Magnetic field distribution in two-dimensional Josephson junctions

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The entrance of a vortex structure into a distributed two-dimensional Josephson junction of arbitrary shape is analyzed theoretically. The two-dimensional sine-Gordon equation that describes in detail the behavior of the vortices in the junction is replaced, by an asymptotic method, with a much shorter "truncated" equation of hydrodynamic type. This equation permits a study of the behavior of the averaged characteristics of vortex structures at various methods of specifying the current and the external magnetic field.

1. INTRODUCTION

Many recent papers (see, e.g., Refs. 1–6) are devoted to theoretical and experimental studies of distributed Josephson junctions. In most cases the theoretical study is limited to a numerical analysis of initial equations of the sine-Gordon type. Yet numerical integration of these equtions, even for relatively small two-dimensional junctions (with dimensions $a \sim \lambda_J$, where λ_J is the Josephson penetration depth) calls for quite long computer time.[5,6]

We have previously[7] succeeded in developing an asymptotic approach to the description of long ($a \gg \lambda_J$) quasi-one-dimensional junctions. This approach has led to construction of a truncated equation that describes the behavior of the averaged characteristics of the vortex structure. We were then able to obtain analytic expressions for the stationary characteristics of a quasi-one-dimensional junction, and in particular for its critical current.

In this paper this asymptotic approach is generalized to include the case of two-dimensional Josephson junctions. The obtained truncated equation permits effective study of large size ($a \gg \lambda_J$) two-dimensional Josephson junctions.

2. THE TRUNCATED EQUATIONS

It is known (see, e.g., Ref. 8) that the physical processes that occur in a Josephson junction are described by the behavior of the Josephson phase φ that is subject to a two-dimensional sine-Gordon equation that takes in the stationary case the form

$$\lambda_J^2 \nabla^2 \varphi = \sin \varphi, \tag{1}$$

where the operator ∇ acts in the plane of the junction ($\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$). The solution of this equation yields the density distribution of the surface currents $J(x, y)$ in the superconducting electrodes, and the magnetic field H in the junction:

$$\mathbf{J} = j_c \lambda_J^2 \nabla \varphi, \quad \mathbf{H} = [\mathbf{n}_z \times \mathbf{J}], \tag{2}$$

where j_c is the critical density of the Josephson current. Equation (1) is supplemented by a condition imposed on the junction boundary Γ and following from relation (2):

$$\left.\frac{\partial \varphi}{\partial n}\right|_{\Gamma} = \frac{J_n(x, y)}{j_c \lambda_J^2}, \tag{3}$$

where $J_n(x, y)$ is the linear density of the current injected into the junction.

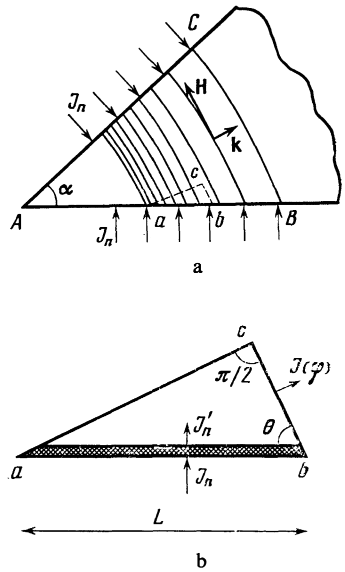

FIG. 1. a) Josephson junction in the form of the a corner α (top view); J_n—distributed density of the current entering the junction. CB—front of domain ABC. The force lines of the magnetic field $\mathbf{H}$ are perpendicular to the density lines of the current $\mathbf{J} \propto \mathbf{k}$. b) Fragment of domain ABC. The magnetic force lines (lines of equal phase z) make an angle θ with the near-boundary layer (shaded).

