Non-linear theory of superfluorescence

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We show in the framework of the Maxwell-Bloch model and neglecting relaxation processes and inhomogeneous broadening that the pulse shape of superfluorescence is self-similar. The shape depends on one random, but experimentally measurable parameter—the individual pulse delay time. There is convincing agreement between the theory and experiments on cesium atoms.

INTRODUCTION

The superfluorescence phenomenon—the generation of coherent electromagnetic radiation in an open laser without mirrors—was predicted by Dicke[1] in 1954 and first observed experimentally by Skribanowitz *et al.*[2] in 1973. A large number of theoretical and experimental papers has up to the present been devoted to the superfluorescence effect (see, e.g., Refs. 3 to 5 and the reviews 6 to 8). In Refs. 3, 4 a consistent quantum-mechanical theory of the initial stage of superfluorescence (linear theory of superfluorescence) was developed and it was shown that one can in fact in all stages of the process use the semiclassical Maxwell-Bloch equations taking into account the quantum nature of the electromagnetic field by specifying at the initial time a small normally distributed polarization of the medium. In a medium with an inverted population this polarization starts the development of an instability as a result of which radiation pulses are formed at the ends of the laser. The linear theory enables one to describe the initial stage of the development of these pulses—to construct a statistics of "delay times." To describe the pulse shape generated for a given initial polarization of the medium it is necessary to go beyond the framework of the linear theory and this is the object of the present paper.

We make somewhat idealized assumptions, the most important of which are homogeneity of the problem (neglect of diffraction) and neglect of the homogeneous line broadening. In addition, we consider only waves traveling in one direction. This, at first sight very restrictive, assumption, can, as we show, easily be justified by the nature of the answer found.

In the present paper we also neglect inhomogeneous broadening. Strictly speaking, we shall assume that the time for inhomogeneous broadening appreciably exceeds the length of the first pulse. In principle it is possible to take inhomogeneous broadening into account but this requires the development of a more detailed theory using the inverse scattering method. Here we present an elementary theory variant which is based upon joining up the solution of the linearized problem with the well known self-similar solution of the Maxwell-Bloch equations. We show that under reasonable restrictions on the length of the laser the superfluorescence pulse shape turns out to be universal, depending self-similarly on the length of the sample, and is determined by a single undetermined parameter—the pulse delay time which is uniquely connected with the value of the polarization fluctuations at the opposite end of the laser. This conclusion is well confirmed by comparison with laboratory experiments on the observation of superfluorescence in cesium vapor[5] and with numerical solutions of the Maxwell-Bloch equation. After that we study the self-similar solution itself which plays an important role not only in superfluorescence problems but also in amplification problems.[8–10] Although the self-similar solution cannot be expressed in terms of elementary functions, in the limiting case of interest for the theory of superfluorescence it is possible to obtain a simple explicit expression which well approximates the first pulse.

§1. BASIC EQUATIONS

We restrict ourselves to considering a very simple model of a laser. The active atoms are randomly distributed in space with an average density N. Initially we assume that there is a transition between non-degenerate states and we later discuss possible effects connected with degeneracy. The atoms have different transition frequencies close to the frequency ν_0; the distribution over the transition frequencies is characterized by the function $g(\nu)$, where $\nu = \omega - \omega_0$, which has a maximum at $\nu = 0$. In that case

$$\int_{-\infty}^{\infty} g(\nu)\, d\nu = 1,$$

and ν_0—the characteristic width of the function $g(\nu)$—is rather small ($\nu_0 \ll \omega_0$).

Let the sample have a length L and cross section S. For moderate Fresnel numbers $F = S/\lambda L \sim 1$, where $\lambda = c/\omega$ is the characteristic wavelength, one can also with adequate accuracy assume that the electromagnetic field depends solely on a single coordinate x directed along the laser axis. We shall assume also that the electromagnetic field is linearly polarized.

For sufficiently large atom densities $\tilde{N}$, when $\tilde{N}S\lambda \gg 1$ the system of atoms can be split up into subsystems each of which is localized in a layer appreciably smaller than the wavelength and which consists of atoms with frequencies which are close, $\Delta\omega \ll \nu_0$. Each of the subsystems is characterized by its own state vector which we can use to calculate its macroscopic characteristics—the average occupation number $n(x,\omega,t)$ of the excited level and the average value $dP(x,\omega,t)$ of the polarization (here d is the dipole moment of a single atom). If such a subsystem interacts with a classical uniform (possibly varying in time) electric field we can ob-

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tain for the quantities n and P the closed equations:[3,4]

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)P=-\frac{d\omega}{\hbar}En, \tag{1.1}$$

$$\frac{\partial n}{\partial t}=\frac{d}{\hbar\omega}E\frac{\partial P}{\partial t}. \tag{1.2}$$

We then have for the electric field the Maxwell equation

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}-\frac{\partial^2 E}{\partial x^2}=\frac{4\pi d}{c^2}\frac{\partial^2}{\partial t^2}\langle P\rangle. \tag{1.3}$$

Here and henceforth the brackets $\langle F\rangle$ indicate the averaging:

$$\langle F\rangle=\int_{-\infty}^{\infty}g(\omega-\omega_0)F(\omega,x,t)\,d\omega. \tag{1.4}$$

The local uniformity of the field is validated by the small, compared to the wavelength, longitudinal dimensions of the subsystem. It is more difficult to validate the requirement that the field be classical. Superfluorescence is essentially a quantum effect and at least in its initial stage quantum effects are the determining ones. This problem is considered in detail in Ref. 3 where it is shown that quantum effects can be taken into account by specifying at the initial time a random polarization distribution.

