boldsymbol{\Omega}\times\mathbf{e}_0^a$ and

$$\dot{N}_0{}^a=[\boldsymbol{\Omega}\times\mathbf{e}_0{}^a]\mathbf{r}=-\mathbf{v}_0\mathbf{e}_0{}^a,$$

where $\mathbf{v}_0 = \boldsymbol{\Omega}\times\mathbf{r}$. Linearization of (9) yields

$$\mathbf{v}_L=\mathbf{v}_0+\dot{\mathbf{u}}+(\mathbf{v}_0\nabla)\mathbf{u}-[\boldsymbol{\Omega}\times\mathbf{u}]-\boldsymbol{\nu}_0\left(\mathbf{v}_0\frac{\partial\mathbf{u}}{\partial z}\right),$$

where $\boldsymbol{\nu}_0$ is a unit vector along the equilibrium direction of the vortices, the latter chosen to be the z axis.

2. HYDRODYNAMICS OF FAST ROTATIONS

In fast-rotation hydrodynamics we must introduce one velocity in a direction perpendicular to the vortices and two independent velocities in the longitudinal direction. Under these conditions it is not convenient to use as the hydrodynamic variable the true superfluid velocity of Eq. (10) and designated by us $\mathbf{V}_s$ in this section. We introduce in its place a single perpendicular velocity $\mathbf{v}(\mathbf{v}\cdot\boldsymbol{\nu} = 0)$ defined by the equality $\mathbf{v} = \mathbf{j}_\perp/\rho$, where $\mathbf{j}_\perp$ is the exact value of the perpendicular momentum. Motion in the longitudinal direction will be described by two velocities, $v_{n\|}$ and $v_{s\|}$, with $v_{s\|} = \mathbf{V}_s\cdot\boldsymbol{\nu}$. We put

$$\mathbf{v}_s=\mathbf{v}+\boldsymbol{\nu}v_{s\|},\quad \mathbf{v}_n=\mathbf{v}+\boldsymbol{\nu}v_{n\|}. \tag{22}$$

We emphasize once more that the velocity $\mathbf{v}_s$ introduced by us does not coincide, generally speaking, with the true superfluid velocity $\mathbf{V}_s$. Nonetheless Eqs. (13) and (14) for the new velocities $\mathbf{v}_n$ and $\mathbf{v}_s$ remain in force. Indeed, by virtue of (22), (12) and the definition of $\mathbf{v}$, the relative velocity $\mathbf{v}_n - \mathbf{v}_s$ and $\mathbf{j}_0$ have only longitudinal components, and the third term of (14) can be written in the form $(v_{n\|} - v_{s\|})dj_{0\|}$, which corresponds precisely to the correct expression for a system that is superfluid only in the longitudinal direction. We emphasize that the one-dimensional densities of the normal (ρ_n) and superfluid ($\rho_s = \rho - \rho_n$) components, defined by the formula $j_0 = \rho_n(v_{n\|} - v_{s\|})$, differ substantially, generally speaking, from the corresponding "microscopic" three-dimensional quantities in expressions (19)–(21) for the elastic moduli of a vortex lattice. All the definitions connected with the kinematics of a vortex lattice, particularly expression (9) for $\mathbf{v}_L$, remain the same as before. Since the number of independent velocity components is two less than in slow-rotation hydrodynamics, the number of equations should be correspondingly less than in (12). In lieu of the last equation of (12) (the three-component equation of the superfluid motion) we need one scalar equation. We derive it by using the formula

$$\mathbf{v}_s\boldsymbol{\nu}=\mathbf{V}_s\boldsymbol{\nu} \tag{23}$$

and relation (11), which in the notation of the present section can be written in the form

$$\dot{\mathbf{V}}_s+\nabla\varphi=[\mathbf{v}_L\times\operatorname{rot}\mathbf{V}_s], \tag{24}$$

where φ is a certain scalar. Differentiating (23) with respect to time, and taking into account (24) and that $\boldsymbol{\nu} = \operatorname{curl}\mathbf{V}_s/$