Let us examine qualitatively the structure of the solutions of the boundary-value problem (1), (3) at $a \gg \lambda_J$. The current generates quantized vortices near the junction boundary (usually near corners; Fig. 1a). On entering the junction, the vortices fill a certain part of the junction plane, forming a single "domain" in which the magnetic field is substantially different from zero. The equilibrium configuration of such a domain is determined by the balance of the forces that act on it. At the junction boundary the domain is acted upon by a Lorentz force proportional to J_n, which pushes the domain into the interior of the junction. Another force, determined by the gradient of the proper energy of the domain, tends to push it out of the junction.

The behavior of the function $\varphi(x, y)$ in the domain is characterized by two different scales. The fast small-scale changes of the phase occur at wavelengths on the order of λ_J, whereas the slow large-scale changes occur on the junction area, i.e., at lengths $a \gg \lambda_J$. It is precisely the presence of these two scales which makes the solution of the boundary-value

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problem (1), (3) difficult; this presence can be eliminated as follows.

Following the procedure of Ref. 7, we consider first a two-dimensional periodic lattice of Josephson vortices. In this case the solution of (1) can be represented in the form $\varphi_0(z,k)$, where z is a dimensionless coordinate that plays the role of an averaged phase φ:

$$z(x,\ y)=(\mathbf{kr})/\lambda_J, \tag{4}$$

and $\mathbf{k}$ is the wave vector of the vortex lattice. Just as in the case of a one-dimensional lattice, the function φ_0 satisfies the condition

$$\varphi_0(z+2\pi,\ k)=\varphi_0(z,\ k)+2\pi. \tag{5}$$

In a real domain, the Josephson vortices form a nonperiodic structure. At $a\gg\lambda_J$, however, this structure can be regarded as locally periodic and its quasiperiod can be taken to be a smooth function of the lattice coordinates. This allows us to seek the solution of Eq. (1) in the form of the following expansion:

$$\varphi=\varphi_0(z,\ k)+\varphi_1(z,\ k)+\varphi_2(z,\ k)+\dots, \tag{6}$$

in terms of the parameer $\varepsilon\sim\lambda_J/a$, with $\varphi_n\sim\varepsilon^n$. The wave vector $\mathbf{k}$ is now no longer a constant, as for a strictly periodic vortex lattice, but is a smooth function of the coordinates and defined by the relation

$$\mathbf{k}=\lambda_J\nabla z. \tag{7}$$

Substituting the expansion (6) in Eq. (1) and linearizing it with respect to φ_1, we arrive in first order in ε, just as in Ref. 7, to the following equation for φ_1:

$$k^2\partial^2\varphi_1/\partial z^2-\varphi_1\cos\varphi_0=\lambda_J f(z), \tag{8}$$

but with a somewhat different right-hand part, namely:

$$f(z)=-\frac{\partial\varphi_0}{\partial z}\operatorname{div}\mathbf{k}-2\frac{\partial^2\varphi_0}{\partial z\,\partial(k^2)}\mathbf{k}\nabla(k^2). \tag{9}$$

We stipulate that the function φ, defined by expansion (6), satisfy a condition analogous to (5). This leads[7] to the following constraint on the function $f(z)$, in which the vector $\mathbf{k}$ is a still undetermined function of the coordinates:

$$\left\langle f(z)\,\frac{\partial\varphi_0}{\partial z}\right\rangle=0. \tag{10}$$

The angle brackes in (10) denote averaging over the 2π-period of the argument z. With account taken of the explicit form (9) of the function $f(z)$, the condition (10) becomes

$$\operatorname{div}(A(k)\mathbf{k})=0, \tag{11}$$

$$A(k)=\langle(\partial\varphi_0/\partial z)^2\rangle. \tag{12}$$

The function $A(k)$ is expressed in the parametric form[7]

$$A(k)=\frac{4}{\pi\gamma k}E(\gamma),\quad k=\frac{\pi}{\gamma K(\gamma)}, \tag{13}$$

where $K(\gamma)$ and $E(\gamma)$ are complete elliptic integrals of the first and second kind, respectively.