We draw attention to yet another fact. Equations (1.1) to (1.3) allow an exact solution

$$P\equiv 0,\quad E\equiv 0,\quad n=n_0(x,\omega). \tag{1.5}$$

Here $n_0(x,\omega)$ is an arbitrary function. From physical considerations it follows that $|n_0(x,\omega)|\leqslant\tilde{N}$. If $n_0(x,\omega)=-\tilde{N}$ all atoms in the medium are in the ground state; if $n_0(x,\omega=\tilde{N}$, all atoms are in the inverted state. This is a coherent quantum-mechanical state. If $|n_0(x,\omega)|<\tilde{N}$ the atomic subsystems are in mixed states. We note that from Eqs. (1.1) to (1.3) there follows the relation

$$n^2+P^2+\omega^{-2}P_t^{\,2}=n_0^{\,2}(x,\omega). \tag{1.6}$$

Physically it is clear that when time goes on all atoms must go over into the ground state with $n_0(x,\omega)=-\tilde{N}$. However, it follows from (1.6) that the quantity $|n(x,\omega)|$, cannot exceed $|n_0(x,\omega)|$ as it relaxes as described by Eqs. (1.1) to (1.3). The set (1.1) to (1.3) therefore describes relaxation to the ground-state occupation only from the coherent completely inverted state with $n_0(x,\omega)=\tilde{N}$. In all other cases it describes partial relaxation to a state with $n=-|n_0(x,\omega)|$. The further relaxation no longer is in the nature of a collective induced process and for its description we need a fuller allowance for the field quantization than by just giving the initial fluctuations. In actual fact the quantity $n_0(x,\omega)$ is determined by the excitation conditions of the system of atoms. We restrict ourselves to the simplest case $n_0(x,\omega)=\tilde{N}$.

In all real situations Eqs. (1.1) to (1.3) contain a small parameter

$$\varepsilon=4\pi d^2\tilde{N}/\hbar\omega_0\ll 1.$$

This makes it possible to change to contracted equations for the enevelopes. We put

$$\begin{aligned}E&=E^+\exp\{i\omega_0(t-x/c)\}+E^-\exp\{i\omega_0(t+x/c)\}+\text{c.c.},\\ P&=P^+\exp\{i\omega_0(t-x/c)\}+P^-\exp\{i\omega_0(t+x/c)\}+\text{c.c.}\end{aligned} \tag{1.7}$$

and neglect second derivatives of the quantities $E^{\pm}$, $P^{\pm}$ with respect to x and t. We also change to dimensionless quantities

$$E^{\pm}\to i\frac{\hbar\omega_0}{d}\varepsilon^{1/2}\mathscr{E}^{\pm},\quad \frac{\partial}{\partial t}\to\omega\varepsilon^{1/2}\frac{\partial}{\partial\tau},$$

$$\frac{\partial}{\partial x}\to\frac{\omega}{c}\varepsilon^{1/2}\frac{\partial}{\partial x},\quad n\to n_0 n,\quad P^{\pm}\to n_0\rho^{\pm}.$$

As a result we get

$$\begin{aligned}&\left(\frac{\partial}{\partial\tau}\pm\frac{\partial}{\partial x}\right)\mathscr{E}^{\pm}=\langle\rho^{\pm}\rangle,\quad \frac{\partial\rho^{\pm}}{\partial\tau}+i\lambda\rho^{\pm}=n\mathscr{E}^{\pm},\\ &\partial n/\partial\tau=-{}^1/_2(\mathscr{E}^{+*}\rho^{+}+\mathscr{E}^{+}\rho^{+*}+\mathscr{E}^{-*}\rho^{-}+\mathscr{E}^{-}\rho^{-*}).\end{aligned} \tag{1.8}$$

In these variables Eq. (1.6) becomes

$$|\rho^+|^2+|\rho^-|^2+n^2=1. \tag{1.9}$$

One must consider Eqs. (1.8) on the section $0\leqslant x\leqslant l$ where $l=(\omega/c)\varepsilon^{1/2}L$ is the dimensionless length of the laser.

The most general mathematical statement of the problem for the system (1.8) consists in giving the initial polarization values

$$\rho^{\pm}(x,t,\lambda)\,|_{t=0}=\rho_0^{\pm}(x,\lambda),\quad 0\leqslant x\leqslant l, \tag{1.10}$$

the sign of the occupation (where an inversely populated medium now corresponds to $n>0$), the initial values of the electric field

$$\mathscr{E}^{\pm}(x,t)\,|_{t=0}=\mathscr{E}_0^{\pm}(x) \tag{1.11}$$

and of the waves incident on the ends

$$\mathscr{E}^{\pm}(x,t)\,|_{x=0,\,l}=\mathscr{E}^{\pm}(t). \tag{1.12}$$

When stating the superfluorescence problem on must assume that there is no radiation incident on the end of the sample: $\mathscr{E}^{\pm}(t)=0$, while the functions $\rho_0^{\pm}(x,\lambda)$ and $\mathscr{E}_0^{\pm}(x)$ are random ones. One must find their correlation properties by solving the exact quantum-mechanical problem[3] when one shows that one must assume that $\mathscr{E}_0^{\pm}\equiv 0$. On the other hand, the functions $\rho_0^{\pm}(x,\lambda)$ are Gaussian with the correlator

$$\langle\rho_0^{\pm}(x,\lambda)\rho_0^{\pm *}(x',\lambda')\rangle=[g(\lambda)N]^{-1}\delta(x-x')\,\delta(\lambda-\lambda').$$

Here and henceforth N is the density of atoms per unit dimensionless length.

