Non-linear relaxation of a beam of relativistic electrons in a plasma: the Langmuir condensate

V. M. Malkin

Institute of Nuclear Physics, Siberian Division of the Academy of Sciences of the USSR
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We find the spectrum of the long-wavelength Langmuir turbulence ("condensate") as a function of the energy flux entering from outside. We study the interaction between a beam of relativistic electrons and the plasma under conditions when the interaction is accompanied by a condensation of Langmuir waves.

1. INTRODUCTION

In the theory of the interaction of relativistic electron beams with a plasma there has arisen up to the present a paradoxical situation. On the one hand, on the basis of weak turbulence theory (WTT) it was possible to describe the interaction of rather dense beams with a plasma for any damping ν_e of the Langmuir waves.[1,2] On the other hand, a consistent description of the relaxation of relatively weak beams[3,4] (to which, apparently, WTT would be the better applicable) is known only when $\nu_e \sim \gamma$ where γ is the growth rate of the beam instability and it is even customary to assume that the analysis of the more interesting case when $\nu_e \ll \gamma$ is impossible on the basis of WTT. "Relatively weak" are here called beams in the stabilization of whose instability an important role is played by the process of lowest order in the energy of the Langmuir waves their induced scattering by ions. According to Ref. 1 such beams are those which satisfy the condition

$$\gamma < (\nu_s/\Omega_s)\, g^3\omega_0. \tag{1}$$

Here $\sigma_0 \sim \omega_p k_0^2 r_D^2$ is the width of the Langmuir spectrum, $k_0^{-1} \sim c/\omega_p$ is the characteristic wavelength of the Langmuir waves excited by the beam, Ω_s and ν_s are the frequency and damping rate of the density fluctuations with spatial scale k_0^{-1}, $g \sim \Omega_s/\omega_0 \ll 1$. Inequality (1) can be satisfied, especially when $\nu_s \sim \Omega_s$, for very powerful beams which are of interest in many applications, amongst which are thermonuclear applications.

The aim of the present paper is the study of the relaxation of such beams in an isotropic plasma for any values of the parameter γ/ν_e. As the case $\gamma/\nu_e \ll 1$ is uninteresting (there is no instability) and the case $\gamma/\nu_e \sim 1$ has been studied, we shall assume below that

$$\gamma/\nu_e \gg 1. \tag{2}$$

In that case the energy flux produced by induced scattering on ions downwards in frequency has no time to be absorbed via collisions and leads to the appearance of long-wavelength Langmuir waves—the condensate.

A large number of papers (see, e.g., Refs. 5 to 8) have been devoted to the study of the condensate and its effect on the spectrum of short-wavelength Langmuir waves. The reasons for the difference between the results of those papers and those obtained below become clear from the contents of the present paper. The scheme of the exposition which follows is: in section 2 we present the theory of a weakly turbulent condensate, in section 3 this theory will be used to find the spectrum of plasma turbulence excited by a beam of relativistic electrons. In order that the crux of the effects considered be not obscured by an abundance of possible spectra and by technical details we limit ourselves to the case of an isothermal plasma

$$\nu_s \sim \Omega_s, \quad \gamma \ll g^3\omega_0 \tag{1'}$$

and shall operate on the level of estimates, while exact calculations will be made only to the extent to which they are necessary to validate for the assumptions used.

2. HOW IS THE CONDENSATE CONSTRUCTED?

In the present section we shall study stationary spectra of long-wavelength Langmuir turbulence for different values of the energy flux Π flowing into the condensate.

We assume that the total energy density of the waves in the condensate is much smaller than the threshold for the modulational instability of the condensate as a whole:

$$W_c/n_0T \ll k_c^2 r_D^2. \tag{3}$$

We shall elucidate in what follows that this condition is equivalent to the assumption that the characteristic wave number k_c in the condensate is small compared to the width of the kernel k_* of the induced scattering by ions:

$$k_c \ll k_* \equiv gk_0 = (m/M)^{1/2} r_D^{-1}. \tag{4}$$

Under condition (3) the turbulence in the largest part of the condensate will be weak and it turns out that the kinetic equation

$$dN_\mathbf{k}/dt = [\gamma_*(\mathbf{k}) - \nu_e]N_\mathbf{k} + \gamma_i(\mathbf{k})N_\mathbf{k} + I_\mathbf{k} \tag{5}$$

