Spectrum of weakly bound states of a particle in external electric fields

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A model-independent theory has been developed for the energy spectrum of weakly bound states of a particle in a short-range potential in external fields. On the basis of this theory we have obtained equations which determine the shifts and widths of shallow levels ($E=\varkappa_0^2/2, \varkappa_0 r_c \ll 1, r_c$ is the radius of the center) in a uniform electric field $F \ll r_c^{-3}$. The cases of weak fields $F \ll \varkappa_0^3$ and strong fields $F \gg \varkappa_0^3$ are studied. For a weak field we have obtained the hyperpolarizabilities and the corrections to the classical expressions for the widths of the states, and in a strong field we have obtained expressions for the shifts and widths. We have solved the problem of ionization of a shallow p-level by the electric field of a circularly polarized wave. Over a wide range of frequency and amplitude of the field we have found analytic expressions for the shifts and widths of all three quasienergy states arising from the initial level under the action of the wave. We have shown that in the antiadiabatic case the three widths have substantially different values. For a strong field we have found the shifts and widths of the levels in the adiabatic case. We have discussed the question of numerical calculations of the width of an s-level. In the high-frequency region we have obtained for the widths of states with arbitrary l expressions which relate them to the asymptotic behavior of the Fourier component of the potential.

The zero-range potential method[1–5] has been used to solve a large number of problems on the effect of external fields on a particle with angular momentum $l=0$ weakly bound in a short-range potential. In Refs. 6–9 attempts have been made to generalize this method to the case $1 \neq 0$. However, in the framework of the approaches proposed it was possible to consider only an extremely limited group of questions.

In this paper we develop a model-independent method of calculating the spectra of weakly bound states of a particle with arbitrary angular momentum l in external fields, and on the basis of this theory we discuss the problems of the shift and broadening of levels in a uniform electric field and in the field of a monochromatic circularly polarized electromagnetic wave (some results on application of the method to the case of an external magnetic field are given elsewhere[10]). The method consists of direct solution of the Schrödinger equation at large distances and at small distances, and for joining the wave functions, which is the most important aspect in a problem with a noncentral external field, we have generalized the approach in Landau and Lifshitz[11] in discussing the scattering of slow particles by a short-range center. Here the spectrum does not depend on the specific form of the potential of the center and is determined by the two parameters—the scattering length and the effective range.

1. EQUATION FOR THE SPECTRUM OF WEAKLY BOUND STATES OF A PARTICLE IN AN EXTERNAL FIELD

We shall consider the problem of determining the energy spectrum of weakly bound states (shallow real or quasidiscrete levels) of a particle characterized by a Hamiltonian ($\hbar=m=e=1$)

$$H=-{}^1/_2\Delta+U(r)+V_f(\mathbf{r})\equiv H_f+U(r), \qquad (1)$$

where $U(r)$ is a short-range central potential of radius r_c and $V_f(\mathbf{r})$ describes both the action of the external field and the interaction with other centers which themselves can bind the particle. By weakly bound we mean states with energy $E=k^2/2 \ll r_c^{-2}$. The interaction V_f in the region $r \lesssim r_c$ is assumed to be weak ($V_f \ll r_c^{-2}$), varying substantially only at distances $L_f \gg r_c$, and to permit exact solution of the Schrödinger equation with the Hamiltonian H_f. Assuming that in the potential U there is a shallow level with arbitrary angular momentum l, we obtain an equation which determines its shift and width under the action of v_f.

In the problem under discussion the Schrödinger equation permits exact solution both at large distances ($r \gg r_c$) where $U \simeq 0$ and at small distances ($r \ll L_f$), where it is possible to neglect V_f, so that the problem becomes spherically symmetric and we can use the idea of the effective-range approximation. The energy spectrum is determined from the condition of matching of the solutions in the region of overlap,

$$r_c \ll r \ll \min\{L_f, k^{-1}\}. \qquad (2)$$

For a solution of the problem of matching which takes into account the specific feature of the problem—the presence in the field U of a shallow level, we introduce a complete system $G^f_{lm}(\mathbf{r},E)$ of solutions of the Schrödinger equation with the Hamiltonian H_f as follows. We shall require that in addition to satisfying the boundary condition at infinity these solutions at $r \to 0$ contain singular terms of the form $r^{-l'-1}Y_{l'm'}(\mathbf{n})$ only with $l'=l$, and $m'=m$; here in the region (2) we have

$$G^f_{lm}=r^{-l-1}Y_{lm}(\mathbf{n})+\ldots+\sum_{l'm'} A^{l'm'}_{lm}(E,f)\,(r^{l'}Y_{l'm'}(\mathbf{n})+\ldots). \qquad (3)$$

These conditions uniquely determine both G^f_{lm} and $A^{l'm'}_{lm}$.

For $r \gg r_c$ the wave function of the described state is

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represented in the form $\Psi = \Sigma C_{l'm'} G^f_{l'm'}$, and in the region (2) it is possible to use the asymptotic behavior (3). At the same time in this region if we approach from the direction of small r we have[11]

$$\Psi=\sum \tilde{c}_{l'm'} Y_{l'm'} [r^{-l'-1}+\ldots+B_{l'}(E)(r^{l'}+\ldots)], \tag{4}$$
$$(2l-1)!!(2l+1)!!B_l=k^{2l+1}\operatorname{ctg}\delta_l(k)\approx-1/a_l+r_lE,$$

where δ_1, a_1, and r_1 are the phase shift, scattering length, and effective range for the potential U. If in $U(r)$ there is a shallow level with angular momentum[1)] l, then in this partial wave the scattering is anomalously large, but the phase shifts $\delta_{l'\neq 1}$ are small and in Eq. (4) we can omit the singular terms proportional to $r^{-l'-1}$ with $l'\neq 1$. Matching the wave functions in the region (2), we find that $c_{l'm'}\neq 0$ only for $l'=1$, and here $c_{lm}=\tilde{c}_{lm}$ and

$$B_l c_{lm}-\sum_{m'} A^{lm}_{lm'} c_{lm'}=0, \quad B_{l'}\tilde{c}_{l'm}=\sum_{m'} c_{lm'} A^{l'm}_{lm'}, \quad l'\neq l.$$

The condition of existence of a nontrivial solution of this system

$$\det\{[-1/a_l+r_lE]\delta_{mm'}-\tilde{A}^{lm}_{lm'}(E,f)\}=0 \tag{5}$$

is the relation which determines the energy spectrum; here

$$\tilde{A}^{lm}_{lm'}=(2l-1)!!(2l+1)!!A^{lm}_{lm'}.$$

We note that Eq. (5) is greatly simplified if $V_f(\mathbf{r})$ has certain symmetry properties. For example, in the case of axial symmetry it breaks up into $2l+1$ independent relations

$$-1/a_l+r_lE=A_{lm}(E,f), \quad \tilde{A}^{lm}_{lm'}\equiv A_{lm}\delta_{mm'}. \tag{6}$$

However, if V_f commutes with the operator I of reflection in the x, y plane, then the condition (5) breaks up into two independent equations which contain values of m and m' of definite identical parity.

Calculation of the spectrum on the basis of Eq. (5) in specific problems requires knowledge of the coefficients $A^{lm}_{l'm'}$ determined by the asymptotic relation (3) of the functions G^f_{lm}. The latter can be found from the known Green's function G^f_0 of the Hamiltonian H_f according to the formula

$$G_{lm}{}^f=\frac{2\pi}{(2l-1)!!}\{Y_{lm}(\nabla')G_0{}^f(\mathbf{r},\mathbf{r}',E)\}_{r'=0}, \tag{7}$$

where the differential operator $Y_{lm}(\nabla)$ is obtained by replacement of n_i by $\partial/\partial x_i$ in the spherical function

$$Y_{lm}(\mathbf{n})=\varepsilon_{i\ldots k}(l,m)n_i\ldots n_k$$

($\varepsilon_{i\ldots k}$ is a tensor of rank l symmetric in any pair of indices with zero trace $\varepsilon_{iij\ldots k}=0$). However, in specific problems it is sometimes possible to express G^f_{lm} directly in terms of $G^f_0(\mathbf{r},0,E)$.