We consider now the instability growth rate γ of an inversely populated medium. To do this we linearized Eq. (1.8) in an unbounded medium on the background stationary solution $n=1$, $\rho=\mathscr{E}=0$. After that we put $\delta\mathscr{E}\propto\exp\{i(\Omega t+Px)\}$ and find the dispersion relation

$$\Omega-P+\frac{1}{2}\int_{-\infty}^{\infty}\frac{g(\lambda)\,d\lambda}{\Omega-\lambda+i0}=0. \tag{1.13}$$

Then $\gamma(P)=\operatorname{Im}\Omega$. A study of Eq. (1.13) shows that there are two limiting cases.

If the width $1/T_2^*$ of the function $g(\lambda)$ is small, i.e., $\gamma(P)T_2^*\gg 1$, we can put $g(\lambda)=\delta(\lambda)$. In that case

$$\gamma(P)={}^1/_2(2-P)^{1/2}. \tag{1.14}$$

In that case $\gamma_{\max}=\gamma(0)=2^{-1/2}\sim 1$. In the opposite case of a broad line when $\gamma(P)T_2^*\ll 1$ we have

$$\gamma(P)=(\pi/2)g(P). \tag{1.15}$$

In that case $\gamma_{max} = \gamma(0) \ll 1$.

Depending on the relation between the quantities γ_{max} and l one can consider three kinds of mirrorless lasers in which superfluorescence may occur. If

$$l\gamma_{max} \ll 1, \tag{1.16}$$

the amplification of a pulse during one passage through the laser is small, one may call such a laser a short one. If

$$1 < l\gamma_{max} \ll {}^1/_2 \ln N, \tag{1.17}$$

the amplification of the pulse during one passage of the sample is large but because of their smallness the initial fluctuations do not manage to grow sufficiently to appreciably change the population during one passage. We shall call such a laser one of moderate length. Finally the case

$$l\gamma_{max} \gg {}^1/_2 \ln N \tag{1.18}$$

corresponds to a long laser. The theory which follows will refer mainly to short and moderately long lasers.

We shall in what follows study exclusively the situation when there are waves present propagating only in one direction (say, to the right). The basis for this, at first sight arbitrary, assumption will be given in the next section.

We now consider the situation when the states between which the transition takes place are degenerate with respect to angular momentum. The degree of degeneracy will be equal to $2J + 1$ where J is the value of the angular momentum. The selection rules for a linearly polarized field allow transitions for which $\Delta J = -1,0,1$; $\Delta m = 0$, where m is the component of the angular momentum along the direction of the polarization vector. In that case every dipole transition contributes to the total current and Eq. (1.8) takes the form

$$\left(\frac{\partial}{\partial\tau} \pm \frac{\partial}{\partial x}\right)\mathscr{E}^{\pm} = \sum_{m=-J}^{J} \langle K_m \rho_m^{\pm} \sigma_m \rangle,$$
$$\partial\rho_m^{\pm}/\partial\tau + i\lambda\rho_m^{\pm} = n_m \mathscr{E}^{\pm} K_m, \tag{1.19}$$
$$\partial n_m/\partial\tau = -{}^1/_2(\mathscr{E}^{+*}\rho_m^{+} + \mathscr{E}^{+}\rho_m^{+*} + \mathscr{E}^{-*}\rho^{-} + \mathscr{E}^{-}\rho^{-*}) K_m.$$

Here $\sigma_m = N_{m\uparrow} - N_{m\downarrow}$ is determined by the pumping conditions, $N_{m\uparrow}$ and $N_{m\downarrow}$ are the initial populations of the excited and the ground states with angular momentum components m,

$$K_m = \begin{cases} [J^2 - m^2]^{1/2}/J, & \Delta J = -1 \\ m/J, & \Delta J = 0. \\ [(J+1)^2 - m^2]^{1/2}/(J+1), & \Delta J = 1 \end{cases}$$

As in the non-degenerate case the superfluorescence problem can be reduced to the solution of the mixed problem for the set of Eqs. (1.19). The initial and boundary values for $\mathscr{E}$ are zero and the initial polarization fluctuations ρ_m are random functions distributed normally with a correlation matrix

$$\langle \rho_i(x,\lambda)\rho_m^{*}(y,\mu)\rangle = \delta_{im}\delta(x-y)\delta(\lambda-\mu)/g(\lambda)N\sigma_m. \tag{1.20}$$

§2. THE LINEAR MODEL, SELF-SIMILARITY OF THE ASYMPTOTIC BEHAVIOR

The Maxwell-Bloch equations for waves moving solely in one direction have in our dimensionless variables the form

$$\partial\mathscr{E}/\partial\tau + \partial\mathscr{E}/\partial x = \langle\rho\rangle, \quad \partial\rho/\partial\tau = -i\lambda\rho + n\mathscr{E},$$
$$\partial n/\partial\tau = -{}^1/_2(\mathscr{E}\rho^{*} + \mathscr{E}^{*}\rho). \tag{2.1}$$