Equation (11) is the sought "truncated" equation that replaces the initial equation (1) for two-dimensional Josephson junctions of large size $(a\gg\lambda_J)$ and describes the slowly varying parameter of the domain—the wave vector $\mathbf{k}$. We emphasize that Eq. (11) no longer has the characteristic scale λ_J, owing to the averaging carried out in (10). The changes of the function $z(x, y)$ take place at a distance of the order of the junction dimension, an advantage of the truncated equation over the initial Eq. (1).

The solution $z(x, y)$ of this equation determines the averaged (over the quasiperiod of the lattice) values of all the physical quantities. In fact, substituting in (2) the expansion (6), confining ourselves to the zeroth approximation in ε, and averaging (2) over the quasiperiod of the vortex structure we find that the vector $\mathbf{k}$ coincides with the averaged surface-current density in the electrodes:

$$\mathbf{k}=\langle\mathbf{J}\rangle/j_c\lambda_J, \tag{14}$$

and its absolute value, by virtue of the second equality in (2), is proportional to the mean value of the magnetic field. The current lines ($\mathbf{k}$) are orthogonal to the equal-phase lines $z(x, y)$ = const, which in turn are parallel to the force lines of the magnetic field $\mathbf{H}$ (Fig. 1a). Spreading over the surface of one of the superconductors into the interior of the domain, the $\mathbf{J}$ current lines are joined through the Josephson current $j=\operatorname{div}\mathbf{J}$ of the domain to the lines of the current flowing in the opposite direction over the surface of the second superconductor. Using Eq. (11), we obtain the average density of the Josephson current:

$$\langle j\rangle=\operatorname{div}\langle\mathbf{J}\rangle=j_c\lambda_J\mathbf{k}\nabla A/A. \tag{15}$$

3. BOUNDARY CONDITIONS

On the "free" boundary of the domain with the "empty" junction region (i.e., on curve CB of Fig. 1a we have

$$z|_\Gamma=0,\quad \left.\frac{\partial z}{\partial n}\right|_\Gamma=0. \tag{16}$$

Let us discuss the boundary conditions for that part of the domain boundary which coincides with the junction boundary. The presence of an abrupt boundary of the Josephson junction leads to formation of a boundary layer (of thickness $\sim\lambda_J$), inside of which the vortices are additionally deformed and Eq. (11) does not hold. Therefore the averaged condition (3)

$$(\partial z/\partial n)|_\Gamma=J_n'/j_c\lambda_J^2 \tag{17a}$$

can be written only for a circuit that lies deep below the boundary, at a distance $\gg\lambda_J$, so that the entering current density J_n' is, generally speaking, not equal to $\langle J_n\rangle$. To find J_n' we consider a small fragment abc of the domain, which contains nonetheless a large number of vortices and whose side ab abuts the junction boundary (see Fig. 1b); the sides bc and ac are parallel respectively to $\mathbf{H}$ and $\langle\mathbf{J}\rangle\propto\mathbf{k}$. We produce a small homogeneous shift of the entire vortex structure (see Fig. 1a) in the direction of the vector $\mathbf{k}$, such that bc intersects dn vortices and the phase φ acquires an increment $d\varphi=2\pi dn$ (the energy of the vortices contained in the contour abc remains unchanged). Equating to zero the work performed on the structure by the currents J_n and $J(\varphi)$ we get

$$\langle J_n\rangle L\frac{\hbar\, d\varphi}{2e}-dn\, L\cos\theta\frac{\hbar}{2e}\int_{(a)}J(\varphi)\,d\varphi=0, \tag{18}$$

where the integral is with respect the period $a(k)=2\pi\lambda_J/k$ of the vortex lattice. Taking into account the first equality of (2), relation (4), and the definition (12) of the function $A(k)$, we have

$$\int_{(a)}J(\varphi)\,d\varphi=2\pi j_c\lambda_J k\langle(\partial\varphi/\partial z)^2\rangle=2\pi\langle J\rangle A(k). \tag{19}$$

