The appearance of stochasticity in the interaction between a modulation soliton and low-frequency waves

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The problem of interaction between a modulation soliton and low-frequency (LF) waves is investigated within the framework of the well-known set of coupled equations for Langmuir and ion-acoustic waves in a nonisothermal plasma. It is shown that in the presence of a sufficiently intense LF wave a stochastization of the envelope soliton may occur. The problem is treated both in the approximation of a prescribed LF wave field, and in the selfconsistent case, i.e., taking account of the radiation field and its reaction on the motion of the soliton.

The use of solitons as elementary objects, on which concepts of relatively complicated phenomena of nonlinear physics are built, assumes a detailed investigation of the properties of individual solitons, properties which manifest themselves in their interaction among themselves and with external fields. The majority of theoretical and experimental papers on the soliton problem is devoted to the analysis of these interactions. In many cases the solitons behave like classical particles[1,2] under such interactions and one sometimes even talks about the "classical mechanics" of solitons or the "kinetic theory" of solitons.

The investigation of the statistical properties of an ensemble (gas) of solitons has recently attracted the attention of many groups of investigators, mainly in connection with attempts to construct a theory of strong plasma turbulence[3–5] and various problems of solid state physics.[1] At the same time we are aware of few papers which consider the mechanisms which lead to the stochastization of the motion of an individual soliton or of a small number of interacting solitons.[6–8]

In the present paper we investigate the dynamics of a modulation soliton (also known as the Schrödinger soliton) in the self-consistent field of a low-frequency wave. Such an interaction is described by the equations

$$i\Psi_t+\Psi_{xx}-n\Psi=0,\quad u_t+n_x=-(|\Psi|^2)_x,\quad n_t+u_x=0, \tag{1}$$

where Ψ is the complex envelope of the high-frequency field; u and n are the components of the low-frequency field. This system of equations, first derived for the analysis of the interaction between Langmuir and ion-acoustic waves in a nonisothermal plasma,[9] also describes the interaction between surface and internal ocean waves, as well as electronic excitations in long helical molecules,[10] and is generally convenient for the analysis of the interaction between high-frequency and low-frequency waves of arbitrary nature.

The system (1) admits the well-known soliton solution

$$\Psi=\Phi^{(0)}(\zeta)\exp\left[i\left(-\frac{V}{2}\zeta+\varphi\right)\right],\quad n^{(0)}=\tilde n-\frac{\Phi^{2(0)}}{1-V^2},$$
$$u^{(0)}=\tilde u-\frac{V\Phi^{2(0)}}{1-V^2},$$
$$\Phi^{(0)}=[2\lambda^2(1-V^2)]^{1/2}\operatorname{ch}^{-1}\lambda\zeta,\quad \zeta=x-Vt,\quad \varphi=\omega t,$$
$$\lambda^2=\omega+\tilde n-\frac{V^2}{4},\quad \tilde n=\text{const},\quad \tilde u=\text{const}. \tag{2}$$

In the case we are interested in, when $\tilde u$, $\tilde n\neq$const, the soliton solution (2) is, in general, no longer valid,however, when $\tilde u(x,t)$ and $\tilde n(x,t)$ are slowly varying functions of x and t (on the scale of the soliton), then one may expect that the soliton will be able to follow the variations of $\tilde n$ and that its speed and amplitude will be slowly varying. When the problem is posed this way it can be solved by an asymptotic method. One looks for the solution of the system (1) in the form of the series

$$\psi=\left[\Phi^{(0)}(\zeta,\rho,\tau)+\sum_{n=1}^{\infty}\varepsilon^n\Phi^{(n)}(\zeta,\rho,\tau)\right]\exp\left[i\left(-\frac{V}{2}\zeta+\varphi\right)\right],$$
$$\begin{Bmatrix} n \\ u\end{Bmatrix}=\begin{Bmatrix} n^{(0)}(\zeta,\rho,\tau) \\ u^{(0)}(\zeta,\rho,\tau)\end{Bmatrix}+\sum_{n=1}^{\infty}\varepsilon^n\begin{Bmatrix} n^{(n)}(\zeta,\rho,\tau) \\ u^{(n)}(\zeta,\rho,\tau)\end{Bmatrix}. \tag{3}$$