The superfluorescence effect is, as was explained in the preceding section, described by the solution of the mixed problem in the region

$$\tau \geqslant 0, \quad 0 \leqslant x \leqslant l \tag{2.2}$$

with the following boundary and initial conditions:

$$\mathscr{E}|_{x=0} = 0, \quad \mathscr{E}|_{\tau=0} = 0,$$
$$\rho|_{\tau=0} = \rho_0(x,\lambda), \quad n|_{\tau=0} = (1-|\rho_0|^2)^{1/2}, \tag{2.3}$$

where one should assume that $\rho_0(x,\lambda)$ is a very small quantity, $\rho_0 \sim N^{-1/2}$, so that $n|_{\tau=0} \approx 1$. One can for such initial and boundary conditions linearize the set of Eqs. (2.1) near the solution $n = 1, \rho = \mathscr{E} = 0$; this gives

$$\partial\mathscr{E}/\partial\tau + \partial\mathscr{E}/\partial x = \langle\rho\rangle, \quad \partial\rho/\partial\tau = -i\lambda\rho + \mathscr{E}. \tag{2.4}$$

Using the conditions (2.3), the solution of the set (2.4) has the form[3]

$$\mathscr{E}(x,t) = \int_0^x dx' \int_{-\infty}^{\infty} d\lambda\, g(\lambda) G(x-x', t, \lambda)\rho_0(x',\lambda), \tag{2.5}$$

where

$$G(x,t,\lambda) = \frac{1}{2\pi i}\int_{-i\infty+\alpha}^{i\infty+\alpha} \frac{\exp[Pt - Px + \varphi(P)x]}{P+i\lambda}\, dP \tag{2.6}$$

and

$$\varphi(P) = \int_{-\infty}^{\infty} \frac{g(\lambda)}{P+i\lambda}\, d\lambda, \tag{2.7}$$

while α is positive. When the function $g(\lambda)$ is a Lorentz distribution:

$$g(\lambda) = \frac{\pi/T_2^{*}}{\lambda^2 + 1/T_2^{*}}, \tag{2.8}$$

we have

$$\varphi(P) = \frac{1}{P + 1/T_2^{*}}$$

and

$$G(x,\tau,\lambda) = \theta(\tau-x)\left\{I_0[2(x(\tau-x))^{1/2}] + (1/T_2^{*} + i\lambda) \times \int_0^{\tau-x} d\tau'\, I_0[2(x\tau')^{1/2}] \times \exp[(i\lambda + 1/T_2^{*})(\tau-\tau'-x)]\right\}\exp(-\tau/T_2^{*}). \tag{2.9}$$

Here $I_0(z)$ is a Bessel function of imaginary argument, and $\Theta(z)$ the Heaviside function. In the limit of infinitely thin lines ($T_2^{*} \to \infty$) we get

$$G_\infty(x,\tau,0) = I_0[2(x(\tau-x))^{1/2}]\,\theta(\tau-x). \tag{2.10}$$

We consider the asymptotic behavior of the solution (2.5) for large times $\tau \gg x$. The main contribution to the integral comes from the region near zero. To begin with we study the asymptotic behavior in the case of an infinitely thin line and afterwards discuss the legitimacy of such a calculation. We are thus interested in the asymptotic behavior, for large $\tau \gg x$, of the integral

$$\mathscr{E}(x,\tau) = \int_0^x \rho_0(y) I_0\{2[(x-y)(\tau-x+y)]^{1/2}\}\, dy \tag{2.11}$$

[in the case of interest to us $\Theta(\tau - x + y) = 1$]. Because of the exponential increase of the Bessel function I_0 with respect to the parameter $[(x-y)/\tau]^{1/2}$, the largest contribution to the integral comes from the region of small values of y. To the main order we have

$$\mathscr{E}(x,\tau) \approx \frac{1}{(x\tau)^{1/4}} \exp\{2(x\tau)^{1/2}x\} \int_0^\infty \rho_0(y) \exp\{-y(\tau/x)^{1/2}\} dy. \tag{2.12}$$

Evaluation of the integral in (2.12) is made difficult by the random nature of the function $\rho_0(x)$. We perform the calculation for the very special case when the realization of $\rho_0(x)$ in the vicinity of the point $x = 0$ is a smooth function. In that case

$$\rho_0(y) = \rho_0(0) + (y/l)\rho_1 + \dots . \tag{2.13}$$

We have now from (2.11)

$$\mathscr{E}(x,\tau) \approx \frac{\rho_0(0)x}{[x(\tau-x)]^{1/2}} I_1[2(x(\tau-x))^{1/2}]$$
$$\times\left\{1+O\left[\max\left[\frac{\rho_1}{\rho_0(0)}\left(\frac{1}{l\tau}\right)^{1/2}, \frac{l^{1/2}}{\tau^{1/2}}\right]\right]\right\}. \tag{2.14}$$

Changing to dimensional variables we find that the asymptotic behavior (2.14) is valid for sufficiently large times t when

$$t \gg (\rho_1/\rho_0(0))^2 \tau_{SF}, \quad t \gg \tau_{fl}^2/\tau_{SF}.$$

(τ_{SF} is the superfluorescence time and τ_{fl} the free flight time). Of course, we assume that for such times the linear approximation is still valid, i.e., $\mathscr{E}(x,\tau) \ll 1$. This is well possible as the initial fluctuations $\rho_0(0)$ are of order $N^{-1/2}$. It is important for us that in that intermediate range of times the answer has the form

$$\mathscr{E}(x,\tau) = x\mathscr{E}(\xi), \quad n(x,\tau) = n(\xi), \quad \rho(x,\tau) = \rho(\xi),$$
$$\xi = 2[x(\tau-x)]^{1/2}. \tag{2.15}$$

In the case of an imhomogeneously broadened line the asymptotic behavior (2.14) is intermediate. We can neglect inhomogeneous broadening if $\tau < T_2^{*2}/\tau_{SF}$. The theory given below refers just to such times.