From (17) and (18) we obtain an expression for the normal component of the average current density $J_n'\equiv\langle J\rangle\cos\theta$ injected into the domain:

$$J_n'=\langle J_n\rangle/A(k). \tag{17b}$$

In the particular case when the vortices are parallel to the junction boundary ($\theta=0$), the condition (17b) yields the known connection[8] between the external magnetic field H_0 at the boundary of a one-dimensional junction ($H_0\propto J_n$) and the mean field $\langle H\rangle$ in the junction ($\langle H\rangle\propto\langle J_n'\rangle$).

It follows from condition (17b) that in the case of a dense vortex lattice ($k\to\infty$, $A(k)\to1$) the difference between J_n and J_n' is inessential. If, however, the lattice is sparse ($k<1$, $A(k)\gg1$), almost the entire current J_n entering junction is shorted by the near-boundary layer ($J_n'\ll\langle J_n\rangle$); this short circuit is complete on the junction boundary outside the domain.

4. CHARACTER OF THE SOLUTIONS OF THE TRUNCATED EQUATION

Consider a Josephson-junction fragment in the form of a corner α (see Fig. 1a). Let the total current I_0 enter the junction through the vertex (point A). The distribution of the phase z in the junction has then axial symmetry and the truncated equation (11) is easily integrated:

$$A(k)\,kr=\text{const}. \tag{20}$$

With increasing r, the product $A(k)k$ in (20) should decrease to its minimum value $4/\pi$. Consequently, the constant in (20) is equal to $4r_0/\pi$, where r_0 is the radius of the domain. Integrating Eq. (15) for the average Josephson-current density over the domain area and equating the result to the current I_0, we obtain

$$r_0=(\pi/4\alpha)\,(I_0/j_c\lambda_J). \tag{21}$$

Figure 2 shows the distributions, which follow from (13), (15), and (20), of the averaged phase z, of the surface current density $\langle J\rangle$, and of the density of the Josephson current $\langle j\rangle$ in the domain. Using (13), we obtain the following asymptotic relations:

$$\langle J(r)\rangle=j_c\lambda_J\cdot\frac{4}{\pi}\frac{r_0}{r}\quad\text{at}\quad r\ll r_0, \tag{22}$$

$$\langle J(r)\rangle=-j_c\lambda_J\cdot2\pi\ln^{-1}\left(1-\frac{r}{r_0}\right)\quad\text{as}\quad r\to r_0,$$

$$\langle j(r)\rangle=j_c\cdot2\left(\frac{\pi}{4}\right)^3\frac{\lambda_J}{r_0}\left(\frac{r}{r_0}\right)^2\quad\text{at}\quad r\ll r_0, \tag{23}$$

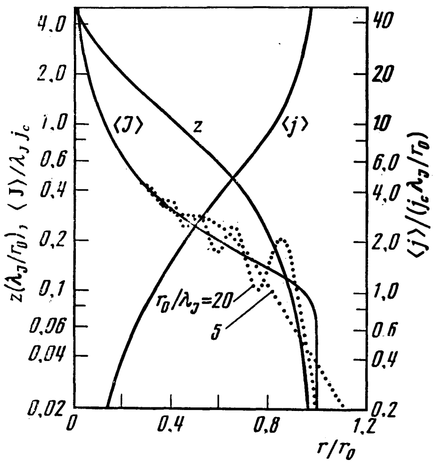

FIG. 2. Dependence of the mean values of the current density $\langle J\rangle$, of the phase z, and of the Josephson current $\langle j\rangle$ on the coordinate r in an axisymmetric domain (solid curves). The points show the behavior of the non-averaged current density for a ratio r_0/λ_J equal to 5 and 20.

$$\langle j(r)\rangle=j_c\cdot2\pi\frac{\lambda_J}{r_0}\left(1-\frac{r}{r_0}\right)^{-1}\times\ln^{-2}\left(1-\frac{r}{r_0}\right)\quad\text{as}\quad r\to r_0.$$