Here $\Phi^{(0)}$, $u^{(0)}$, and $n^{(0)}$ are the leading terms of the solution, having the form (2); $V=V(\tau)$, $\lambda=\lambda(\rho,\tau)$ are slowly varying functions of time, $\rho=\varepsilon x$, $\tau=\varepsilon t$, with ε a small parameter of the same order of magnitude as the ratio of the scales $\tilde u$ and $\tilde n$ of the soliton and the specified sound wave. Substituting the series (3) into Eq. (1) and equating to zero the coefficients of equal powers of ε we obtain the following system of equations for the successive approximations:

$$\left[\frac{d^2}{d\zeta^2}-\lambda^2\right]\Phi^{(n)}=\sum_{k=0}^{n}n^{(k)}\Phi^{(n-k)}-i\frac{d}{dt}\Phi^{(n-1)}$$
$$+\frac{V_\tau}{2}\zeta\Phi^{(n-1)}-2\Phi_{\zeta\rho}^{(n-1)}-\Phi_{\rho\rho}^{(n-2)},$$
$$\frac{d}{d\zeta}(-Vu^{(n)}+n^{(n)})=H_u^{(n)}=-u_\tau^{(n-1)}-n_\rho^{(n-1)}$$
$$-\sum_{k=0}^{n}[(\Phi^{(n-k)}\Phi^{*(k)})_\zeta+(\Phi^{(n-k)}\Phi^{*(k-1)})_\rho], \tag{4}$$
$$\frac{d}{d\zeta}(-Vn^{(n)}+u^{(n)})=H_n^{(n)}=-n_\tau^{(n-1)}-u_\rho^{(n-1)}.$$

Here $d/dt=\partial/\partial\tau+V\partial/\partial\rho$. The equations for $u^{(n)}$ and $n^{(n)}$ can be directly solved, and the real and imaginary parts of the high-frequency envelope $\Phi^{(n)}$ are determined from the independent system of second-order equations:

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$$\hat{L}_1 \operatorname{Re}\Phi^{(n)} = \left[\frac{d^2}{d\zeta^2} - (\lambda^2+3n^{(0)})\right] \operatorname{Re}\Phi^{(n)} = \operatorname{Re} H^{(n)},$$

$$\hat{L}_2 \operatorname{Im}\Phi^{(n)} = \left[\frac{d^2}{d\zeta^2} - (\lambda^2+n^{(0)})\right] \operatorname{Im}\Phi^{(n)} = \operatorname{Im} H^{(n)},$$

$$H^{(n)} = -i\frac{d}{dt}\Phi^{(n-1)} + \frac{V_t}{2}\zeta\Phi^{(n-1)} - 2\Phi_{\zeta\rho}^{(n-1)} - \Phi_{\rho\rho}^{(n-2)} - C_n^{(n)}(\rho,\tau)\Phi^{(0)} - \frac{1}{1-V^2}\Phi^{(0)}\Phi^{(n-k)}\Phi^{*(k)} + \sum_{k=1}^{n} n^{(k)}\Phi^{(n-k)}$$
$$- \frac{\Phi^{(0)}}{1-V^2}\int_0^{\zeta} d\zeta'\left\{u_\tau^{(n-1)} + n_\rho^{(n-1)} + V(n_\tau^{(n-1)} + u_\rho^{(n-1)}) + \sum_{k=0}^{n}(\Phi^{(n-k)}\Phi^{*(k-1)})_\rho\right\}. \quad (5)$$

It was shown in Ref. 2 that a necessary and sufficient condition for the non-growth of the corrections $\Phi^{(n)}$ is the orthogonality of $H^{(n)}$ to eigenfunctions of the adjoints to $\hat{L}_1$ and $\hat{L}_2$ which decrease to zero as $\zeta \to \pm\infty$. It is easy to verify that these orthogonality conditions can be written in the form

$$\int_{-\infty}^{+\infty} \Phi_\zeta^{(0)} \operatorname{Re} H^{(n)}\, d\zeta = 0, \qquad \int_{-\infty}^{+\infty} \Phi^{(0)} \operatorname{Im} H^{(n)}\, d\zeta = 0. \quad (6)$$