$|\mathrm{curl}\,\mathbf{V}_s|$, we obtain

$$\mathbf{v}\dot{\mathbf{v}}_s=-\mathbf{v}\nabla\varphi-\dot{\mathbf{v}}(\mathbf{V}_s-\mathbf{v}_s). \tag{25}$$

An expression for $\dot{\mathbf{v}}$ can be easily derived from (24):

$$\dot{\mathbf{v}}+(\mathbf{v}_L\nabla)\mathbf{v}=(\mathbf{v}\nabla)\mathbf{v}_L-\mathbf{v}(\mathbf{v},\ (\mathbf{v}\nabla)\mathbf{v}_L). \tag{26}$$

Substituting it in (25) we obtain after simple transformations

$$\mathbf{v}\{\dot{\mathbf{v}}_s+\nabla(\mu+v_s^2/2+\psi)-[\mathbf{v}_L\times\mathrm{rot}\,\mathbf{v}_s]\}=0,$$

where

$$\psi=\varphi-\mu-v_s^2/2-\mathbf{v}_L(\mathbf{V}_s-\mathbf{v}_s).$$

It is convenient to choose Eq. (26) with the as yet undetermined scalar ψ as the sought scalar equation that makes up together with the first three equations of (12) the complete system.

Differentiating, as usual, the first equation of (13) with respect to time and using the aforementioned complete system, we reduce the equation for E to the form

$$\dot{E}=-\mathrm{div}(\mathbf{Q}_0+\mathbf{q}+v_{nk}\pi_{ki}+v_{Lk}h_{ab}e_i^a e_k^b+(\mathbf{j}-\rho\mathbf{v}_n)\psi)+R+\frac{\mathbf{q}\nabla T}{T}+\pi_{ik}\frac{\partial v_{ni}}{\partial x_k}+\psi\,\mathrm{div}(\mathbf{j}-\rho\mathbf{v}_n)+\{\mathbf{v}_L-\mathbf{v},\mathbf{F}+(\mathbf{j}-\rho\mathbf{v}_n,\mathbf{v})[\mathrm{rot}\,\mathbf{v}_s\times\mathbf{v}]\},$$

where the expresssions for $\mathbf{Q}_0$ and π_{ik} formally coincide with those given in the preceding section after Eq. (15). From this we find the dissipation function

$$R=-\frac{\mathbf{q}\nabla T}{T}-\pi_{ik}\frac{\partial v_{ni}}{\partial x_k}-\psi\,\mathrm{div}(\mathbf{j}-\rho\mathbf{v}_n)-\{\mathbf{v}_L-\mathbf{v},\mathbf{F}+(\mathbf{j}-\rho\mathbf{v}_n,\mathbf{v})[\mathrm{rot}\,\mathbf{v}_s\times\mathbf{v}]\}. \tag{27}$$

Confining ourselves, as in the preceding section, to consideration of only the last term in (27), we write down the expression for the relative velocity of the vortices and of the matter in the following general form:

$$(\mathbf{v}_L-\mathbf{v})_\alpha=-\hat{B}_{\alpha\beta}G_\beta,$$

where

$$\mathbf{G}=\mathbf{F}+(\mathbf{j}-\rho\mathbf{v}_n,\ \mathbf{v})[\mathrm{rot}\,\mathbf{v}_s\times\mathbf{v}],$$

α and β are two-dimensional spatial indices in a plane perpendicular to $\mathbf{v}$. The matrix of the coefficients $\hat{B}_{\alpha\beta}$ satisfies the Onsager relations

$$\hat{B}_{\alpha\beta}(\mathbf{v})=\hat{B}_{\beta\alpha}(-\mathbf{v}).$$

Therefore

$$(\mathbf{v}_L-\mathbf{v})_\alpha=(mg^{1/2}/2\rho\pi\hbar-B')[\mathbf{G}\times\mathbf{v}]_\alpha-B_{\alpha\beta}G_\beta, \tag{28}$$

where $B_{\alpha\beta}$ is the symmetric part of $\hat{B}_{\alpha\beta}$, $B_{\alpha\beta}(T=0)=B'(T=0)=0$. For an arbitrarily deformed lattice $B_{\alpha\beta}$ is an arbitrary symmetric tensor. We point out that the second term in the expression for $\mathbf{G}$, in contrast to the analogous term in the slow-rotation hydrodynamics, is of second order in the deviations from the state of uniform rotation. Therefore $B_{\alpha\beta}$ has in fast-rotation hydrodynamics the meaning of the diffusion coefficient of the vortices. The coefficient B' describes an effect of the Hall type in diffusion. In an undeformed triangular lattice the tensor B reduces to a scalar.