Equtions (22) and (23) as well as Fig. 2, reflect the general character of the vortex-domain structure: the bulk of the Josephson current flows in the immediate vicinity of the domain "front," i.e., near its boundary with the empty region of the junction (in our case at $r\sim r_0$). The reason is that inside the domain ($r<r_0$) the vortices are strongly compressed, so that their shape is close to symmetric and the average Josephson current is close to zero. On the contrary, the vortices located near the front are strongly asymmetric and make the main contribution to the average Josephson current.

5. DOMAINS AND CRITICAL CURRENT OF SQUARE JUNCTION

In the general case, to find the functions $z(x,y)$, $\langle J(x,y)\rangle$ and $\langle j(x,y)\rangle$ we must solve Eqs. (7) and (11) on the domain area with account taken of the boundary conditions (16) and (17a,b) at a known distribution of the current density $\langle J_n\rangle$ over the boundary of the Josephson junction. The boundary of the domain with the empty region of the junction is not known beforehand. Such a problem belongs to the class of nonlinear two-dimensional problems with unknown (free) boundary (see e.g., Ref. 9) and, except for rare cases (see Sec. 4) admits only of numerical solution. The principal computational algorithms for the solution of free-boundary problems are based on the basis of the method used to solve the Poisson equation in Ref. 10.

We denote by L the nonlinear elliptic operator corresponding to (7) and (11). According to this method it is necessary to solve not the equation $Lz=0$ inside a domain with unknown boundary, but a corresponding boundary-value problem for the "perturbed" equation

$$Lz_\varepsilon+C_\varepsilon(z_\varepsilon)\,z_\varepsilon=-f_\varepsilon(z_\varepsilon) \tag{24}$$

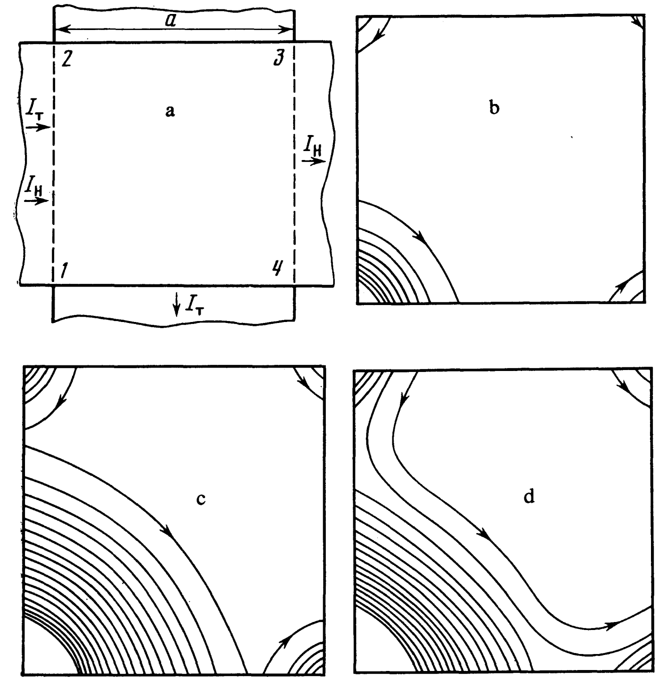

FIG. 3. Square Josephson junction made up in the overlap region of two superconducting films: a) top view, b, c) static domains in the junction for a current I_T equal respectively to $j_c a\lambda_J$ and $1.8 j_c a\lambda_J$ ($I_H = 0$). The lines of equal phase z ($z\lambda_J/a = 0$; 0.05; 0.10;...) show the picture of the mutual location of the vortices in the junction; d) initial stage of coalescence of three unpolar domains $I_M = 1.86 j_c a\lambda_J$; $I_H = 0$).