In the first approximation these conditions lead to the system of equations

$$\frac{d}{dt}\lambda(1-V^2) = 0,$$

$$\lambda(1-V^2)\left(\frac{dV}{dt} + 2\frac{\partial\tilde{n}}{\partial x}\right) + \frac{8}{3}\frac{d}{dt}(\lambda^3 V) = 0. \quad (7)$$

Here the values of λ, $\tilde{n}$, and $\tilde{n}_x$ are taken for

$$x = X_s = \int_0^t V\, dt'.$$

When the conditions (6) are satisfied, the corrections $\Phi^{(n)}$ can be considered localized [in particular, the first corrections $\Phi^{(1)}$ tend to zero like $\zeta^2\exp(-\lambda|\zeta|)$] as $\zeta \to \pm\infty$.

In order that the corrections to the LF field $n^{(n)}$ and $u^{(n)}$ not increase with ζ it is necessary that the so-called algebraic orthogonality relations (Ref. 2) be satisfied. In the case under discussion these conditions have the form

$$H_n^{(n)}|_{\zeta\to\pm\infty} = H_u^{(n)}|_{\zeta\to\pm\infty} = 0$$

and, on account of the fact that $\Phi^{(n)}(\zeta) \to 0 (\zeta \to \pm\infty)$, lead to the obvious result:

$$(u_\pm)_\tau^{(n)} + (u_\pm)_\rho^{(n)} = 0, \qquad (n_\pm)_\tau^{(n)} + (u_\pm)_\rho^{(n)} = 0. \quad (8)$$

The quantities $n_\pm^{(n)}$ and $u_\pm^{(n)}$, treated as radiation fields of the nth order, are defined as follows

$$n_\pm^{(n)} = C_n^{(n)} + \frac{1}{1-V^2}\int_0^{\pm\infty} d\zeta\,(H_n^{(n)} + VH_u^{(n)}),$$

$$u_\pm^{(n)} = C_u^{(n)} + \frac{1}{1-V^2}\int_0^{\pm\infty} d\zeta\,(H_u^{(n)} + VH_n^{(n)}). \quad (9)$$

The solution of the equations (8) is determined independently in the regions $\zeta \to +\infty$ and $\zeta \to -\infty$ and are matched by means of the boundary conditions implied by Eq. (9):

$$[n^{(n)}] = n_+^{(n)} - n_-^{(n)}|_{\rho\to X_s} = \frac{1}{1-V^2}\int_{-\infty}^{+\infty} d\zeta\,(H_n^{(n)} + VH_u^{(n)}),$$

$$[u^{(n)}] = u_+^{(n)} - u_-^{(n)}|_{\rho\to X_s} = \frac{1}{1-V^2}\int_{-\infty}^{+\infty} d\zeta\,(H_u^{(n)} + VH_n^{(n)}). \quad (10)$$

This concludes the formal procedure of constructing the approximate solutions for the problem under consideration.

We now turn to an investigation of the equations of motion in the first approximation (7). We first note that the first of the equations (7) is a consequence of the conservation of the number of "quanta," whereas the second is a consequence of the law of variation of the total wave momentum in the original system of equations (1).

The system (7) can be replaced by a single second-order equation for the coordinate $X_s = \int_0^t V dt$ of the center of the soliton:

$$\frac{d^2X_s}{dt^2} = -2\left[1 + \frac{8}{3}C^2\frac{1+5\dot{X}_s^2}{(1-\dot{X}_s^2)^4}\right]^{-1}\frac{\partial\tilde{n}}{\partial x}. \quad (11)$$