is applicable. Here $N_\mathbf{k}$ is the spectral density of the Langmuir waves;

$$\gamma_i(\mathbf{k}) = \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_p^2}{8n_0T} \int_{k_1 \ll k_*} d^3\mathbf{k}_1 \left(\frac{\mathbf{k}\mathbf{k}_1}{kk_1}\right)^2 \frac{\omega_{k_1} - \omega_k}{|\mathbf{k}_1 - \mathbf{k}|\, v_{T_i}} N_{\mathbf{k}_1} \tag{6}$$

is the growth rate of the induced scattering on ions for the waves in the condensate with one another;

$$\omega_k = {}^3/_2\,\omega_p k^2 r_D^2$$

is the dispersive correction to the frequency of the Langmuir waves;

$$I_{\mathbf{k}}=\frac{\pi}{2^6}\frac{\omega_p^4}{(n_0T)^2}\int d^3\mathbf{k}_1d^3\mathbf{k}_2d^3\mathbf{k}_3\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3)$$
$$\times\delta(\omega_k-\omega_{k_1}+\omega_{k_2}-\omega_{k_3})\left[\frac{(\mathbf{kk}_1)(\mathbf{k}_2\mathbf{k}_3)+(\mathbf{kk}_3)(\mathbf{k}_1\mathbf{k}_2)}{kk_1k_2k_3}\right]^2$$
$$\times(N_{\mathbf{k}_1}N_{\mathbf{k}_2}N_{\mathbf{k}_3}-N_{\mathbf{k}}N_{\mathbf{k}_1}N_{\mathbf{k}_2}+N_{\mathbf{k}}N_{\mathbf{k}_1}N_{\mathbf{k}_3}-N_{\mathbf{k}}N_{\mathbf{k}_2}N_{\mathbf{k}_3}) \quad (7)$$

is the collision term describing the four-plasmon interaction in the condensate;

$$\gamma_*(\mathbf{k})=\frac{\omega_p^2}{2n_0T}\int\limits_{k_1\gtrsim k_*}d^3\mathbf{k}_1\left(\frac{\mathbf{kk}_1}{kk_1}\right)^2U_{\mathbf{k}_1-\mathbf{k},\,\omega_{k_1}-\omega_k}N_{\mathbf{k}_1} \quad (8)$$

is the growth rate of induced scattering on ions of waves from the region $k\gg k_*$ in the condensate,

$$U_{\mathbf{k}\omega}=\mathrm{Im}\frac{\rho_{\mathbf{k}\omega}}{1+\rho_{\mathbf{k}\omega}},\quad \rho_{\mathbf{k}\omega}=T\int d^3p\frac{\mathbf{k}(\partial f/\partial\mathbf{p})}{\omega-\mathbf{kv}+i0},$$

and $f(\mathbf{p})$ is the ion momentum distribution function normalized to unity.[1)]

In the region $k\ll k_*$ we can nelgect the k-dependence of the kernel $U_{\mathbf{k}_1-\mathbf{k},\omega_{k_1}-\omega_k}$ and assume that the growth rate $\gamma_*(\mathbf{k})$ depends solely on the direction of $\mathbf{k}$. The nature of the angular dependence of $\gamma_*(\mathbf{k})$ depends on the shape of the Langmuir spectrum in the region $k_1\sim k_*$ which is the region over which, in fact, the integration in (8) takes place. If the spectrum in that region is anisotropic or even singular, $\gamma_*(\mathbf{k})$ is a function with an anisotropy of the order unity. For an isotropic spectrum $\gamma_*(\mathbf{k})$ may be assumed to be constant.

We start the study of the condensate with the most interesting case $\gamma_*\gg\nu_e$, when the collisions are unimportant and the absorption of the waves can be only non-linear. As will be made clear in what follows it occurs in the region $k\sim k_m\ll k_c$ through the Langmuir collapse and the collapsing cavitons hardly affect the weak turbulence spectrum in the region $k\gg k_m$ which because of that can be found from Eq. (5).