in the entire plane of the junction. Here $C_\varepsilon(z_\varepsilon)$ and $f_\varepsilon(z_\varepsilon)$ are discontinuous functions identically equal to zero inside the domain, but outside a domain, whose boundary is determined by the condition $z_\varepsilon(x, y) = 0$, we have $C_\varepsilon(z_\varepsilon) = \varepsilon^{-2}$, and $f_\varepsilon(z_\varepsilon) = \text{const}$. ε is the iteration parameter; the boundary-value problem for Eq. (24) was numerically calculated for relatively large $\varepsilon(\sim 1)$, after which ε was decreased by a fixed number of times, and the preceeding solution $z_\varepsilon(x, y)$ of Eq. (24) was used as the initial approximation. At sufficiently small ε ($\sim 10^{-2}$) the function $z_\varepsilon(x, y)$ comes with good accuracy (on the order of several percent) to the solution of the initial problem. The optimal characteristics of such an iteration algorithm were investigated by us experimentally with model calculations (e.g., we verified Eq. (21) with the current I_0 into the junction corner specified). For a concrete solution of (24) at each fixed ε the system of difference equation was solved by a buildup method using a longitudinal–transverse scheme.[11]

This numerical method enabled us to solve the problem for the important case when a Josephson junction was formed over the entire area of superposition of two superconducting film strips of equal width a (see Fig. 3a), and consequently was square in shape. For this geometry, the density of the current J_n injected into the junction is distributed in a complicated manner along its boundary. This distribution was obtained in Ref. 6 for different ways of specifying the field and the current.

Figure 3b shows the numerically calculated shape of the static domains induced by a transport current I_T ($I_H = 0$) for two values of I_T smaller than the critical I_M. The bulk of the current enters the junction in the region of the corner 1, therefore the size of the domain located in this corner is a maximum. With increasing current I_T, the domains increase in size and when the critical current is reached the unipolar domains come in contact (the polarities of the domains is marked by arrows in Figs. 3b–d). At the instant of contact, the domains merge into a single domain with concave boundary. This domain shape is energywise unfavorable, and the domain goes into motion and shortens the length of the boundary. The motion of the domain gives rise to a resistive state (see Fig. 3d).[1)]

The solid lines in Figs. 4a and 4b show the dependence of the critical current I_M respectively on the current I_H over one of the films and on the external homogeneous magnetic field H parallel to the side of the junction. In the first case the specified current I_H is not uniformly distributed over the films[6]; in the second case the uniform magnetic field H induces a Meissner current $I_H = aH$, which is uniformly distributed over one of the films. The points in Figs. 4a and 4b show the results of the numerical calculation for the initial equation (1) at $a = 6\lambda_J$, which, in turn, agree quite well with the experimental results.[6] It can be seen that in the limit as $a/\lambda_J \to \infty$ (solid curves) there are no secondary maxima of $I_M(I_H)$ at all. On the contrary, in the region of the central maxima one observes satisfactory agreement between the asymptotic and exact curves. Let us discuss these regions separately.

6. CENTRAL MAXIMUM

We shall show that in this region ($|I_H| < 2j_c a\lambda_J$; see Fig. 4a) an even simpler approximate description of the $I_M(I_H)$ dependence is possible. Injection of the currents I_T and I_H along the sides of the Josephson junction is effected predominantly in the region of its corners in the respective ratios 3/4:1/4: − 1/4:1/4 and 1/2: − 1/2: − 1/2:1/2 (see Ref. 6). We replace the distributed injection by specifying the currents exactly into the corners, at the same ratios. With this approximation of the boundary conditions, the domain