We recall that the complete system of equations consists of the first three equations of (12) and of Eq. (26).

By linearizing the equations near the uniform rotation we obtain the following set of equations that describes the oscillations of the temperature and the associated oscillations of the relative velocity $\delta w=\delta v_n-\delta v_s$ along the vortices:

$$\begin{gathered}\delta\dot{T}+(\mathbf{v}_0\nabla)\delta T+\frac{TS\rho_s}{C\rho^2}(\mathbf{v}_0\nabla)\delta w+\frac{\mathrm{div}\,\mathbf{q}}{\rho C}=0,\\ \delta\dot{w}+(\mathbf{v}_0\nabla)\delta w+\frac{S}{\rho_n}(\mathbf{v}_0\nabla)\delta T=0,\end{gathered} \tag{29}$$

where C is the heat capacity per unit mass and the liquid is assumed incompressible: $\rho_n\delta v_n+\rho_s\delta v_s=0$.

Equations (29) are reduced to equations with constant coefficients by transforming to a rotating coordinate frame. This corresponds to the substitutions $\partial/\partial t\to\partial/\partial t-(\mathbf{v}_0\nabla)$ for scalars and $\partial/\partial t\to\partial/\partial t-(\mathbf{v}_0\nabla)-\boldsymbol{\Omega}\times$ for vectors. The heat flux in (29) can be set equal to $\mathbf{q}=\varkappa_\perp\nabla_\perp T$, since the equations contain other considerably larger terms with $\partial/\partial z$. As a result we get the dependence of the frequency in the rotating system on the wave vector:

$$\omega=-i\frac{\varkappa_\perp k_\perp^2}{2\rho C}\pm\left[c_2^2k_z^2-\left(\frac{\varkappa_\perp k_\perp^2}{2\rho C}\right)^2\right]^{1/2},$$

where $c_2^2=TS^2\rho_s/C\rho^2\rho_n$. The oscillations of the temperature propagate in the form of second sound only along the axis of the vortices, while in perpendicular directions they are ordinary damped thermal waves (the second root corresponds at small k_z to $\delta T\to 0$ and $\delta w=\mathrm{const}$).

By way of another application of the derived equation we consider the oscillations of the transverse component $\delta\mathbf{v}_\perp$ of the velocity and of the displacement $\mathbf{u}$ in a state with simultaneous uniform rotation and a uniform heat flux $Q=TSv_n$ along the vortex lines. This problem is of interest because a substantial role is played in it by the second term of the expression for $\mathbf{G}$. Confining ourselves for simplicity to the case of low temperatures $\rho_n\ll\rho$ and neglecting in (28) the terms with B and B', we rewrite this equation in the rotating frame in the form

$$\mathbf{u}-\mathbf{v}_\perp=(2\Omega\rho)^{-1}[\mathbf{F}\mathbf{v}_0]-iv_nk_z\mathbf{u}+\frac{i}{2\Omega}v_n\left\{k_z[\mathbf{v}_0\times\delta\mathbf{v}_\perp]+[\mathbf{v}_0\times\mathbf{k}_\perp]\frac{\mathbf{k}_\perp\delta\mathbf{v}_\perp}{k_z}\right\}, \tag{30}$$

where $\mathbf{F}$ is defined in (21). The second equation that connects $\mathbf{u}$ with $\mathbf{v}_\perp$ is obtained by projecting on the (xy) plane the second equation of (12) and excluding from it the pressure using the incompressibility condition $\mathrm{div}\,\mathbf{v}_s=0$. We have

$$\delta\dot{\mathbf{v}}_\perp+[2\boldsymbol{\Omega}\times\delta\mathbf{v}_\perp]-\frac{\mathbf{k}_\perp}{k^2}(\mathbf{k}_\perp[2\boldsymbol{\Omega}\times\delta\mathbf{v}_\perp])+\frac{1}{\rho}\mathbf{F}-\frac{\mathbf{k}_\perp}{\rho k^2}(\mathbf{k}_\perp\mathbf{F})=0. \tag{31}$$

If $\mathbf{k}$ is small enough, Eqs. (30) and (31) describe in the principle approximation the independent oscillations of $\delta\mathbf{v}_\perp$ and $\mathbf{u}$.