2. PARTICLE IN AN EXTERNAL FIELD $V_f=Fz, Fr_c^3\ll 1$

For an electric field the general relation (7) is simplified since from the integral representation for the Green's function it follows that $\partial/\partial x'=-\partial/\partial x$, $\partial/\partial y'=-\partial/\partial y$, $\partial/\partial z'=-\partial/\partial z-F\partial/\partial E$, and therefore

$$G_{lm}{}^f=(-1)^l\frac{2\pi}{(2l-1)!!}Y_{lm}(\tilde{\nabla})G_0{}^f(\mathbf{r},0,E),$$
$$\tilde{\nabla}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}+F\frac{\partial}{\partial E}\right),$$
$$G_0{}^f(\mathbf{r},0,E)=(8\pi^3 i)^{-1/2}\int_0^\infty Q(t,\mathbf{r},E,F)t^{-3/2}\,dt, \tag{8}$$
$$Q=\exp\left[i\left(Et+\frac{r^2}{2t}-\frac{1}{2}Fzt-\frac{1}{24}F^2t^3\right)\right].$$

1. The function G^f_{lm} is expressed particularly simply for the case $m=l$:

$$G_{ll}{}^f=N_l\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^l G_0{}^f$$
$$=N_l i^l(8\pi^3 i)^{-1/2}(x+iy)^l\int_0^\infty Q(t,\mathbf{r},E,F)t^{-l-3/2}\,dt, \tag{9}$$

where N_l is a normalization factor; since in an electric field the energies of states differing only in the sign of the projection of the angular momentum m are identical and the wave functions are connected by the relation $\Psi_{E,-m}(\mathbf{r})=\Psi_{E,m}(x,-y,z)$, in what follows we shall consider only states with $m\geqslant 0$.

Separating from the integral in (9) in accordance with (3) and (6) the constant term in its expansion in powers of $z^k r^n$, we find A_{ll} and in accordance with (6) obtain an equation which determines the spectrum:

$$-\frac{1}{a_l}+r_lE=A_{ll}$$
$$=(-2i)^{l+1/2}\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-3/2}\frac{d^{l+1}}{dt^{l+1}}\exp\left[i\left(Et-\frac{1}{24}F^2t^3\right)\right]dt. \tag{10}$$

a) We shall consider first the case of a weak electric field $F\ll\varkappa_0^3$; here $E^{(0)}_l=-\varkappa_0^2/2$ is the energy of the level in the absence of the field. Calculating the integral in (10) by the method of steepest descent (this method was used in Ref. 12 for $l=0$), we obtain the equation $[\varkappa=(-2E)^{1/2}$, $|\arg\varkappa|<\pi/6]$

$$-\frac{1}{a_l}+r_lE=(-1)^{l+1}\varkappa^{2l+1}$$
$$\times\left(1-\frac{1}{3}g_l^{(1)}F^2\varkappa^{-6}+\frac{1}{9}g_l^{(2)}F^4\varkappa^{-12}+\ldots\right) \tag{11}$$
$$+i(2l+1)!!\left[1-\frac{1}{12}(3l^2+15l+17)F\varkappa^{-3}+\ldots\right]$$
$$\times\frac{F^{l+1}}{(2\varkappa)^{l+2}}\exp\left(-\frac{2\varkappa^3}{3F}\right),$$
$$g_l^{(1)}=\frac{1}{8}(2l-3)(4l^2-1),$$
$$g_l^{(2)}=\frac{1}{128}(2l+1)(2l-1)\ldots(2l-9).$$

The first term is determined by the beginning of the integration contour, and the second by the saddle point. Representing the energy in the form

$$E_{lm}=E_l^{(0)}-{}^1/_2\beta_1(l,m)F^2-{}^1/_4\beta_2(l,m)F^4-{}^1/_2i\Gamma_{lm}$$

and solving (11) by an iterative method, we find the polarizabilities, the hyperpolarizabilities of second order, and the widths of states with $m=l$:

$$\beta_1(l,l)=\frac{2}{3}(-1)^l g_l^{(1)}C_{\varkappa l}^2\varkappa_0^{-4}, \qquad (12)$$

$$\beta_2(0,0)=\frac{1}{32}(105-10C_{\varkappa0}^2+C_{\varkappa0}^4)C_{\varkappa0}^2\varkappa_0^{-10}. \qquad (13)$$

$$\beta_2(l,l)=\frac{4}{9}(-1)^{l+1}g_l^{(2)}C_{\varkappa l}^2\varkappa_0^{-10}, \quad l\neq0, \qquad (14)$$

$$\Gamma_{ll}=\left[1-\beta_1(l,l)F\varkappa_0-\frac{1}{12}(3l^2+15l+17)F\varkappa_0^{-3}\right]\Gamma_{ll}^{(0)}. \qquad (15)$$

Here $\Gamma_{ll}^{(0)}$ coincides with the result of a calculation of the width in the quasiclassical approximation[13,14]

$$\Gamma_{lm}^{(0)}=\varkappa_0^2C_{\varkappa l}^2\frac{(2l+1)(l+m)!}{2^{2m+1}m!(l-m)!}\left(\frac{F}{\varkappa_0^3}\right)^{m+1}\exp\left(-\frac{2\varkappa_0^3}{3F}\right). \qquad (16)$$

For $l=0$ and 1 the expressions for the polarizability coincide with the well known results.[15,6,8] For $l\geqslant3$, as follows from (12) (see also Footnote 1), $\beta_1\propto(\varkappa_0 r_c)^{2l-5}\to0$ as $\varkappa_0\to0$; this means that for these l values the polarizability already is not related to large distances[16] but is determined by the action of the field on the particle in the region of localization of the wave function $r\lesssim r_c$, has an order of magnitude $(me^2/\hbar^2)r_c^4$, and depends on the specific form of the potential $U(r)$.

From Eq. (13) for the case $C_{\varkappa_0}^2=1$ we obtain the well known result for the hyperpolarizability in a potential of zero range.[17] Since $\beta_2\propto(\varkappa_0 r_c)^{2l-11}$, the features of the behavior of weakly bound states in an electric field appear more distinctly in the hyperpolarizability values than for the polarizability values also for larger values of angular momentum ($l\leqslant5$).

b) Up to the present time we have been discussing a weak field. The case of a strong field $F\gg\varkappa_0^3$ (but as before $Fr_c^3\ll1$) also permits analytic study. For such fields the shift and width of a level with $l\neq0$ satisfy the condition ΔE, $\Gamma\ll F^{2/3}$ ($\Delta E,\Gamma\sim F^{2/3}$ for $l=0$), so that in calculation of the integral in (10) we can make use of an expansion in powers of $E/F^{2/3}$. We shall give the final results:

$$\Gamma_{ll}=\frac{(2l-1)!!\sqrt{\pi}}{2^{l-1}3^{(2l+1)/6}\Gamma((2l+1)/6)}C_{\varkappa l}^2\left(\frac{F^{1/3}}{\varkappa_0}\right)^{2l+1}\varkappa_0^2, \quad l\neq0, \qquad (17)$$

$$\Delta E_{1,1}=-\frac{3^{1/3}\Gamma(5/6)}{4\sqrt{\pi}}C_{\varkappa1}^2\left(3+C_{\varkappa1}^2\frac{F}{\varkappa_0^3}\right)F^{2/3}\varkappa_0, \qquad (18)$$

$$\Delta E_{2,2}=-\frac{3^{2/3}\sqrt{\pi}}{4\Gamma(5/6)}C_{\varkappa2}^2\left(\frac{F^{1/3}}{\varkappa_0}\right)^5\varkappa_0^2. \qquad (19)$$

For $l\geqslant3$ the contribution to the level shift from large distances ($\Delta E_1\sim(r_c F^{1/3})^{2l-5}F^2r_c^4$), as in the case of a weak field, is unimportant in comparison with $\Delta E_l\sim F^2r_c^4$—the contribution from the region $r\lesssim r_c$.