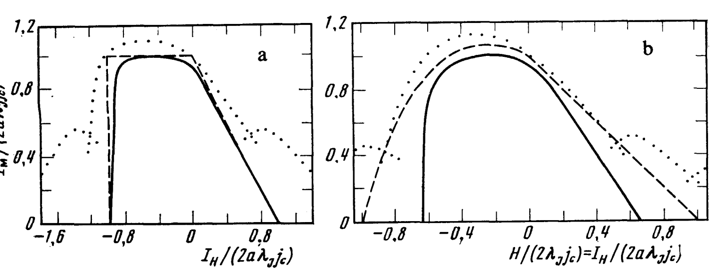

FIG. 4. Dependence of critical current I_M on: a) the current I_H (unevenly distributed over the films), b) on the external homogeneous magnetic field H ($I_H = aH$ is the Meissner current uniformly distributed over one of the films). Solid curves—result of numerical calculation by Eqs. (7) and (11). Points—numerical calculations for Eq. (1) and $a = 6\lambda_J$ (Ref. 6). Dashed curves—dependences of critical current for an axisymmetric approximation of the real boundary conditions.

becomes axially symmetric and the solution of the problem is substantially simplified, since we can now use the result (21), where the current I_0 is a sum of the currents I_T and I_H multiplied by the coefficients indicated above. The critical value of the current can now be obtained by equating the sums of the radii of each pair of neighboring domains to the length a of the side of the junction.

In the upshot we obtain a set of straight line on the (I_T, I_H) plane, which delimit the region of stationary states (dashed curve on Fig. 4a). It can be seen that even so simple an approximation provides a reasonable description of the true $I_M(I_H)$ dependence as $a/\lambda_J \to \infty$, since the real injection is close to that through corners.

Such an "axial domain approximation" works poorly only when the vortices are induced by a uniform external field H exactly parallel to a side of the junction. In this case the Meissner current induced by the field is injected into the junction uniformly on two of its sides.[6] Replacement of such a uniform injection by corner injection (in this case account is taken of only that fraction of the current that can flow into the interior of vortex domain) leads to the dependence shown by the dashed curve in Fig. 4b. Comparison of this dependence with the exact result shows that satisfactory agreement takes place only in the region of the peak of the central maximum where the uniform current injection is small.

7. SECONDARY MAXIMA

We obtain now the asymptotic behavior of the heights of the secondary maxima at large but finite values of the parameter a/λ_J in the region $(|I_H| \gg j_c a\lambda_J$. To this end we must find the maximum admissible value (I_M) of the integral

$$I_T = j_c \int_0^a dx \int_0^a dy \sin\varphi \tag{25}$$

in the case when the vortices fill the entire Josephson junction.

Under the integral sign of (25) is a rapidly oscillating function, so that the value of this integral is determined by the behavior of the phase in the region of the junction boundary. This is physically clear, since nonzero values of I_T in this field region correspond to pinning of the vortices by the junction boundary. This pinning takes place wherever the vortex approaches the boundary in such a way that its vortex current makes a nonzero contribution to I_M.

Figures 5a–5c show typical possible distributions of the magnetic field in the junction. If the external magnetic field

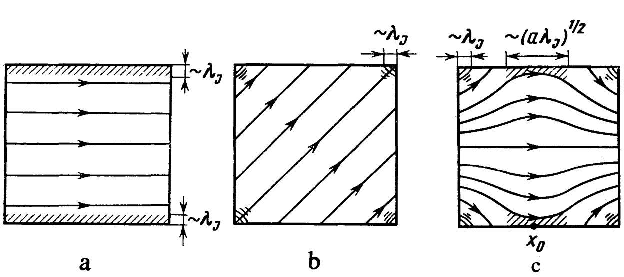

FIG. 5. a–c Characteristic distribution of the magnetic field in a junction at $|I_H| \gg j_c a\lambda_J$. The pinning regions are shaded.