§3. SELF-SIMILAR SOLUTIONS

In the case of exact resonance, $g(\lambda) = \delta(\lambda)$, the Maxwell-Bloch Eqs. (2.1) allow the self-similar substitution[11,14]

$$\mathscr{E}(x,\tau) = x\mathscr{E}(\xi), \quad n(x,\tau) = n(\xi), \quad \rho(x,\tau) = \rho(\xi), \tag{3.1}$$

where ξ is the self-similar variable. We note that such a space-time dependence of the $\mathscr{E}, n, \rho$ was already met with in the preceding section. Just such a form has the asymptotic behavior (2.15) of the linear problem for large τ. We must thus continue the solution (2.5) into the non-linear region through a self-similar solution (3.1) in which the functions $\mathscr{E}, n, \rho$ satisfy the following set of equations:

$$\xi\mathscr{E}' + 2\mathscr{E} = 2\rho, \quad 2\rho' = \xi n\mathscr{E}, \quad 2n' = -\xi\rho\mathscr{E}. \tag{3.2}$$

We note that the self-similar variable may take on both real ($\tau > x$) and imaginary ($\tau < x$) values. Imaginary ξ corresponds to the non-causal region in the x,τ coordinates which is unimportant for the superfluorescence problem; in what follows we shall thus be interested in the case of real ξ.

The set (3.2) has a one-parameter family of solutions which are not singular at zero and which are completely determined by the value $\mathscr{E}_0 = \mathscr{E}(0)$ and by the sign of $n_0 = n(0)$. Indeed, from the given $\mathscr{E}_0$ we determine $\rho_0 = \mathscr{E}_0$ and from the conservation law (1.9) we find $n_0 = \pm(1-\rho_0^2)^{1/2}$.

It is convenient for the further analysis to reduce the set (3.2) to a single second-order equation. Putting

$$n = \cos\varphi, \quad \rho = \sin\varphi, \quad \mathscr{E} = (2/\xi)\varphi',$$

we get

$$\varphi'' + \xi^{-1}\varphi' = \sin\varphi. \tag{3.3}$$

This equation has self-similar solutions of the sine-Gordon model which can be expressed in terms of classical Painlevé transcendentals. The solution of (3.3) determined by the initial conditions $\varphi_0 = \varphi_0(0)$, $\varphi_\xi(0) = 0$ describes the motion of a Newtonian particle with non-stationary friction in the potential $U(\varphi) = \cos\varphi$. It has the form shown in Fig. 1.

The qualitative behavior of the solutions of Eq. (3.3) depends on φ_0 which in our case is asymptotically small ($\varphi_0 \propto N^{-1/2}$). The corresponding solution will have the shape of a succession of damped pulses and the maximum of the first of them is logarithmically far from the origin, $\xi \sim \ln(1/\varphi_0)$. For that reason neglecting in Eq. (3.3) the second ("friction") term we get a formula for the approximate description of the first pulse:

$$\varphi(\xi) = 4\,\mathrm{arctg}\,[\exp(\xi - \xi_0)], \tag{3.4}$$
$$E(\xi) = 2/\xi\,\mathrm{ch}\,(\xi - \xi_0), \tag{3.5}$$

where $\xi_0 \sim \ln(1/\varphi_0)$ is the coordinate of the maximum of the first pulse.

In laboratory coordinates x,τ there correspond two values of the coordinate x, viz.,

$$x_{1,2} = {}^1/_2[\tau \pm (\tau^2 - 4\xi^2)^{1/2}] \tag{3.6}$$

to each value of the self-similar variable ξ and fixed τ. We assume that the population n (for the first time) changes sign in a point ξ_0. This means in laboratory coordinates that the population first changes sign at time $t_0 = 2\xi_0$ at the point $x_0 = \xi_0$. In accordance with Eq. (3.6) two waves, on the front of which $n = 0$, move from the point x_0 in different directions when t increases. We show in Fig. 2 the function $n(x)$ at different times $\tau_1 < \tau_2 < \tau_3$. The position of the point x_0 in which n first changes sign depends logarithmically on the self-similar parameter φ_0. The smaller the value of φ_0 the

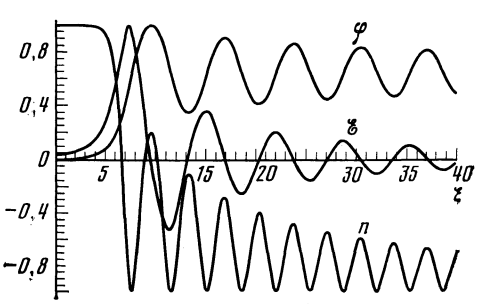

FIG. 1. Characteristic shape of the self-similar solution. The maxima of the functions $\varphi(\xi)$, $\mathscr{E}(\xi)$, and $n(\xi)$ are normalized to unity.