The properties of the width which follow from (17)—its independence of the binding energy (see the first footnote) and the power dependence on the field—reflect the quasiclassical nature of the situation. In fact

$$\Gamma\sim\frac{v}{r_c}\exp\left\{-2\int_{\sim r_c}^{a}\left[\frac{(l+1/2)^2}{z^2}+\varkappa^2-2Fz\right]^{1/2}dz\right\}$$

$$\sim r_c^{-2}(r_cF^{1/3})^{2l+1}$$

(the term $\varkappa^2$ in the square root is a correction, in contrast to the weak-field case when the centrifugal barrier is unimportant). Note that our discussion of a strong field assumes the weakly bound nature of the state ($\varkappa_0^3\ll F\ll r_c^{-3}$). In the opposite case the condition $F\gtrsim\varkappa_0^3$ corresponds to atomic fields, which destroy the system in an atomic time $\sim r_c^2$.

2. In the case $m=l-1$ in accordance with (8) we have

$$G_{l,l-1}^f=N_l\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^{l-1}\left(\frac{\partial}{\partial z}+F\frac{\partial}{\partial E}\right)G_0^f$$

$$=(-1)^{l+1}\left(\frac{2l+1}{8\pi^2 i(2l-1)!}\right)^{1/2} \qquad (20)$$

$$\times(x+iy)^{l-1}\int_0^\infty\left(z+\frac{1}{2}Ft^2\right)t^{-l-3/2}Q(t,\mathbf{r},E,F)\,dt.$$

The equation for the spectrum can then be converted to the form

$$-\frac{1}{a_l}+r_lE=A_{l,l-1}=\left(1+3F\frac{\partial}{\partial F}\right)A_{ll}. \qquad (21)$$

Using the expression obtained previously for A_{ll}, we find from (21) in case a) of a weak field and case b) of a strong field:

$$\text{a)}\ \beta_1=7\beta_1(l,l),\ l=1,2;\quad \beta_2=13\beta_2(l,l),\ 1\leqslant l\leqslant5, \qquad (22)$$

$$\Gamma_{l,l-1}=[1-{}^1/_{12}(3l^2-3l-7)F\varkappa_0^{-3}]\Gamma_{l,l-1}^{(0)}; \qquad (23)$$

$$\text{b)}\ \Delta E_{1,0}=2\Delta E_{1,1},\quad \Delta E_{2,1}=6\Delta E_{2,2},\quad \Gamma_{l,l-1}=2(l+1)\Gamma_{ll}$$

[see Eqs. (17)–(19)].

3. In the case $m=l-2$ we find

$$A_{l,l-2}=\left(1+\frac{9}{2}F^2\frac{\partial^2}{\partial F^2}+\frac{3}{2l-1}FE\frac{\partial^2}{\partial F\,\partial E}\right)A_{ll} \qquad (24)$$

and according to (6) and (11) for weak and strong fields we obtain

$$\text{a)}\ \beta_1(2,0)=9\beta_1(2,2),\quad \beta_2=\frac{98l-121}{2l-1}\beta_2(l,l),\quad 2\leqslant l\leqslant5, \qquad (25)$$

$$\Gamma_{l,l-2}=\left[1-\frac{3l^3-24l^2+32l-11}{12(l-1)}F\varkappa_0^{-3}\right]\Gamma_{l,l-2}^{(0)};$$

$$\text{b)}\ \Delta E_{2,0}=6\Delta E_{2,2},\quad \Gamma_{l,l-2}=l(2l-1)\Gamma_{ll}. \qquad (26)$$

We can carry out a similar discussion of the case of other values of l and m. However, to find the hyperpolarizability such calculations are unnecessary. In fact, since $\beta_2(l,m)=a+bm^2+cm^4$, on the basis of the three values of this expression calculated above for $m=l$, $l-1$, and $l-2$ we can find a, b, and c and with them we can find also the values of β_2 for the remaining m values.

3. PARTICLE IN THE FIELD OF A CIRCULARLY POLARIZED WAVE

The Hamiltonian $H(t)$ of a charged particle acted on by a central potential U in the field of a monochromatic circularly polarized electromagnetic wave in the dipole approximation on transformation to a rotating coordinate system takes the form[18]

$$H_\omega=-{}^1/_2\Delta+U(r)-\omega\mathbf{L}+Fx,\quad V_f=-\omega L_z+Fx, \tag{27}$$

where ω and F are the frequency and amplitude of the electric field of the wave and the z axis is directed along $\boldsymbol{\omega}$. The spectrum of complex quaisenergy levels of the operator $H(t)$ coincides with the spectrum of ordinary quasidiscrete levels of the static Hamiltonian H_ω. Therefore in determining the shift (splitting) and width of a shallow real level with angular momentum l in the potential U under the influence of the wave field one can use the method developed in Section 1.

1. The case $l=0$. Taking into account the form of the Green's function of the operator $H_f = -1/2\Delta + V_f$

$$G_0^f(\mathbf{r},0,E)=i^{-1/2}\int_0^\infty (2\pi t)^{-3/2} \times \exp\left\{i\left[Et+\frac{r^2}{2t}+\varphi_1(t)x+\varphi_2(t)y+\varphi(t)\right]\right\}dt,$$

$$\varphi_1=(\cos\omega t-1)F/\omega^2t,\ \varphi_2=(\omega t-\sin\omega t)F/\omega^2 t, \quad \varphi=[1-\cos\omega t-{}^1/_2(\omega t)^2]F^2/\omega^4 t, \tag{28}$$

according to Eq. (6) we obtain the equation for the energy of an s-level

$$-\frac{1}{a_0}+r_0E=(2\pi i)^{-1/2}\int_0^\infty t^{-3/2}\{\exp[iEt+i\varphi(t)]-1\}dt. \tag{29}$$

The results obtained from Eq. (29) for $F\ll\varkappa_0^3$ reproduce to a significant degree the corresponding results for a state with $l=0$ in a short-range potential obtained by various methods in Refs. 19–25, 13, and 17. However, a number of statements which exist in the literature require clarification.

Let us discuss first the question of calculation of the total width of the level $\Gamma=\Sigma_n\Gamma_n$. The expression for the width of n-photon ionization which follows from Eq. (29) (Refs. 20, 13, 17, and 22)

$$\Gamma_n=\varkappa_0\omega^2C_{\varkappa0}^2F^{-1}\int_0^B J_{2n}(z)\,dz, \tag{30}$$

$$B=2^{3/2}(n\omega-\tilde{E})^{1/2}F/\omega^2,\ \tilde{E}=-E_0-\Delta E+F^2/2\omega^2,$$

on expansion of the Bessel function in series takes the form

$$\Gamma_n=\Gamma_n^{(0)}\left\{1-\frac{(n-K_0+\delta)}{2n+3}\frac{2F^2}{\omega^3}+\frac{(n-K_0+\delta)^2}{4(n+1)(2n+5)}\left(\frac{2F^2}{\omega^3}\right)^2+\dots\right\}, \tag{31}$$

where

$$\Gamma_n^{(0)}=\frac{2\sqrt{2}\varkappa_0\omega^{1/2}}{(2n+1)!}C_{\varkappa0}^2\left(\frac{2\varkappa_0^2}{\omega\gamma^2}\right)^n(n-K_0+\delta)^{n+1/2},$$

$$K_0=\left[\left(-E_0-\Delta E+\frac{F^2}{2\omega^2}\right)\omega^{-1}+1\right], \tag{32}$$

$$\delta=K_0+\left(E_0+\Delta E-\frac{F^2}{2\omega^2}\right)\omega^{-1},$$

where ΔE is the level shift, K_0 is the threshold number of absorbed photons, and $\gamma=\omega\varkappa_0/F$ is the Keldysh parameter.