When $k\sim k_c$ the following estimates are valid for the non-linear growth rates in Eq. (5):

$$\gamma_i(k_c)\sim\omega_p\frac{W_c}{n_0T}\frac{k_c}{k_*},$$
$$\gamma_f(k_c)\equiv\frac{I_{k_c}}{N_{k_c}}\sim\omega_p\left(\frac{W_c}{n_0T}\right)^2\frac{1}{(k_cr_D)^2}. \quad (9)$$

The assumption used here that the energy of the waves with $k_c\ll k\ll k_*$ is small (which ultimately turns out to be correct) is feasible if the time for those waves to reach the main part of the condensate due to induced scattering on ions does not exceed the time for the four-plasmon interaction. Letting k approach k_c from above we find

$$\gamma_i(k_c)\gtrsim\gamma_f(k_c). \quad (10)$$

At the same time when k decreases by several times in the region $k\sim k_c$ the non-linear growth rate $\gamma_i(k)$ changes by an order of magnitude its characteristic value (and even changes sign). To guarantee stationarity this change must be compensated by four-plasmon processes. These are possible only when

$$\gamma_i(k_c)\lesssim\gamma_f(k_c). \quad (11)$$

It follows from conditions (10) and (11) that

$$\gamma_i(k_c)\sim\gamma_f(k_c). \quad (12)$$

This estimate in combination with (9) gives a connection between W_c and k_c:

$$\frac{W_c}{n_0T}\sim(k_cr_D)^2\frac{k_c}{k_*}. \quad (13)$$

To establish the connection between γ_* and k_c we note that the four-plasmon interaction conserves $E=\int d^3\mathbf{k}\omega_kN_\mathbf{k}$ and the induced scattering on ions decreases this integral by a factor two in a time $\sim\gamma_i^{-1}(k_c)$. In the stationary regime E is maintained at a constant level due to waves leaving the region $k\gg k_*$ and hence

$$\gamma_*\sim\gamma_i(k_c)\sim\omega_p(k_cr_D)^2(k_c/k_*)^2. \quad (14)$$

Substituting the estimates (13), (14) into the formula $\Pi\sim\gamma_*W_c$ for the energy flux flowing into the condensate we can express k_c in terms of Π:

$$k_c\sim k_*(\Pi/\Pi_*)^{1/7}. \quad (15)$$

Assumptions (3), (4) are valid when

$$\Pi\ll\Pi_*\equiv\omega_pn_0T(k_*r_D)^4. \quad (16)$$

It will become clear in what follows that in the relaxation problem of interest to us the inequality $\Pi<\Pi_*$ is the same as (1′). We shall therefore assume that condition (16) is satisfied and turn to the discussion of other problems which remained open when we derived the estimates (13) to (15).

To elucidate how fast the spectrum reaches the interior of the region $k\gg k_c$ we note that in that region the growth rate $\gamma_i(k)$ is much larger than γ_* since

$$\gamma_i(k)\sim\omega_p(W_c/n_0T)(k/k_*)\sim(k/k_c)\gamma_*\gg\gamma_*.$$

The shape of the "tail" is therefore determined by the condition for the balance between induced scattering on ions inside the condensate and four-plasmon processes. One can guess the result without calculations. Indeed, when there is no energy flux from the short-wavelength ($k\gg k_*$) region into the condensate the quantity k_c decreases by a factor two in a time $\sim\gamma_i^{-1}(k_c)$ which is large in comparison with the time of the non-linear interaction of the waves of the "tail." The decrease in k_c can be interpreted as the cooling of the gas of plasmons due to their friction by the "thermostat," the role of which is played by the plasma ions. In view of the slowness of the cooling in the above mentioned sense the distribution of the "tail" plasmons must manage to approach the equilibrium one. It is obvious to assume that the latter turns out to be a Boltzmann distribution[2)]:

$$N_k\propto\exp(-k^2/k_c^2). \quad (17)$$

An exact calculation described in the Appendix confirms this guess.

We note that the problem solved in the Appendix of the search for the short-wavelength asymptotic behavior of the condensate was already discussed earlier in Ref. 11 and the result was different: the spectrum decreased as a power law. The reason for this divergence consists in the error allowed by the authors of Ref. 11 when averaging over the angles of the kernel of the four-plasmon interaction. The subsequent use of the same incorrect formula for averaging the kernel led to the conclusion that there exists a stationary condensate without waves arriving from the region $k \geqslant k_c$ (see Ref. 5, p. 193).[3)] The impossibility of such a stationary situation even when there are no collisions follows from the already mentioned fact that E is conserved in the four-plasmon interaction and that that quantity decreases in the induced scattering on ions.