The former have a frequency

$$\omega = (2\boldsymbol{\Omega}\mathbf{k})/k, \tag{32}$$

and are the known inertial waves of ordinary hydrodynamics in a rotating liquid.

The second mode constitutes oscillations of the displacement vector $\mathbf{u}$ and are peculiar to fast-rotation hydrodynamics. The frequency of this mode is $\omega = v_n k_z$. It must be emphasized that this mode exists only at finite temperatures. It vanishes at $T = 0$, i.e., in total absence of the normal component. It is here that the essential difference between fast-rotation hydrodynamics at finite temperatures differs from slow-rotation hydrodynamics. In the latter, in contrast to the former, the velocity $\mathbf{v}_s$ and the displacement $\mathbf{u}$ are not independent variables but are connected by the additional condition (10). In fast-rotation hydrodynamics the same takes place at $T = 0$, when the difference between $\mathbf{v}_s$ and $\mathbf{V}_s$ vanishes. The presence of the root $\omega = 0$ means here simply compatibility of the hydrodynamic equations with the supplementary condition (10).

A remark is in order also with respect to (32). Although the mode (32) has zero gap, its frequency, generally speaking, does not tend to zero as $\mathbf{k}\to 0$. This highlights the distinction of fast-rotation hydrodynamics, for the validity of which the condition $\mathbf{k}\to 0$ is generally speaking not sufficient and one more condition is required. To determine this condition we note that the frequency in a rotating coordinate system can be regarded as an eigenvalue of the operator

$$i\left\{\frac{\partial}{\partial t} + ([\boldsymbol{\Omega}\times\mathbf{r}], \nabla) - \boldsymbol{\Omega}\times\right\} = i\frac{\partial}{\partial t} - \Omega J_z, \tag{33}$$

where $J_z = L_z + S_z$, $L_z = -\mathbf{r}\times i\nabla_\perp$ is the orbital angular momentum, and $S_z = -i\varepsilon_{z\alpha\beta}$ is the spin of the vector field $\delta\mathbf{v}_\perp$. The derivatives $\partial\omega/\partial\Omega$, as can be seen from (33) is equal to $-\langle J_z\rangle$. For the frequencies of all the modes in fast-rotation hydrodynamics to tend to zero we must satisfy besides the condition $\mathbf{k}\to 0$ also $\langle J_z\rangle \to 0$. The latter is equivalent for the spectrum (32) to the condition $k_z/k\to 0$.

3. OSCILLATIONS OF A VORTEX FILAMENT

We consider one vortex filament aligned with the axis of a cylindrical vessel of radius R. It is known that such a state corresponds to thermodynamic equilibrium at $\Omega \sim (\hbar/mR^2)\ln(R/a)$. As shown by Hall,[16] the equation that describes the filament oscillations is reduced to a Schrödinger equation with mass $m^* = m/\ln(\hbar/pa)$ (p is the momentum of the oscillation) by introducing the wave function $\psi = \mathrm{const}(u_x + iu_y)$, where $\mathbf{u}$ is the two-dimensional vector of filament displacement in a plane perpendicular to the rotation axis. If we choose $\mathrm{const} = (\pi\rho_s/m)^{1/2}$, the energy of the oscillations also becomes identical with the Schrödinger expression

$$E = \frac{\hbar^2}{2m^*}\int\left|\frac{\partial\psi}{\partial z}\right|^2 dz. \tag{34}$$