Analysis of Eqs. (31) and (32) shows that in calculation of the total width it is convenient to introduce three different overlapping regions of values of the parameters ω and γ in which ionization channels corresponding to different numbers of absorbed photons n contribute substantially to $\Gamma(\omega,\gamma)$.

1) The asymptotic region[2)]:

$$\omega\lesssim\varkappa_0^2,\ \gamma\gg1;\ \omega\ll\varkappa_0^2,\ 2\ln\gamma>K_0\gg1. \tag{33}$$

For an arbitrary fixed frequency and $\gamma\to\infty$ the width of the level is completely determined by the ionization channel with $n=K_0=[\varkappa_0^2/2\omega+1]$, so that $\Gamma\approx\Gamma_{K_0}^{(0)}\propto\gamma^{-2K_0}\propto F^{2K_0}$, which is the result of perturbation theory in the first nonvanishing order in the electric field. However, the approach of $\Gamma(\omega,\gamma)$ to this asymptotic regime usually is highly protracted. This is due to the kinematic suppression of the n-photon ionization channels $\Gamma_n\propto p_n^{2n+1}$ with decrease of n (p_n is the momemtum of the photoelectron in the corresponding channel). This suppression leads to the result that at a large but finite value of γ absorption processes with $n>K_0$ which occur in higher orders of perturbation theory can contribute substantially to the total width. Here, as can be seen from Eq. (31), $\Gamma_n\approx\Gamma_n^{(0)}$, which corresponds essentially to the possibility of calculating Γ_n in the first nonvanishing order of perturbation theory. It is easy to see that under the conditions (33) we have $\Gamma_{n+1}\ll\Gamma_n$ for $n>K_0$, so that a kinematic enhancement can appear only for $n=K_0+1$ and the total width is

$$\Gamma\approx\Gamma_{K_0}^{(0)}+\Gamma_{K_0+1}^{(0)}. \tag{34}$$

We note that for $K_0\sim1$ (few-photon ionization) a suppression of the dominant channel with $n=K_0$ can occur only near threshold, i.e., at frequencies for which $\delta\ll1$ [in the opposite case in Eq. (34) it is possible to restrict the discussion to the one term with $n=K_0$]. Here for values $\gamma\sim\delta^{-(2K_0+1)/4}\equiv\gamma_0\gg1$ the two terms in (34) are of the same order. The approach to the asymptotic region $\Gamma\approx\Gamma_{K_0}^{(0)}$ occurs for $\gamma\gg\gamma_0\gg1$. For values $1\ll\gamma\ll\gamma_0$ in (34), on the other hand, the term $\Gamma_{K_0+1}^{(0)}$ is dominant.

2) The intermediate region:

$$\omega\ll\varkappa_0^2,\ \gamma\gg1,\ \text{but}\ \varepsilon K_0\lesssim2\ln\gamma<K_0,\ \text{where}\ \varepsilon\ll1. \tag{35}$$

For a fixed frequency ω with decrease of γ the kinematic enhancement of channels with larger n values leads to an increase of their role. Here there is an increase both in the most probable number of absorbed photons n_0 and in the number Δn of ionization channels in the vicinity of n_0 which contribute substantially to the toal width. As long as the number of such channels is small, the efficient method of calculating the total width is direct summation of the contributions of these channels. However, the case $\Delta n\gg1$ corresponds to the quasiclassical situation.

3) The quasiclassical region:

$$\omega\ll\varkappa_0^2,\ \gamma\lesssim1,\quad\text{or}\quad\gamma\gg1,\ \text{but}\ 2\ln\gamma\ll K_0. \tag{36}$$

In this region of parameters a large number of ionization channels contribute to the total width. However, in calculating $\Gamma(\omega,\gamma)$ there is no need to sum the partial widths, since Eq. (29) can be used directly. Calculating the integral in this equation by the saddle point method, as in the case of an electrostatic field, we obtain for the total width of the level

$$\Gamma=\frac{\omega\gamma C_{\varkappa_0}^2}{2\tau_0}[\mathrm{ch}\,2\tau_0-(1+\gamma^2)]^{-1/2}\exp\left[-\frac{\varkappa_0^2}{\omega}f(\gamma)+\frac{2\Delta E\tau_0}{\omega}\right], \tag{37}$$

where

$$f(\gamma)=2\tau_0(1+\gamma^{-2})-\gamma^{-2}\,\mathrm{sh}\,2\tau_0, \tag{38}$$

$$\gamma^2=\tau_0^{-1}\,\mathrm{sh}\,2\tau_0+(1-\mathrm{ch}\,2\tau_0)/2\tau_0^2-1,$$

and $t_0=2i\tau_0/\omega$ is the saddle point. These expressions in parametric form solve the problem of determining the total width of an s-level. For $\omega\ll\varkappa_0^2$ the level shift is $\Delta E\approx-C_{\varkappa 0}^2F^2/8\varkappa_0^4$ and it is easy to see that $|\Delta E|\tau_0\ll C\omega$; here Eqs. (37) and (38) coincide with the results obtained in Refs. 20 and 13.

As an illustration of the dependence of the level width and the role of the various ionization channels on the amplitude of the electric field of the wave, we have shown in Fig. 1 the results of a calculation of the total width as a function of γ for $\omega=0.06\varkappa_0^2$ [the results of a numerical calculation of the width of an s-level according to Eq. (29) with $r_0=0$ which are shown in Figs. 2 and 3 in Ref. 17 must be considered erroneous]. The straight line 1 gives the extrapolation of the relation $\Gamma_{\text{extrap}}=\Gamma_{K_0}^{(0)}\propto\gamma^{-2K_0}$ from the region of large values of γ and was calculated on the basis of Eq. (32) with $K_0=9$ and $\delta=2/3$. The portion of curve 2, which represents the total width, for values $\gamma<5$ was constructed according to Eqs. (37) and (38), and for $\gamma>5$ the curve was calculated on the basis of the formula $\Gamma=\Sigma_n\Gamma_n$. For $\gamma=5$ (here $\Gamma/\Gamma_{\text{extrap}}\approx 12$) there is a substantial contribution to the total width from the ionization channels with $n=10$–13, the relative probabilities of which are

$$\Gamma_9:\Gamma_{10}:\Gamma_{11}:\Gamma_{12}:\Gamma_{13}:\Gamma_{14}=2\cdot10^{-4}:0.48:1:0.44:0.08:0.01$$

[we note that for this value of γ in calculation of the partial widths Γ_n in (31) it is necesary to take into account corrections proportional to F^2]. Although in the present case $\Delta n=3$–4 the result of the quasiclassical calculation differs from the exact calculation by about 25%.

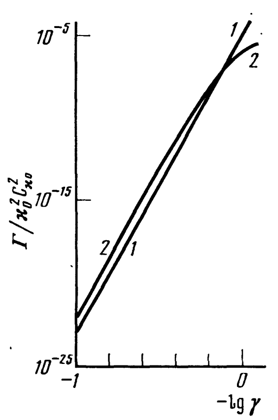

FIG. 1.