Here $C = \lambda(1-V^2) = \text{const}$ is to be interpreted as the number of "quanta." The equation (7) is the starting point of the analysis that follows. For small velocities ($|\dot{X}_s| \ll 1$) this equation coincides with the equation of motion of a classical nonrelativistic particle in an external field. It is known[8,11,12] that in this case it is sufficient for the stochastization of the motion that the spectrum of $\tilde{n}(x,t)$ should contain, in addition to the component corresponding to one propagating harmonic wave, a small "admixture" of components corresponding to counterpropagating waves. If the magnitude of the additional components is small, then only those particles become stochastic which have velocities close to the phase velocity of the fundamental wave, i.e., $|\dot{X}_s| \approx 1$ so that for "nonrelativistic" solitons the equations (1) are a contradictory condition. The restrictions on the synchronism conditions are lifted if among the additional components of $\tilde{n}$ there is a counterpropagating wave of magnitude comparable to the fundamental wave. In order to investigate this case numerically on a computer we have integrated the equation

$$d^2X_s/dt^2 = E\sin X_s \sin t, \quad (12)$$

which describes the motion of solitons in the field of a standing wave. Here $E = (3/4)N_0/C^2$, N_0 is the amplitude of the sound wave. Fig. 1 represents the Poincaré map of the section $t = 2\pi/E^{1/2}$ onto itself. It is clear that the motion in this case is stochastic. The quantity λ^+ is the Kolmogorov entropy characterizing the mean divergence of trajectories over the invariant set and equals 0.0355 ($E = 1.0$).

Equations of the type (12) have been investigated analytically in Ref. 13; here we restrict our attention to the simplest estimates obtained in Refs. 8 and 13. We rewrite Eq. (12) in the form

$$d^2X_s/dt^2 = E_1\cos(X_s - t) + E_2\cos(X_s + t).$$

If the amplitudes of the two waves which propagate in opposite directions are very different, $|E_1| \gg |E_2|$, then the

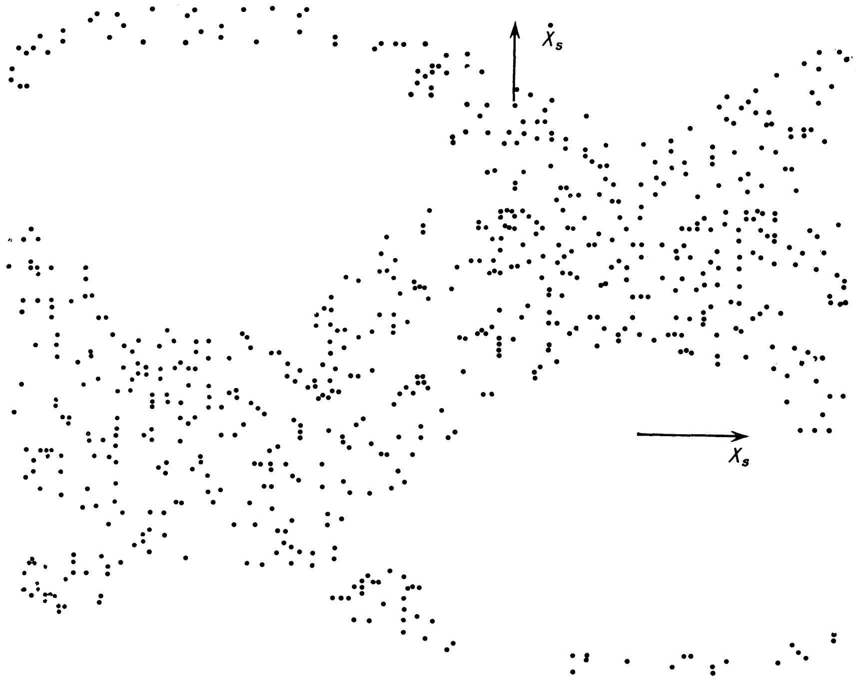

FIG. 1.

equation can be considered as a perturbed pendulum equation

$$d^2y/dt^2=E_1(\cos y+\varepsilon \cos(y+2t)),$$

where $y = X_s - t$, $\varepsilon = E_2/E_1 \ll 1$. In this case the width of the region of stochasticity is determined by means of the Mel'nikov function[8] which describes the distance between the separatrices

$$\Delta(t_0) \approx 8E_2\pi \sin\frac{2t_0}{E_1^{1/2}}\left[\operatorname{ch}^{-1}\frac{\pi}{E_1^{1/2}} - \operatorname{sh}^{-1}\frac{\pi}{E_1^{1/2}}\right] + O(\varepsilon^2).$$