Rotation about the z axis through an angle φ is in this case simultaneously the wave-function gauge transformation $\psi\to\psi e^{i\varphi}$. The generator of the gauge transformation is the operator $-\hbar_N$, where N is the particle-number operator and is expressed in the usual manner through the second-quantized operator ψ. It coincides therefore in this case with the operator J_z of the z-component of the angular momentum, $J_z = -\hbar N$. The filament-oscillation quanta have thus, independently of their momentum, an angular momentum equal to $-\hbar$.[16]

Since the equilibrium statistical distribution depends on the energy in the rotating system, the oscillations have an equilibrium Planck distribution function with argument

$$\varepsilon(p) = \hbar\Omega + \frac{p^2}{2m}\ln\left(\frac{\hbar}{pa}\right). \tag{35}$$

The presence of the energy gap $\hbar\Omega$ causes the spectrum (35) to satisfy the Landau superfluidity criterion. The critical velocity is

$$v_c = \left(\frac{\hbar\Omega}{m}\ln\frac{\hbar}{2m\Omega a^2}\right)^{1/2}. \tag{36}$$

For the same reason, the divergences at small momenta, which are customary for one-dimensional systems, are absent in this case. Indeed, let us calculate the mean squared fluctuation displacement of the filament from the equilibrium position, assuming satisfaction of the condition $T\gg\hbar\Omega$:

$$\langle\mathbf{u}^2\rangle = \frac{m}{\pi\rho_s}\langle|\psi|^2\rangle = \frac{m}{\pi\rho_s}\int\frac{dp}{2\pi\hbar}n_0,$$

where n_0 is the equilibrium distribution function which we can set equal in this case to T/ε. We have

$$\langle\mathbf{u}^2\rangle = mT/\pi\hbar\rho_s v_c. \tag{37}$$

The ratio of this quantity to the square of the vessel radius is of the order of

$$\frac{\langle\mathbf{u}^2\rangle}{R^2} \sim \frac{a}{R}\frac{Tma^2}{\hbar^2}\ln^{-1}\frac{\hbar}{2m\Omega a^2} \ll 1.$$

The contribution of the filament oscillations to the one-dimensional density of the normal component is calculated in similar fashion:

$$\rho_n = -\int p^2\frac{\partial n_0}{\partial\varepsilon}\frac{dp}{2\pi\hbar} = \frac{2T\Omega}{v_c^3}. \tag{38}$$

With decreasing Ω the density ρ_n varies in proportion to $\Omega^{-1/2}$. We write down also an expression for the heat capacity C_Ω per unit filament length at constant rotation velocity:

$$C_\Omega = C_0(T) + 2\Omega/v_c, \tag{39}$$

where

$$C_0(T) = \frac{3\zeta(3/2)}{4\hbar}\left\{\frac{mT}{\pi\ln(\hbar^2/mTa^2)}\right\}^{1/2}.$$

In contrast to (37) and (38), in the case of the heat capacity only the second correction term in (39) depends on the rotation frequency at $T\gg\hbar\Omega$.

The thermal oscillations delocalize the filament. As a result, the average velocity curl differs from a δ function and is determined by the probability distribution of the values of the distance r of the filament from the vessel axis. Since this distribution is obviously Gaussian at $T\gg\hbar\Omega$, we have in accord with (37)

$$\langle |\mathrm{rot}\,\mathbf{v}_s(r)|\rangle=\frac{2\hbar}{m\langle \mathbf{u}^2\rangle}\exp\left(-\frac{r^2}{\langle \mathbf{u}^2\rangle}\right).$$

The fact that the filament oscillations have an intrinsic angular momentum produces a unique effect of angular-momentum transport by the heat flux the absence of matter flux. The angular-momentum flux is

$$L=-\hbar|\psi|^2 v_n=-\frac{T}{v_c}v_n=-\frac{Q}{Sv_c},$$

where $Q = TSv_n$ is the heat flux.

The torque carried by the heat flux between solid surfaces perpendicular to the rotation axis is at $v_n \sim v_c$ of the order of $\mathcal{N}T$, where $\mathcal{N}$ is the total number of vortex filaments. At $T\sim 1$ K the torque in dyn·cm is of the order of $10^{-12}\Omega$ [sec^{-1}]S_0 [cm^2], where S_0 is the area of the solid surface. Although this is a small quantity, it seems to be experimentally observable.

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