For $\gamma=10$ the relative contributions of the important channels to the width are

$$\Gamma_9:\Gamma_{10}:\Gamma_{11}:\Gamma_{12}=0.06:1:0.29:0.02$$

and $\Gamma/\Gamma_{\text{extrap}}\approx 6.2$. For $\gamma=40$ only the contributions of two channels are important, and here $\Gamma_9/\Gamma_{10}\approx 1.9$, while $\Gamma/\Gamma_{\text{extrap}}\approx 1.4$. With further increase of γ the relative contribution of the channel with $n=10$ decreases in proportion to γ^{-2}, and the level width in accordance with (34) reaches the asymptotic region $\Gamma\approx\Gamma_{\text{extrap}}(1+C/\gamma^2)$ with

$$C=\left[\frac{(K_0+1-\varkappa_0^2/2\omega)^{K_0+1/2}}{(K_0+1)(2K_0+3)(K_0-\varkappa_0^2/2\omega)^{K_0+1/2}}-\frac{2K_0+1}{4(K_0-\varkappa_0^2/2\omega)}\right]\times\frac{\varkappa_0^2}{\omega}$$

and in this case $C\approx 680$. This large numerical value of C reflects the kinematic enhancement mentioned above of the channel with $n=K_0+1$ for $K_0\gg 1$.

Note that the quasiclassical formula (37) correctly reproduces the dependence of the width on the field over a wide range of γ: the ratio $\Gamma/\Gamma_{\text{qu}}$ rises only to 2.5 on change of γ from 1 to 1000, while the value of Γ itself changes by 54 orders of magnitude.

Finally we note that the relation for the total width of an s-level $\Gamma\propto\omega^{-5/2}$ for $\omega\gg\varkappa_0^2$, obtained in the zero-range potential model[17,22] reflects the specific properties of the model, since it is strongly related to the behavior of the wave function $\Psi\propto 1/r$ at small distances and has a natural region of applicability $\omega\ll r_c^{-2}$ [see Eq. (44) below].

2. The case $l=1$ and $m=0$. We shall turn to discussion of the influence of a circularly polarized wave on a level with angular momentum $l=1$. This triply degenerate level in the potential U will correspond to three quasidiscrete levels of the Hamiltonian (27). Since the reflection operator $I(z\to-z)$ commutes with the operator H_f, according to Eq. (5) one of these levels is characterized by $m=0$, and the other two are superpositions of states with $m=\pm1$.

The function $G_{1,0}^f$ is expressed in terms of the Green's function (28):

$$G_{1,0}^f=-i(3\pi)^{1/2}\frac{\partial}{\partial z}G_0^f(\mathbf{r},0,E).$$

Separating from it for $r\to0$ the term proportional to $rY_{10}\propto z$, we obtain an equation which determines the energy of the quasidiscrete level with $l=1$ and $m=0$:

$$-\frac{1}{a_1}+r_1E=-3i(2\pi i)^{-1/2}\int_0^\infty\{\exp[iEt+i\varphi(t)]-1-iEt\}t^{-5/2}\,dt. \tag{39}$$

Still restricting the discussion to the case of a weak field $F\ll\varkappa_0^3$, in the lowest approximation in F^2 we obtain

$$E_{1,0}=E_1^{(0)}-\frac{C_{\varkappa 1}^2F^2}{5\varkappa_0\omega^4}\left[\varkappa_0^5+\frac{15}{2}\varkappa_0\omega^2-\frac{1}{2}(2\omega+\varkappa_0^2)^{5/2}+\frac{i}{2}(2\omega-\varkappa_0^2)^{5/2}\right]. \tag{40}$$

From this for $\omega \ll \varkappa_0^2$ it follows that

$$E_{1,0}-E_1^{(0)} \approx -\frac{C_{\varkappa 1}{}^2F^2}{8\varkappa_0{}^4}\left(1+\frac{\omega^2}{2\varkappa_0{}^4}\right) \tag{41}$$

[for $\omega = 0$ Eq. (41) coincides with the shift in an electrostatic field of a level with angular-momentum projection on the field direction[3)] $|m| = 1$].

For $\omega > \varkappa_0^2/2$ there is an imaginary part in $E_{1,0}$ which determines the width of one-photon ionization of the state considered:

$$\Gamma_{1,0} \approx \frac{C_{\varkappa 1}{}^2F^2}{5\varkappa_0\omega^4}(2\omega-\varkappa_0{}^2)^{5/2}. \tag{42}$$

The behavior of the width $\Gamma_{1,0} \propto p_k^5$, where $p_k = (2\omega - \varkappa_0^2)^{1/2}$ is the momentum of the emitted electron, is related to the transmission of the centrifugal barrier $\propto p_k^{2l_k+1}$, since an electron in the state with $l = 1$ and $m = 0$, having absorbed a dipole photon with $j_z = 1$, has angular momentum $l_k = 2$.

In the high-frequency region $\omega \gg \varkappa_0^2$ we have according to (40)

$$\Delta E_{1,0} = \frac{2^{3/2}C_{\varkappa 1}{}^2F^2}{5\varkappa_0\omega^{3/2}} \propto \omega^{-3/2}, \qquad \Gamma_{1,0} = \frac{2^{5/2}C_{\varkappa 1}{}^2F^2}{5\varkappa_0\omega^{3/2}} \propto \omega^{-3/2}. \tag{43}$$

However, these relations have a limited region of applicability: $\varkappa_0^2 \ll \omega \ll r_c^{-2}$. The physical reason for this is that a particle with low energy, having absorbed a photon of frequency $\omega \gtrsim r_c^{-2}$, acquires an energy $E \gtrsim r_c^{-2}$ and is no longer slow, so that the assumptions of Section 1 are not satisfied.

At frequencies $\omega \sim r_c^{-2}$ the shift and width of the level will depend substantially on the form of the wave function in the region $r \lesssim r_c$. However, for $\omega \gg r_c^{-2}$ simple regularities again appear. According to the perturbation-theory formula for the width

$$\Gamma_{lm} = \frac{\pi}{2}F^2\sum_s |\langle s|x+iy|lm\rangle|^2\delta(\omega-\omega_{0s})$$

[the summation is carried out over the complete set of states in the potential $U(r)$] we obtain for $\omega \gg r_c^{-2}$

$$\Gamma_{lm} = 2^{3/2}\frac{l^2+l+m^2-1}{(2l-1)(2l+3)}\omega^{-1/2}F^2\chi_l^2(\sqrt{2\omega})$$
$$= \frac{2}{\pi^2}\frac{l^2+l+m^2-1}{(2l-1)(2l+3)}F^2R_l^2(0)\left[p^{l-5/2}\left(\frac{\partial}{p\partial p}\right)^l\tilde{U}(p)\right]^2_{p=\sqrt{2\omega}}, \tag{44}$$

where $\varphi_{lm}(\mathbf{p}) = \chi_l(p)Y_{lm}(\mathbf{p}/p)$ is the wave function of the state under discussion in the momentum representation and R_l is related to the function $\Psi_{lm}(\mathbf{r}) = r^lR_l(r)Y_{lm}(\mathbf{n})$. In derivation of (44) it is necessary, after making use of the relation $i\omega_{sn}\langle s|\mathbf{r}|n\rangle = \langle s|\mathbf{p}|n\rangle$, to replace the wave function $|s\rangle$ in the matrix element by a plane wave (this substitution can be carried out just in the matrix element of the momentum, but not of the coordinate), to integrate over the angles of $\mathbf{p}$, and to take into account the relation which connects the asymptotic behavior of the wave function in the momentum representation with the Fourier component of the potential $\tilde{U}(p)$ in the case in which it falls off according to a power law as $p \to \infty$.[26]

By continuing the iterations it is possible to find the shift and width of the level in higher order in F^2. In particular, the expression for the width of n-photon ionization has the form

$$\Gamma_{1,0;n} = \frac{3C_{\varkappa 1}{}^2\omega^6}{4\varkappa_0F^3}\int_0^B dz\int_0^z J_{2n}(z')dz' = \Gamma_{1,0;n}^{(0)}\left\{1-\frac{(n-K_0+\delta)}{2n+5}\frac{2F^2}{\omega_3}\right.$$
$$\left.+\frac{(2n+3)(n-K_0+\delta)^2}{4(n+1)(2n+5)(2n+7)}\left(\frac{2F^2}{\omega^3}\right)^2+\ldots\right\}, \tag{45}$$

where

$$\Gamma_{1,0;n}^{(0)} = \frac{2^{5/2}3\omega^{3/2}C_{\varkappa 1}{}^2}{(2n+3)(2n+1)!\varkappa_0}\left(\frac{2F^2}{\omega^3}\right)^n(n-K_0+\delta)^{n+3/2}, \tag{46}$$

and the values of B, K_0, and δ are determined by Eqs. (30) and (32).