This estimate is valid for nonrelativistic solitons, i.e., it is not directly applicable to Eq. (11). However such a result is of interest in connection with the analysis of the behavior of the soliton of the sine-Gordon equation in the field of a wave packet

$$\Phi_{tt}-\Phi_{xx}+\sin\Phi=\varepsilon F_{\text{ext}}(\varepsilon x, \varepsilon t),$$

since in this situation the phase velocity of the perturbations can be arbitrary, and synchronous motions of the soliton and wave are possible.[6,8]

For an approximate description of the motion of a modulation soliton, Eq. (12) can only be used in the case when there is no global overlap of resonances (in this case the trajectory cannot get into the region $|\dot{X}_s| > 1$). The method of overlap of resonances[11] it follows that for $(2E)^{1/2} < 1$ there is no overlap of resonances from the two forced components. Such an estimate is quite acceptable, since more accurate estimates obtained, e.g., by means of numerical methods, or by means of the renormalization method yield the value: $(2E)^{1/2} < 0.7$.

Since for chaotic motion the velocity of the soliton can, in general, reach large values ($|\dot{X}_s| \approx 1$), it is interesting to consider also the influence of relativistic effects on the stochastization picture. For this purpose a numerical integration was carried out for the system:

$$\frac{d^2X_s}{dt^2} = E\frac{(1-\dot{X}_s^2)^4}{1+5\dot{X}_s^2}\sin X_s \sin t, \tag{13}$$

which corresponds to the motion of a relativistic soliton for a value $C = 1$ [1)] of the constant C. The Poincaré map for this case (Fig. 2, where $E = 1.0$) bears witness to the fact that being relativistic does not inhibit the stochastization of the motion of the soliton, although it modifies the structure of the point mapping. The portrait of the map shows that the average of the velocity of the soliton over a period of the wave is close to the limiting speed (speed of sound), i.e., most of the time the soliton moves with relativistic velocities.

It is difficult to apply the method of resonance overlap directly to Eq. (13). For this case the width of the region filled by stochastic trajectories can be approximately estimated in the following manner. We rewrite Eq. (13) in the form

$$\frac{d^2X_s}{dt^2} = \frac{(1-\dot{X}_s^2)^4}{1+5\dot{X}_s^2}[E_1\sin(X_s-t)+E_2\sin(X_s+t)].$$

If even one of the quantities E_1 or E_2 vanishes then the equation admits a first integral, which we will not write out explicitly, since it is rather bulky. We consider the Poincaré map of the section $t = 2\pi$ for the following two cases: a) $E_1 \neq 0$, $E_2 = 0$, b) $E_2 \neq 0$, $E_1 = 0$ (Fig. 3). For this mapping the straight lines $\dot{X}_s = \pm 1$ remain invariant for arbitrary E_1 and E_2. A numerical analysis of Eq. (13) shows that for $E > 1$, $|\dot{X}_s|$ is almost always close to 1. Therefore, expanding the integral near $\dot{X}_s = \pm 1$ it is easy to obtain the following estimates for the trajectory with the largest amplitude of oscillations along the $\dot{X}_s$ axis.

The deviations from the unperturbed state are equal to

$$d_1 \approx 1-[8(E_1/6)^{1/2}]^{-1}, \quad d_2 \approx 1-(E_2/2)^{-1/2}.$$

Taking the interaction of both components into account has the effect that the trajectories turn into "stochastic belts". On account of the symmetry for $E_1 = E_2$ the width of the