Turning to a discussion of the shape of the spectrum in the region $k \ll k_c$ we assume that there the energy flux is practically independent of k up to some $k_m \ll k_c$ and is produced by the local four-plasmon interation. Let $W(k)$ be the total energy density of waves with wavelengths of order k^{-1}:

$$W(k) \sim \omega_p k^3 N_k.$$

The following estimate is valid for the inverse time of the local four-plasmon interaction:

$$\gamma_f(k) \sim \omega_p \left[\frac{W(k)}{n_0 T}\right]^2 \frac{1}{(kr_D)^2}. \tag{18}$$

From the condition that the energy flux along the spectrum be constant, $\gamma_f(k) W(k) \sim \Pi \sim \gamma_* W_*$, we find[4)]:

$$W(k) \sim W_c \left(\frac{k}{k_c}\right)^{2/3}, \quad N_k \sim \frac{W_c}{\omega_p k_c^3}\left(\frac{k_c}{k}\right)^{7/3}; \quad \gamma_f(k) \sim \gamma_* (k_c/k)^{2/3}. \tag{19}$$

In accordance with the assumptions made above the four-plasmon interaction turns out to be very fast when $k \ll k_c$:

$$\gamma_f(k) \gg \gamma_i(k) \sim \gamma_*.$$

Substituting the spectrum (19) into the four-plasmon collision term one checks easily that the assumption about the local nature of the interaction with respect to k is also satisfied (see Ref. 9): the main contribution to the integral comes from the region $k_1 \sim k_2 \sim k_3 \sim k$. Finally the energy flux $\delta\Pi(k) \sim \gamma_* W(k)$ entering from the region $k \gtrsim k_*$ to the scale $k \ll k_c$ is much smaller than the flux $\Pi \sim \gamma_* W_c$ transferred along the spectrum and the latter can therefore be assumed to be constant. Following the scheme proposed in Ref. 12 one shows easily that the flux Π is, indeed, directed towards the long-wavelength region and one also evaluates easily the coefficients in Eqs. (19).

The lower limit k_m of the spectrum (19) is determined from the condition for its modulational instability which for waves of wavelength of order k^{-1} has the form

$$W(k) < W_*(k) \sim n_0 T (kr_D)^2.$$

The ratio

$$\frac{W(k)}{W_*(k)} \sim \frac{W_c}{W_*(k_c)}\left(\frac{k_c}{k}\right)^{4/3}$$

increases with decreasing k and when $k \sim k_m$ becomes of the order of unity. Using (13) and (15) we find

$$k_m \sim k_* (\Pi/\Pi_*)^{1/4}. \tag{20}$$

When $k \sim k_m$ the growth rate of the four-plasmon process is approximately equal to the dispersive correction to the wave frequency:

$$\gamma_f(k_m) \sim \omega_m \sim \omega_p (k_m r_D)^2.$$

The growth rate of the modulational instability is a quantity of the same order when $W(k)/W_*(k) - 1 \sim 1$. A threshold excess ratio of the order unity is clearly reached already when $k \sim k_m$ after which the whole energy flux is absorbed through collapse. The cavitons present at the initial stage of the collapse have a spatial size k_m^{-1} and densely fill the whole of space. The scattering of waves with $k \gg k_m$ by those cavitons is weaker than the four-plasmon interaction:

$$\gamma_m(k) \sim \omega_m (k_m/k)^3 \ll \gamma_f(k)$$

and therefore hardly affects the weak turbulence spectrum in the region $k \gg k_m$. The effect of deeper cavitons turns out to be unimportant as they occupy a small fraction of the plasma volume: for cavitons of size $k^{-1} \ll k_m^{-1}$ this fraction is of order (k_m/k).[5]