The calculation of the total width in the regions of variation of the parameters ω and γ 1)–3) given above is carried out in exactly the same way as in the case $l = 0$. Here in the quasiclassical region the calculation of the integral in (39) by the method of steepest descent gives for the width

$$\Gamma_{1,0} = \frac{3\gamma\omega^2}{4\tau_0{}^2\varkappa_0{}^2}C_{\varkappa 1}{}^2[\operatorname{ch}2\tau_0-(1+\gamma^2)]^{-1/2}\exp\left[-\frac{\varkappa_0{}^2}{\omega}f(\gamma)\right] \tag{47}$$

[see Eqs. (37) and (38); the level shift is negligible in this case].

3. The case $l = 1$ and $m = \pm 1$. To determine the parameters of the quasidiscrete levels corresponding to $l = 1$ and representing superpositions of states with angular-momentum projections $m = \pm 1$ on the direction of propagation of the wave, it is necessary to know the functions $G_{1,\pm 1}$. These functions can be expressed directly in terms of the Green's function (28). For this purpose we introduce the operators

$$\Pi_\pm = \frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y} \pm \frac{F}{\omega}, \qquad [\Pi_\pm, H_f] = \pm\omega\Pi_\pm, \tag{48}$$

so that for $r \neq 0$

$$(H_f - E \pm \omega)\Pi_\pm G_0^f(\mathbf{r}, 0, E) = 0.$$

The functions $\tilde{G}^f_{1,\pm 1} = \Pi_\pm G_0^f(\mathbf{r}, 0, E\pm\omega)$ differ from $G^f_{1,\pm 1}$ only in one respect: they contain an extra singular term $\propto 1/r$ corresponding to $l = 0$. After compensating this term by addition to $\tilde{G}^f_{1,\pm 1}$ of a term proportional to G_0^f, we obtain

$$G^f_{1,\pm 1}(\mathbf{r}, E) = \pm i\left(\frac{3\pi}{2}\right)^{1/2}\left\{\Pi_\pm G_0^f(\mathbf{r}, 0, E\pm\omega) \mp \frac{F}{\omega}G_0^f(\mathbf{r}, 0, E)\right\}. \tag{49}$$

The quantities $A^{1m'}_{1m}$ which enter into Eq. (5) are found from Eq. (3) and have the form

$$A^{1,1}_{1,1}(E, F, \omega) = A^{1,-1}_{1,-1}(E, F, -\omega)$$
$$= \left(\frac{i}{2\pi}\right)^{1/2}\int_0^\infty t^{-5/2}\left\{1+i(E+\omega)t\right.$$
$$\left.-e^{iEt+i\varphi(t)}\left[e^{i\omega t}-\frac{iF^2}{2\omega^4 t}(1+i\omega t-e^{i\omega t})^2\right]\right\}dt,$$
$$A^{1,-1}_{1,1}(E, F, \omega) = A^{1,1}_{1,-1}(E, F, \omega)$$

$$=\frac{1}{(8\pi i)^{1/2}}\frac{F^2}{\omega^4}\int_0^\infty t^{-1/2}e^{iEt+i\varphi(t)}(1+i\omega t-e^{i\omega t})(1-i\omega t-e^{-i\omega t})\,dt. \tag{50}$$

The energies of the levels are given by the equation

$$\left[-\frac{1}{a_1}+r_1(E+\omega)-3A_{1,1}^{1,1}(E,F,\omega)\right]\left[-\frac{1}{a_1}+r_1(E-\omega)-3A_{1,1}^{1,1}(E,F,-\omega)\right]=9[A_{1,1}(E,F,\omega)]^2, \tag{51}$$

the difference of which from Eq. (5) (the terms $\pm r_1\omega$) corresponds to the fact that in solution of the Schrödinger equation in the region $r\lesssim r_c$ in the Hamiltonian H_ω only the term Fx has been neglected, while the term $-\omega L_z$ has been taken into account. This refinement of Eq. (5) permits inclusion in the discussion of the high-frequency case $\omega\gtrsim\varkappa_0^2$. For a weak field a solution of Eq. (51) can be obtained in the form of an expansion in powers of $F/\varkappa_0^3$.

For $\omega\ll\varkappa_0^2$ in the lowest approximation in this parameter we obtain

$$A_{1,1}^{1,1}=\frac{2^{3/2}i}{3}(E+\omega)^{3/2}-\frac{iF^2}{6\varkappa_0^3},\quad A_{1,1}^{1,-1}=\frac{iF^2}{8\varkappa_0^3}$$

and in accordance with Eq. (51) we obtain

$$E_1^{(\pm)}=E_1^{(0)}+\frac{1}{r_1}\left[\frac{1}{2}F^2\varkappa_0^{-3}\pm\left(r_1^2\omega^2+\frac{9}{64}F^4\varkappa_0^{-6}\right)^{1/2}\right]. \tag{52}$$

Hence for $\omega\ll F^2\varkappa_0^{-3}r_c\ll\varkappa_0^2$ it follows that

$$E_1^{(\pm)}=E^{(0)}+\frac{1}{r_1}\left[\frac{4\pm3}{8}F^2\varkappa_0^{-3}\pm\frac{4}{3}r_1^2\varkappa_0^3\omega^2F^{-2}\right] \tag{53}$$

[for $\omega=0$ Eq. (53) goes over into the results for an electrostatic field], and for $F^2\varkappa_0^{-3}r_c\ll\omega\ll\varkappa_0^2$ we obtain

$$E_1^{(\pm)}=E_1^{(0)}\mp\omega+\frac{1}{2}r_1^{-1}\varkappa_0^{-3}F^2\mp\frac{9}{128}r_1^{-2}\omega^{-1}\varkappa_0^{-6}F^4 \tag{54}$$

[for the condition $\omega\ll(r_c\varkappa_0)\varkappa_0^2$ inclusion of the term proportional to F^4 is legal in spite of the fact that the result was obtained in lowest order in $(F/\varkappa_0^3)^2$]. The last two terms in Eq. (54) determine the dynamic polarizabilities $\beta_1^{(\pm)}=(|r_1|\varkappa_0^3)^{-1}$, which are identical for the two states (for a discussion of the difference in these terms see below) and which have the order of magnitude of the static polarizability and hyperpolarizability $\beta_2^{(\pm)}=\pm9(r_1^2\omega\varkappa_0^6)^{-1}/32$, the value of which greatly exceeds the static value.