is parallel with good accuracy (slope $\lesssim \lambda_J/a \ll 1$) to one of the sides of the junction, the pinning region consists of two strips of length a and width $\sim\lambda_J$, located along two sides of the junctions (shaded strips on Fig. 5a). For such a one-dimensional geometry the expression for the envelope at the secondary maxima is well known[12–13]:

$$I_M/2j_c a\lambda_J \leqslant (|I_H|/j_c a\lambda_J)^{-1}. \tag{26}$$

If the current I_H that enters the junction is produced by an external uniform magnetic field H oriented parallel to a junction side, the vortices are pinned only at the corners of the junction, i.e., on a considerably smaller area $\sim\lambda_J^2$ (see Fig. 5b), and the pinning is considerably weaker. In fact, in this case we can use the known (see, e.g., Ref. 14) asymptotic representation of the integral (25), which is valid when the phase varies rapidly along the coordinate axes and when there are no stationary points on the junction area. This yields directly

$$I_M/2j_c a\lambda_J \leqslant (4\lambda_J/a)\,(I_H/2j_c a\lambda_J)^{-2}. \tag{27}$$

The third type of vortex structure configuration (Fig. 5c) can be realized, for example, when the field H is given by the current I_H flowing through the film. In this case the current injection is not uniform, the vortices are bent, and on both sides of the junction there appear sections where the vortices touch the sides of the junction. The length of these sections (where the vortex is separated from the side of the junction by a distance $\lesssim\lambda_J$) is of the order of $(a\lambda_J)^{1/2}$, therefore the pinning force should in this case be $(a/\lambda_J)^{1/2}$ times stronger than at the corners. From the mathematical point of view the points of tangency of the vortices to the junction sides are stationary points of the phase, and the asymptotic expansion of the integral (25) is determined in this case by the behavior of the phase precisely at these points (see Ref. 14). For the envelope of the lateral maxima we obtain

$$I_M/2j_c a\lambda_J \leqslant \left(f^{-1}(x)\,\tilde{z}_{xx}^{-1/2}\right)_{\substack{x=x_0\\ y=0}} (\pi\lambda_J/4a)^{1/2} (|I_H|/2j_c a\lambda_J)^{-3/2}. \tag{28}$$

The function $z(x, y)$ is the phase $z(x, y)$ normalized to the current I_H and calculated at $I_T = 0$, while the function $f(x)$ is its derivative $\partial\tilde{z}/\partial y$ on the junction boundary, so that the factor in the first round brackets, taken at the stationary-phase point, is a constant independent of a/λ_J and I_H (numerical calculation found it to be equal to 1.6).

Comparing the obtained equations (26)–(28) with one another, we see that at the vortex configuration shown in Fig. 5a we get the strongest pinning (26), and for Fig. 5b we get the weakest pinning (27). The pinning assumes intermediate values in case of tangency of the vortices to the boundary of the Josephson junction [Eq. (28)]. It is of special importance that the relative heights of the secondary maxima at any manner of specifying the magnetic field (except when the field is parallel to one of the sides of the junction) tends to zero with increase of the parameter a/λ_J. Thus, at sufficiently large values of the ratio a/λ_J the form of the $I_M(I_H)$ dependence is indeed given by the basic approximation of our asymptotic theory (solid line in Fig. 4a).

8. CONCLUSION

We developed in this paper an asymptotic approach to the description of two-dimensional Josephson junctions of arbitrary shape and of large size ($a \gg \lambda_J$). As a result we obtained the truncated equation (11), which replaces the initial two-dimensional sine-Gordon equation (1). The truncated equation is superior to the initial one in that it does not contain the characteristic scale λ_J, so that it becomes much easier to analyze. Comparison of the results with a numerical calculation[6] performed for the initial equation (1) has shown that the truncated equation describes satisfactorily the region of the central maxima of the $I_M(I_H)$ dependences even for relatively small junctions ($a/\lambda_J \sim 6$). We have shown that the lateral maxima of the $I_M(I_H)$ dependences, which are absent from the first approximation of our method, actually decrease without limit with increasing ratio a/λ_J).

[1)] Although the truncated equation describes only the static distribution of the domains in the junction, the algorithm used by us nevertheless enabled us to simulate qualitatively also the dynamic state connected with the motion of the vortex domains.

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