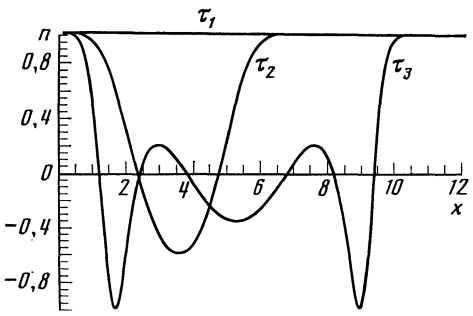

FIG. 2. Self-similar solution. The function $n(x)$ at different times $\tau_1<\tau_2<\tau_3$.

larger x_0. In our case the condition that the laser is of moderate length is equivalent to the condition that this point lies outside the limits of the sample so that we are interested in a wave moving to the left. As $\tau\to\infty$ the front of that wave moves with its right-hand end according to the law

$$x=\xi_0^2/\tau+O(\xi_0^4/\tau^3). \tag{3.7}$$

The second root corresponds to a wave moving in the opposite direction. It generates a field collecting the energy of the excited atoms. The front of this wave is logarithmically far from the light cone. As $\tau\to\infty$ it is pushed towards the light cone $x=\tau-O(\xi_0^2/\tau)$. Such a situation is realized in the case of a quantum amplifier.

We elucidate the picture which a measuring device will register at the end of the sample. To do this we fix $x=1$, where 1 is the reduced length of the sample. We measure the time from the moment of pumping. In the time interval $t\approx\frac{1}{4}\ln^2(1/\varphi_0)$ there occurs a slow growth of the field $E(\tau,1)$. Afterwards there arises a succession of pulses which decrease in amplitude. The population then changes sign and oscillates approaching the ground state. An estimate of the energy stored in the first, most powerful pulse as a ratio of the total energy in the sample gives a quantity of order $4/\ln N$. The power of the radiation is thus of order $N^2\ln N$.

The intensity $I(\tau)=|(1,\tau)|^2$ of the radiation from the sample into the first pulse can, as follows from (3.5), be approximated by the formula

$$I(\tau)=4\tau_{SF}\left[\tau\operatorname{ch}^2\left(2\frac{\tau^{1/2}-\tau_D^{1/2}}{\tau_{SF}^{1/2}}\right)\right]^{-1}, \tag{3.8}$$

where τ_D is the delay time of an individual pulse which must be determined experimentally.

§4. THE CASE OF DEGENERATE LEVELS

In the case of degenerate transitions we shall operate using the scheme described in the preceding sections. In fact, our scheme can be applied in those cases when the system, firstly, has a self-similar solution, secondly, when the solution of the linear set asymptotically goes over into a self-similar solution, and finally, when there is a region where the asymptotic region overlaps with the region where the linear approximation is applicable.

Neglecting inhomogeneous broadening, Eqs. (1.19) for waves moving in one direction have the form

$$\frac{\partial\mathscr{E}}{\partial\tau}+\frac{\partial\mathscr{E}}{\partial x}=\sum_{m=-J}^{J}K_m\sigma_m\rho_m,$$

$$\frac{\partial\rho_m}{\partial\tau}=K_m n_m\mathscr{E},\quad \frac{\partial n_m}{\partial\tau}=-{}^1/_2K_m(\rho_m\mathscr{E}^*+\rho_m^*\mathscr{E}). \tag{4.1}$$

This set has a set of first integrals:

$$n_m^2+|\rho_m|^2=1. \tag{4.2}$$

When stating the superfluorescene problem we assume as before that $\mathscr{E}(t,0)=\mathscr{E}(0,x)=0$ while the initial value for $\rho_m(0,x)$ is random and small. As in section 2 we linearize the set of equations near the solution $\mathscr{E}=\rho_m=0$, $n_m=1$ and evaluate the solution of that linearized set:

$$\mathscr{E}(x,\tau)=\int_0^x dx'\,\theta(\tau-x-x')\sum_{m=-J}^{J}\sigma_m\rho_m(x')$$

$$\times I_0\left\{2\left[(x-x')\sum_{m=-J}^{J}K_m^2\sigma_m(\tau-x+x')\right]^{1/2}\right\},$$

$$\rho(x,\tau)=\rho_m(x,0)+K_m\int_0^\tau\mathscr{E}(\tau')\,d\tau'. \tag{4.3}$$

For large $\tau\gg x$ the asymptotic expansion of $\mathscr{E}(x,\tau)$ starts with the term

$$\mathscr{E}(x,\tau)\approx\frac{2x}{\xi}\sum_{m=-J}^{J}\sigma_m\rho_m(0)I_1(\xi), \tag{4.4}$$

where

$$\xi=2\left[\sum_{m=-J}^{J}K_m^2\sigma_m x(\tau-x)\right]^{1/2}. \tag{4.5}$$

As in the preceding case the main term of the asymptotic behavior is self-similar with the self-similarity variable ξ given by Eq. (4.5). The non-singular self-similar solutions of the set of Eqs. (4.1) depend on the $2J+1$ parameters $\rho_m(0)$. Putting formally $\xi=0$ we find in the first term of the asymptotic expansion for $\rho(x,\tau)$

$$\rho_m(0)=K_m\sum_{i=-J}^{J}K_i\sigma_i\rho_i(0,0)\Big/\sum_{i=-J}^{J}K_i^2\sigma_i. \tag{4.6}$$