However, at frequencies $\omega\gg r_c\varkappa_0^{-3}F^2$ (this region overlaps with the frequency region $\omega\ll\varkappa_0^2$ discussed above) the right-hand side of Eq. (51) can be set equal to zero. This corresponds to the fact that the states considered in the zero approximation already correspond to definite values of the angular-momentum projection $m=\pm1$ on the direction of propagation of the wave. After calculating $A_{1,1}^{1,1}$ we find from (51)

$$E_{1,1}=E_1^{(0)}-\omega+\frac{F^2}{10r_1\omega^4}[15\omega^2\varkappa_0-10\omega\varkappa_0^3+4\varkappa_0^5-(2\omega+\varkappa_0^2)^{1/2}(3\omega^2-2\omega\varkappa_0^2+2\varkappa_0^4)+i2(2\omega-\varkappa_0^2)^{5/2}], \tag{55}$$

$$E_{1,-1}=E_1^{(0)}+\omega+\frac{F^2}{10r_1\omega^4}[15\omega^2\varkappa_0+10\omega\varkappa_0^3+4\varkappa_0^5-2(2\omega+\varkappa_0^2)^{5/2}+i(2\omega-\varkappa_0^2)^{1/2}(3\omega^2+2\omega\varkappa_0^2+2\varkappa_0^4)].$$

Since the regions of applicability of the formulas (52) and (55) overlap, they completely solve the problem of the influence of the wave field on the energy of the states with accuracy to terms proportional to F^4. Here (55) permits improvement of (54) and enables one to find the difference of the polarizabilities $\Delta\beta^{(\pm)}\approx\pm3/4\omega/r_1\varkappa_0^5$. This result corresponds to the known smallness of the antisymmetric polarizability $\beta_1^{(a)}\propto\omega$ for $\omega\ll\varkappa_0^2$.[24] In the frequency region $\omega\sim\varkappa_0^2$ according to (55) the polarizabilities are substantially different; for example, for $\omega=\varkappa_0^2/2$ we have $\beta_1^{(+)}/\beta_1^{(-)}\approx0.2$.

For frequencies $\omega>\varkappa_0^2/2$ there appear in $E_{1,\pm1}$ imaginary parts which determine the probabilities of one-photon ionization of the corresponding states:

$$\Gamma_{1,1}=\frac{2F^2}{5|r_1|\omega^4}(2\omega-\varkappa_0^2)^{5/2},\qquad \Gamma_{1,-1}=\frac{F^2}{5|r_1|\omega^4}(3\omega^2+2\omega\varkappa_0^2+2\varkappa_0^4)(2\omega-\varkappa_0^2)^{1/2}. \tag{56}$$

These formulas do not reflect the shift $\Delta E_{1,\pm1}$ of the decay level and the change of the free energy of the particle by $F^2/2\omega^2$ in the wave field, which are effects of higher order in F^2 (it must be kept in mind that part of the shift of the levels in $E_{1,\pm1}$, equal to $\mp\omega$, is a purely kinematic effect associated with the transformation to a rotating coordinate system and has no direct relation to the change of the binding energy of the particle). The threshold behavior of the widths $\Gamma_{1,1}\propto p_k^5$, $\Gamma_{1,-1}\propto p_k$ as $p_k\to0$ is consistent with the relation $\Gamma\propto p_k^{2l_k+1}$, where l_k is the angular momentum of the electron in the final state [compare with Eq. (42)].

In the frequency region $\varkappa_0^2\ll\omega\ll r_c^{-2}$ [regarding frequencies $\omega\gg r_c^{-2}$ see Eq. (44)] it follows from (55) and (56) that

$$E_{1,\pm1}=E_1^{(0)}\mp\omega-\frac{(11\mp5)}{2^{3/2}5}\frac{F^2}{r_1\omega^{3/2}},\qquad \Gamma_{1,1}=\frac{8}{3}\Gamma_{1,-1}=\frac{2^{7/2}}{5}\frac{F^2}{|r_1|\omega^{3/2}}. \tag{57}$$

Let us turn to discussion of ionization processes with $n>1$. Rather simple expressions for the partial widths can be obtained for frequencies $\omega\gg r_c\varkappa_0^{-3}F^2$ (this condition is equivalent to $\gamma^2\gg(r_c\varkappa_0)\omega/\varkappa_0^2$, so that it includes the region $\gamma\gtrsim1$). In this case, as was pointed out above, the angular momentum projection m is a good quantum number, Eq. (51) for the energy breaks up, and in the first nonvanishing approximation in F^2 we obtain

$$\Gamma_{1,1;n}^{(0)}=(n+1)\Gamma_{1,0;n}^{(0)}, \tag{58}$$

$$\Gamma_{1,-1;n}^{(0)}=\left[n+1-\frac{n(2n+3)}{n-K_0+\delta}+\frac{n(2n+1)(2n+3)}{4(n-K_0+\delta)^2}\right]\Gamma_{1,0;n}^{(0)},$$

where K_0, δ and $\Gamma_{1,0;n}^{(0)}$ are given by the expressions (32) and (46).

The general reasoning regarding the possibility of using these expressions for calculation of the total ionization width in regions 1) and 2) of the parameters ω and γ are exactly the same as those given above in relation to the ionization of states with $l=0$ and $l=1$, $m=0$. In the case of multiphoton ionization ($K_0 \gg 1$) for $\gamma \to \infty$ it follows from (46) and (58) that

$$\Gamma_{1,0} : \Gamma_{1,1} : \Gamma_{1,-1} = 1 : K_0 : K_0^3/\delta^2. \tag{59}$$

We note that this substantial difference of the widths of the three states is preserved in order of magnitude over the entire region of variation of the parameters $\gamma > 1$, $K_0 \gg 1$ (a proof of this for the quasiclassical region will be given below) and indicates the limited applicability of the quasiclassical approach developed in Ref. 21 to the photoionization of weakly bound states with $1 \neq 0$.

To determine the width in the quasiclassical region it is necessary to carry out a calculation of the imaginary parts of $A_{1m}^{1m'}$ in (50) by the method of steepest descent. After this we obtain from Eq. (51) the width of the levels [their shift $\Delta E_1^{(\pm)}$ is given by (52)]

$$\Gamma_1^{(\pm)} = \frac{3}{|r_1|} \left\{ \operatorname{Im}(A_{1,1}^{1,1} + A_{1,-1}^{1,-1}) \mp \left(r_1^2\omega^2 + \frac{9}{64}\varkappa_0^{-6}F^4 \right)^{-1/2} \times \left[\frac{3}{4}\varkappa_0^{-3}F^2 + \operatorname{Im} A_{1,1}^{1,-1} + r_1\omega \operatorname{Im}(A_{1,1}^{1,1} - A_{1,-1}^{1,-1}) \right] \right\}, \tag{60}$$

where

$$\operatorname{Im} A_{1,\pm1}^{1,\pm1} = \frac{\omega^2\gamma}{8\varkappa_0\tau_0^2} \left[e^{\pm 2\tau_0} + \frac{\varkappa_0^2}{4\omega\tau_0\gamma^2}(1 \pm 2\tau_0 - e^{\pm 2\tau_0})^2 \right] S,$$

$$\operatorname{Im} A_{1,1}^{1,-1} = -\frac{\omega\varkappa_0\gamma}{8\tau_0} S, \tag{61}$$

$$S = [\operatorname{ch} 2\tau_0 - (1+\gamma^2)]^{-1/2} \exp\left[2\frac{\Delta E_1^{(\pm)}}{\omega}\tau_0 - \frac{\varkappa_0^2}{\omega} f(\gamma) \right]$$

[τ_0 and $f(\gamma)$ are given by (38)].

Let us discuss the properties of the widths $\Gamma_1^{(+)}$. In the case $|r_1|\omega \ll F^2/\varkappa_0^3$, which is equivalent to the condition $\gamma \ll (r_c\varkappa_0)(F\varkappa_0^{-3}) \ll 1$, Eq. (60) goes over into the results for an electrostatic field (16). Here the state which corresponds to the energy $E_1^{(+)}$ corresponds to angular momentum projection $m=0$ on the field direction, and the corresponding $E_1^{(-)}$ corresponds to superposition of states (with identical amplitudes) with $m = \pm 1$ and $\Gamma_1^{(+)} \gg \Gamma_1^{(-)}$.