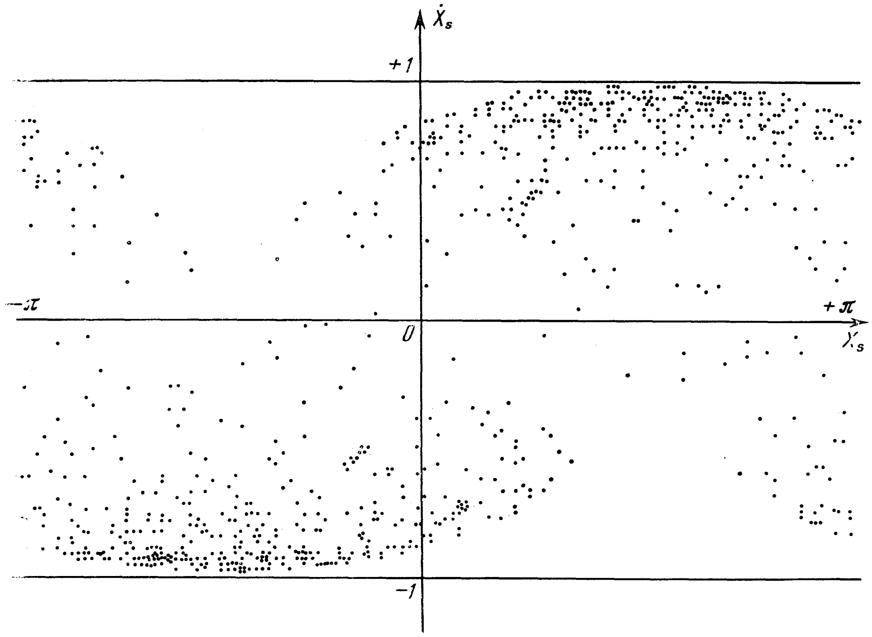

FIG. 2.

stochasticity region will be approximately $2d_1$, since for $E_{1,2} > 1$ $d_1 > d_2$ and the stochastic layers which appear on account of the destruction of the trajectories in the cases a) and b) will always overlap. A comparison of the formula obtained for the maximal velocity of the soliton with the value which was obtained numerically yields results which are close to each other. For instance, for $E = 2$ (E is the amplitude of the standing wave) $\dot{X}^{\max}_{\text{snum}} = 0.72$ and $\dot{X}^{\max}_{\text{stheor}} = 0.70$. The fact that the soliton has almost all the time a velocity close to the speed of sound has the following simple physical explanation: for $|\dot{X}_s| \sim 1$ the acceleration of the soliton tends to zero, and therefore more time is needed to change its speed, i.e., something like the "relativistic increase of mass" occurs.

Thus, for an approximate description of the motion of a soliton one can make use of the simplified equation (12) only for very small amplitudes of the sound wave ($E < 0.25$); otherwise one must investigate the full relativistic equation (13).

The results presented here need further explanation. The equation (11) within the framework of which the stochastization of the motion of the soliton was established is, in general, valid only over a restricted time interval ($\sim\varepsilon^{-1}$). It is important to note that over a time interval of the same order of magnitude there occurs a significant spreading

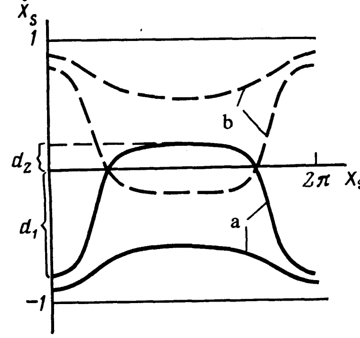

FIG. 3.

apart of the trajectories of Eq. (7). Therefore the fact of stochastization of the motion of a soliton must be verified within the range of validity of equations which are more precise than (7), equations that take into account quantities of the order ε^2, and are thus valid over times longer than $\sim\varepsilon^{-1}$. Such equations can be obtained from the orthogonality relations (6) for $H = H^{(2)}$ have the form:

$$\frac{d}{dt}\lambda'(1-V^2)=0, \quad \lambda'(1-V^2)\left(\frac{dV}{dt}+2\frac{\partial\tilde{n}}{dx}\right) + \frac{8}{3}\frac{d}{dt}(\lambda^3 V) = 2\lambda\left(\frac{dC_u^{(1)}}{dt}+V\frac{dC_n^{(1)}}{dt}\right). \tag{14}$$