We note that all $\rho_m(0)$ have the same phase and only differ in the real factor K_m. This enables us to reduce the set of nonlinear equations in $6J+5$ real functions ρ_m, n_m, $\mathscr{E}$ describing the self-similar solution to a single second-order equation. Such a reduction is possible even at the level of the set of Eqs. (4.1) if one assumes that the initial and boundary conditions, and hence also the solution of the system as a whole have the same phase (for instance, are real). In that case, putting

$$n_m=\cos\theta_m,\quad \rho_m=\sin\theta_m, \tag{4.7}$$

we find that

$$\partial\theta_m/\partial\tau=\mathscr{E}K_m, \tag{4.8}$$

and, hence, θ_m can be written in the form

$$\theta_m(x,\tau)=K_m\varphi(x,\tau)+\theta_m^0(x), \tag{4.9}$$

and then

$$\mathscr{E}(x,\tau)=\partial\varphi/\partial\tau. \tag{4.10}$$

Substituting (4.10) and (4.9) into (4.1) we obtain

$$\frac{\partial^2\varphi}{\partial\tau^2}+\frac{\partial^2\varphi}{\partial\tau\,\partial x}=\sum_{m=-J}^{J}K_m\sigma_m\sin[K_m\varphi+\theta_m{}^0(x)]. \qquad (4.11)$$

Equation (4.11) is equivalent to the set (4.1) for the above stipulated reality condition. One must then put

$$\varphi(x,0)=0,\quad \partial\varphi(x,0)/\partial\tau=\mathscr{E}(x,0),$$
$$\rho_m(x,0)=\sin\theta_m{}^0(x),$$
$$n_m(x,0)=\cos\theta_m{}^0(x),\quad \mathscr{E}(0,\tau)=\partial\varphi(0,\tau)/\partial\tau.$$

We are interested in the self-similar solution of Eq. (4.11) for extremely small values of $\Theta^0_m\sim 1/(N\sigma_m)^{1/2}$. In that case the answer depnds weakly on the direction of the vector $(\Theta^0_{-J},\Theta^0_{-J+1},\ldots\Theta^0_J)$ and the first pulse of the electric field is well approximated by the solution of the equation

$$\sum_{m=-J}^{J}K_m{}^2\sigma_m\varphi''(\xi)=\sum_{m=-J}^{J}K_m\sigma_m\sin[K_m\varphi(\xi)] \qquad (4.12)$$

with asymptotic behavior $\varphi'/\xi\to 0$ as $|\xi|\to\infty$.

As an example we consider the case when the levels corresponding to the transition are degenerate with respect to the angular momentum and both correspond to multiplets with $J=2$ so that we are dealing with a so-called Q transition.

Under the same stipulations as above the contracted Maxwell-Bloch model can be reduced to the equation[11]

$$\frac{\partial^2\varphi}{\partial\tau^2}+\frac{\partial^2\varphi}{\partial\tau\,\partial x}=a\sin\varphi+\frac{b}{2}\sin\frac{\varphi}{2}; \qquad (4.13)$$

here $a=\sigma_{-2}+\sigma_2$, $b=\sigma_{-1}+\sigma_1$. The corresponding self-similar equation has the form of Newton's equation with "friction"

$$\varphi''(\xi)+\frac{1}{\xi}\varphi'(\xi)=\frac{4}{4a+b}\left(a\sin\varphi+\frac{b}{2}\sin\frac{\varphi}{2}\right)\equiv-\frac{\partial u(\varphi)}{\partial\varphi}. \qquad (4.14)$$

In contrast to the non-degenerate case the potential $u(\varphi)$ has under conditions $4a>b>0$ in the region $0\leqslant\varphi\leqslant 4\pi$ two minima. The envelope field $\mathscr{E}(\xi)$ then is a choppy function and in our case can have two maxima. The initial condition φ_0 of Eq. (4.14) determines in which of the two minima the "Newtonian" particle moving with friction in the potential $u(\varphi)$ falls. By virtue of the random nature of the fluctuations the trajectory of the system for different realizations will reach the neighborhoods of different minima.

The region of small self-similar parameters and large ξ is of interest for the superfluorescence problem so that the first pulse can rather well be approximated by the solution of the equation

$$\varphi''=\frac{4}{4a+b}\left(a\sin\varphi+\frac{b}{2}\sin\frac{\varphi}{2}\right),\quad \varphi\to 0\,(\mathrm{mod}\,4\pi),\quad |\xi|\to\infty, \qquad (4.15)$$

and the appropriate solution of this equation has the form

$$\varphi=2\arccos\left[1-\frac{4(4a+b)}{b\,\mathrm{ch}(\xi-\xi_0+\ln b)+8a+b}\right]. \qquad (4.16)$$

The field intensity at the end of the sample will for $\tau\gg\tau_{SF}$ be

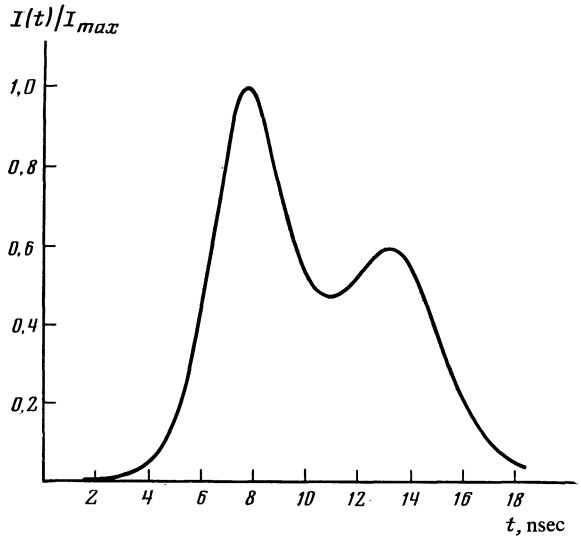

FIG. 3. Characteristic shape (4.17) of the radiation intensity when two-fold degeneracy of the levels is taken into account.