However, already for $|r_1|\omega \sim F^2/\varkappa_0^3$ (but such that we still have $\gamma \ll F/\varkappa_0^3 \ll 1$) the static relation between the widths is strongly violated, although the difference of the argument of the exponential in S in (61) from the static value can be neglected. Now

$$\Gamma_1^{(\pm)} = \frac{3F}{4|r|} \left[1 \pm \left(1 + \frac{64}{9} r_1^2\omega^2\varkappa_0^2F^{-4} \right)^{-1/2} \right] \exp\left(-\frac{2\varkappa_0^3}{3F} \right), \tag{62}$$

so that $\Gamma_1^{(+)}$ and $\Gamma_1^{(-)}$ are quantities of the same order and $\Gamma_1^{(+)} + \Gamma_1^{(-)} \approx \Gamma_{1,0}^{\text{stat}}$, i.e., the value of the width aveaged over the states ($\pm$) is unchanged.

On further increase of the frequency, when $|r_1|\omega \gg \varkappa_0^{-3}F^2$ [i.e., $\gamma \gg (r_c\varkappa_0)(F\varkappa_0^{-3})$], the states ($\pm$) correspond already to definite angular-momentum projections $m = \pm 1$, the widths $\Gamma_1^{(\pm)}$ become equalized, and amount to $(1/2)\Gamma_{1,0}^{\text{stat}}$ so long as the argument of S in (61) can be restricted to its static value [i.e., for $\gamma^2/15 \ll F\varkappa_0^{-3}$ (Refs. 20 and 13)]. At larger values of γ

$$\Gamma_{1,\pm1} \approx \frac{3}{16} \frac{\omega\varkappa_0}{|r_1|\gamma\tau_0^3} \frac{[e^{\pm\tau_0} - (1 \pm 2\tau_0)e^{\mp\tau_0}]^2}{[\operatorname{ch} 2\tau_0 - (1+\gamma^2)]^{1/2}} \exp\left[-\frac{\varkappa_0^2}{\omega} f(\gamma) \right], \tag{63}$$

so that $\Gamma_{1,-1}$ begins to exceed $\Gamma_{1,1}$ and for $\tau_0 \gg 1$ we have

$$\Gamma_{1,0} : \Gamma_{1,1} : \Gamma_{1,-1} \approx 1 : K_0 : 4\tau_0^2 K_0 \tag{64}$$

[$\Gamma_{1,0}$ is given by (47)]. As follows from (38), for $\tau_0 \gg 1$ we have $\ln\gamma \approx \tau_0$ and for values $2\ln\gamma \sim K_0$ (i.e., in the transition region) Eq. (64) agrees qualitatively with (59).

Let us consider now a strong field: $F \gg \varkappa_0^3$, $F \gg \omega^{2/3}$ (but as before $F \ll r_c^{-3}$). In calculation of the integrals for $A_{1m}^{1m'}$, as in the case of an electrostatic field we can make use of an expansion in the small parameters $E/F^{2/3}$ and $\omega/F^{2/3}$. For states with $l=1$, $m=0$ according to Eq. (39) we find

$$\Gamma_{1,0} = \frac{F}{3^{1/2}|r_1|} \left[1 - \frac{\Gamma(1/_6)}{3^{4/3}20\pi^{1/2}} \frac{\omega^2}{F^{4/3}} \right],$$

$$\Delta E_{1,0} = \frac{3^{1/6}\Gamma(5/_6)}{4\pi^{1/2}r_1} \left[\left(3\varkappa_0^2 - \frac{F}{r_1} \right) F^{1/3} - \frac{\Gamma^2(1/_6)}{3^{5/3}20\pi} \frac{\omega^2}{F^{1/3}} \right] \tag{65}$$

(in $\Gamma_{1,0}$ we have taken into account the correction only in the small parameer $\omega^2/F^{4/3}$). For $\omega = 0$ these expressions go over to the results (17) and (18) for an electrostatic field in the case $l=1$, $|m|=1$ (see the third footnote).

Calculating the integrals in (50) by this means, we have

$$A_{1,1}^{1,1} = \frac{5iF}{3^{1/2}4} - (3^{1/2} - i)\frac{3^{5/6}\Gamma(5/_6)}{4\pi^{1/2}} EF^{1/3},$$

$$A_{1,1}^{1,-1} = -\frac{iF}{3^{1/2}4} + (3^{1/2} - i)\frac{\Gamma(5/_6)}{3^{1/6}4\pi^{1/2}} EF^{1/3}. \tag{66}$$

Substituting these expressions into (51), we obtain in explicit form an equation for determination of the energies of the two quasienergy levels corresponding to superposition of states with $m = \pm 1$ [in the expressions (66) we have omitted corrections of order $\omega/F^{2/3}$, since in the case considered the most important dependence on frequency is determined by the terms $\pm r_1\omega$ in (51)]. Analysis of this equation shows that for frequencies $|r_1|\omega \ll F$ the expressions for the shifts and widths of these levels are described by the formulas (17), (18), and (23) for a static field. With increase of the frequency the values of the widths come together:

$$\Gamma_1^{(\pm)} \approx \frac{1}{2\sqrt{3}} [5 \mp (9 - 48r_1^2\omega^2F^{-2})^{1/2}] \frac{F}{|r_1|}. \tag{67}$$

However, it must be kept in mind that according (65) and (67) at $\omega \sim r_cF$ the lifetime of the state is comparable with the period of the oscillations, and the probability of ionization by a pulse of electromagnetic wave will depend substantially on the means of turning on the field and requires discussion of the specific experimental situation.

In conclusion we note that the method developed permits also discussion of the effect of an electric field on quasidiscrete levels with $l \geqslant 1$ in a potential $U(r)$, and also discussion of the case in which in U there are several shallow levels with different l values. In this connection we shall make several remarks.

1) In a strong electric field the shift and width of an unperturbed quasidiscrete level

$$E_l^{(0)} \approx (r_l a_l)^{-1} > 0, \quad \Gamma_l^{(0)} \approx 2^{l+3/2} (a_l r_l)^{-l-1/2} / |r_l|$$

(now $a_1 < 0$) in a potential U are described by Eqs. (17)–(19), (23), and (26) with the substitution

$$-\varkappa_0^2/2 \to E_l^{(0)} > 0, \quad \varkappa_0^{1-2l} C_{\varkappa l}^2 \approx |r_l|^{-1} \to {}^1/_2 \Gamma_l^{(0)} (2E_l^{(0)})^{-l-1/2}. \tag{68}$$

Here the electric field, which contracts the centrifugal barrier, broadens the initial quasistationary states.

$$\Gamma_{lm} \sim \Gamma_l^{(0)} (F^{2/3}/E_l^{(0)})^{l+1/2} \gg \Gamma_l^{(0)}.$$

2) However, in the case of a weak field the situation becomes complicated, since in addition to the levels which exist in the potential $U(r)$ the electric field generates new quasidiscrete levels. The appearance of these levels is due to the fact that the combined action of the centrifugal potential and the potential of the electric field leads to appearance near the center of an effective potential well (as in the one-dimensional problem of a particle in an electric field, the motion of which is restricted by an inpenetrable wall, the role of the latter is played by the centrifugal potential).

3) In the case of existence in the potential $U(r)$ simultaneously of two shallow levels, the influence of the field on the particle is described by the well known formulas for an isolated two-level system. Here the approach which has been developed permits calculation of the phenomenological parameters of such a model with application to the problems considered.

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[1)]Here r_1 can be expressed in terms of a dimensionless coefficient in the asymptotic expression for the normalized wave function of a state with $E = -\varkappa^2/2$:

$$C_{\varkappa l}^{-2} = -r_l \varkappa^{1-2l} + (-1)^l (2l+1) + O(\varkappa r_c)^{3-2l}).$$

[2)]At frequencies $\omega \lesssim \varkappa_0^2$ values $\gamma \lesssim 1$ correspond to a strong field.

[3)]In states with $l = 1$, $m = 0$ the projection of the angular momentum on any direction perpendicular to the z axis takes on only values $m_\perp = \pm 1$.

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