Here $\lambda' = \lambda(\tilde{n} + C_n^{(1)}, V, \omega)$ and thus, the first equation is simply the result of the renormalization of λ by the magnitude of the radiation field in the soliton center. The expression in the right-hand side of the second equation (14) is just the spatial derivative of the magnitude of the radiation field at the position of the soliton, calculated to accuracy ε^2. Therefore, the second equation (14) can be written in the same form as Eq. (11):

$$\frac{d^2X_s}{dt^2} = -2\left[1+\frac{8}{3}C^2\frac{1+5\dot{X}_s^2}{(1-\dot{X}_s^2)^4}\right]^{-1}\frac{\partial n}{\partial x},$$
$$n_x = \tilde{n}_x - \frac{1}{1-V^2}\left(\frac{dC_u^{(1)}}{dt}+V\frac{dC_n^{(1)}}{dt}\right), \tag{15}$$

where the magnitude of the acoustic field is no longer prescribed but is determined by solving the radiation problem (8), (10), for $n = 1$:

$$(u_\pm)_\tau^{(1)} + (n_\pm)_\rho^{(1)} = 0, \quad (n_\pm)_\tau^{(1)} + (u_\pm)_\rho^{(1)} = 0,$$
$$[u^{(1)}]|_{\rho=x_s} = V\lambda V_t\frac{3+V^2}{(1-V^2)^2}, \quad [n^{(1)}]|_{\rho=x_s} = \lambda V_t\frac{1+3V^2}{(1-V^2)^2}. \tag{16}$$

It is simplest to solve such a problem in the case of a system without boundaries. In this case the radiation field is represented by waves diverging from the soliton:

$$n_+^{(1)} = u_+^{(1)} = u_+^{(1)}(x-t), \quad n_-^{(1)} = -u_-^{(1)} = -u_-^{(1)}(x+t), \tag{17}$$

the profiles of which can be obtained from the junction conditions (16):

$$n_\pm^{(1)}|_{\rho=X_s} = \frac{\lambda V_t}{(1-V^2)^2}[V(3+V^2) \pm (1+3V^2)]. \tag{18}$$

The Equations (9), (10) and (17) directly determine the parameters $C_u^{(1)}$ and $C_n^{(1)}$:

$$C_n^{(1)}|_{\rho=X_s} = \frac{1}{2}[u^{(1)}] = V\lambda V_t \frac{3+V^2}{(1-V^2)^2},$$
$$C_u^{(1)}|_{\rho=X_s} = \frac{1}{2}[n^{(1)}] = \lambda V_t \frac{1+3V^2}{(1-V^2)^2}. \tag{19}$$

Substituting (19) into Eq. (15) we obtain a closed-form equation for X_s

$$\frac{d^2X_s}{dt^2} = -2\left[1+\frac{8}{3}C^2\frac{1+5\dot{X}_s^2}{(1-\dot{X}_s^2)^4}\right]^{-1}$$
$$\times\left\{\tilde{n}_x - \frac{C}{(1-\dot{X}_s^2)^4}\right.$$
$$\left.\times\left[(1+6\dot{X}_s^2+\dot{X}_s^4)\dddot{X}_s + 3\dot{X}_s(\ddot{X}_s)^2\frac{5+10\dot{X}_s^2+\dot{X}_s^4}{(1-\dot{X}_s^2)^2}\right]\right\}. \tag{20}$$

The last terms in Eq. (20), which distinguish it from Eq. (11), determine the reaction of the radiation field on the motion of the soliton and have a dissipative character: the work done by the radiation field on the soliton equals (with opposite sign) the energy carried away by this field (cf. the similar situation in classical electrodynamics, Ref. 14).

Thus, taking into account the second approximation is equivalent to taking account of a small amount of dissipation which, in general, does not lead to the disappearance of stochasticity, but only modifies the threshold levels of the external field amplitude and initial velocity of the soliton for the onset of stochasticity. In this case the nature of the chaotic motion of the soliton will already be related to the presence of strange attractor for the system (20).

[1)] We note that this case is characteristic for the regime of fully developed Langmuir turbulence in a plasma.

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