given by the formula

$$I=\frac{32(4a+b)\,b\,(\mathrm{ch}\,P+1)\,\tau_{SF}}{\tau(b\,\mathrm{ch}\,P+8a+b)^2}, \qquad (4.17)$$

where

$$P=2\frac{\tau^{1/2}-\tau_D{}^{1/2}}{\tau_{SF}{}^{1/2}}+\left(\mathrm{arcch}\frac{8a-b}{b}-\ln b\right)(4a+b)^{-1/2}$$

and τ_0 is the delay time of the first maximum (Fig. 3).

In those cases when J is large the first pulse becomes more choppy. The nature of the choppiness is determined by the means of populating the sublevels. The largest number of maxima (equal to J) is reached when their populations are approximately equal.

The development of the instability leads to phase mismatching of ρ_m and $\mathscr{E}$ which restricts the applicability of the model (4.11). In the superfluorescence problem the initial phase mismatch is small [in the main asymptotic order there is none, (4.6)] so that for several early pulses the reduction (4.11) of the model (4.1) is valid.

§5. COMPARISON WITH EXPERIMENT

A number of experiments have been performed about the observation of the superfluorescence phenomenon in cesium vapors and beams at a wavelength of $\lambda=3\,\mu$m. These experiments were just performed under the conditions described in the Introduction, i.e.., the Fresnel number was ~ 1, the gas approximation was valid and, most importantly, the time for longitudinal and transverse homogeneous broadening was large: $T_1\approx 70$ ns, $T_2\approx 80$ ns. In the case of beams one managed to increase the inhomogeneous broadening time which was basically due to the Doppler effect to $T_2^*\approx 32$ ns. In that case $\tau_{SF}\approx 0.5$ ns $\langle\tau_D\rangle\approx 10$ ns. The degeneracy of the levels was lifted by a uniform magnetic field. The experimental situation must thus be described by the equations given in the first section.

We have made a comparison between the graphs of individual pulses given in Refs. 5, 12 and the corresponding self-

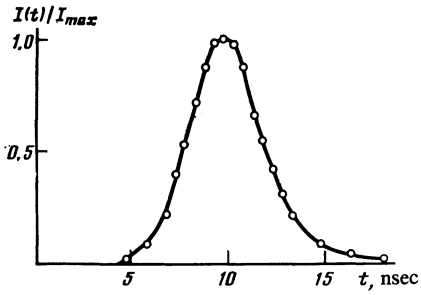

FIG. 4. The time dependence of the radiation intensity at the end of the sample: points—experiment;[12] curve—radiation intensity according to Eq. (3.8). Here $\tau_D \approx 7$ ns, $\tau_{SF} \approx 0.8$ ns, $\tau_0 \approx 3$ ns [the parameter $\tau_0 \approx \pm$ (2 to 3) ns is introduced to take into account possible systematic errors].

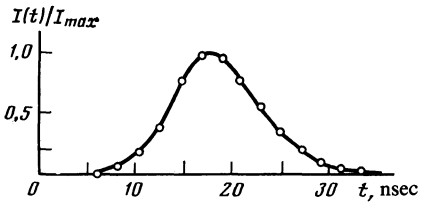

FIG. 5. The same as Fig. 4. The points—experiment[12], the curve evaluated using Eq. (3.8). Here $\tau_D \approx 15.7$ ns, $\tau_{SF} \approx 1.9$ ns, $\tau_0 \approx 3.3$ ns.

similar solutions (Fig. 4, 5).[13] It is clear from these figures that one can consider the agreement to be good.

We now discuss the assumptions made by us. One of them is the neglect of the interaction between waves propagating in opposite directions. This assumption can be justified both by the nature of the answer obtained and by a comparison with a numerical calculation of the exact Eqs. (1.8) on a computer. Indeed, it follows from (1.8) that the interaction between waves propagating in opposite directions occurs because they draw energy from the same source—the inversely populated medium. It is therefore clear that one can neglect this interaction as long as there is no intersection of relaxation waves propagating in opposite directions. One can estimate the average time for intersection of the relaxation waves as $\langle \tau_1 \rangle \approx 8 \langle \tau_D \rangle$. Our theory which is based upon equations which take into account only waves propagating in one direction thus correctly describes the first pulses. Moreover, there is a statistical factor which justifies the assumptions made by us. The fact is that the delay time of the first pulse is a random quantity and if one analyzes the distribution function of the difference of the ignition times of the pulse on the left-hand and the right-hand ends of the sample the dispersion of the times will be almost of the same order as $\langle \tau_D \rangle$.

The fact that we are interested only in the first pulses enables us to justify the mathematically inexact assumption that the function $\rho(\xi)$ is smooth near the point $\xi = 0$. A typical realization of the function $\rho(\xi)$ is not smooth. This leads to the solution not being strictly self-similar, but "quasi-self-similar" in the spirit of Ref. 10. In other words, the self-similar parameter $\mathscr{E}_0$ turns out to be a slow and random function of x/t. However, the difference between the quasi-self-similar and the self-similar solutions hardly affects the form of several of the early pulses and is unimportant for us. The agreement between the theory and experiment for the first pulse clearly illustrates this statement.

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