We have thus consistently expressed all parameters of the condensate in terms of the energy flux Π flowing in it and moreover have established between the parameters γ_* and Π, which are external to the condensate, the relation necessary to determine uniquely the spectrum in the region $k \gtrsim k_*$:

$$\gamma_* \sim \omega_p (k_* r_D)^2 (\Pi/\Pi_*)^{4/7}. \tag{21}$$

These results were obtained assuming that $\gamma_* \gg \nu_e$ which is satisfied for not too weak energy fluxes[5)]:

$$\Pi \gg \Pi_\nu \equiv (\nu_e/g^2\omega_0)^{7/4} \Pi_* \sim \omega_p n_0 T (k_* r_D)^{1/2} (\nu_e/\omega_p)^{7/4}. \tag{22}$$

When the flux Π decreases all estimates remain adequate as long as the difference $\gamma_* - \nu_e$ entering in Eq. (5) is positive and approximately equal to γ_*, i.e., as long as $\gamma_* - \nu_e \gtrsim \nu_e$. The opposite case

$$\delta\gamma_* \equiv \max \gamma_* - \nu_e \ll \nu_e$$

corresponds to a small energy flux along the spectrum and is, as will become clear below, possible only for extraordinarily weak beams. We shall therefore not describe that case in detail but simply give the main estimates.

The total energy density of the waves in the condensate is at once determined from the energy balance condition

$$W_c \sim \Pi/\nu_e. \tag{23}$$

Reasoning in the same sense as when we derived (12) and (14), we easily establish that the spectrum is concentrated in a region where

$$\gamma_* - \nu_e \sim \delta\gamma_*, \tag{24}$$

and satisfies the relation

$$\gamma_i(k_c) \sim \gamma_f(k_c) \sim \delta\gamma_*. \tag{25}$$

Condition (24) gives the connection between the angular width of the spectrum ψ and $\delta\gamma_*$ and conditions (25) together with (23) enable us to express these quantities and k_c in terms of the flux Π. The result, clearly, depends on the symmetry of γ_*.

If the growth rate γ_* is isotropic, the condensate is also isotropic. The estimate (9) then remains valid and also Eq. (13). Determining k_c from (13) and (23) we can then use (25) to evaluate $\delta\gamma_*$ which gives

$$\delta\gamma_* \sim \nu_e (\Pi/\Pi_\nu)^{4/3}. \tag{26}$$

If γ_* does not possess any symmetry properties its maximum is reached for some direction **e** and the spectrum is concentrated in a cone with opening

$$\psi \sim (\delta\gamma_*/\nu_e)^{1/2} \tag{27}$$

close to the given direction. The first estimate (9) remains valid also for such a spectrum and on the right-hand side of the second estimate there occurs an additional factor[6] $\ln \psi^{-1}$. As a result a similar factor appears also in the estimate (26):

$$\delta\gamma_* \sim \nu_e \left(\frac{\Pi}{\Pi_\nu} \ln \frac{\Pi_\nu}{\Pi}\right)^{4/3}. \tag{28}$$

The stability criterion for the condensate as a whole with respect to modulation in directions at right angle to the direction **e** has the form

$$W_c/n_0T < (\psi k_c r_D)^2. \tag{29}$$

Due to the factor ψ^2 on the right-hand side, this criterion gives a lower bound for the flux Π:

$$\frac{\Pi}{\Pi_\nu} \ln \frac{\Pi_\nu}{\Pi} > \left(\frac{\nu_e}{g^2\omega_0}\right)^{3/4}.$$

Nelgecting the logarithmic factor it follows from this that

$$\Pi > \Pi_m \equiv \omega_p n_0 T (\nu_e/\omega_p)^2. \tag{30}$$

We shall show below that when the turbulence is excited by a beam this last inequality is satisfied automatically and with a large margin due to (2).

3. TURBULENCE EXCITED BY A BEAM OF RELATIVISTIC ELECTRONS

The kinetics of Langmuir turbulence in the short-wavelength ($k \gg k_*$) region might in principle be determined by the following processes[7]: induced scattering of the waves on the ions, scattering on induced density fluctuations with a spatial size $k^{-1} \ll k_*^{-1}$, and scattering on long-wavelength density fluctuations which are connected with the condensate. The following estimates hold for the reciprocal times of the processes enumerated here:

$$\gamma_i(k) \sim \omega_p \frac{W(k)}{n_0T} g^2(k), \quad \gamma_f(k) \sim \omega_p \left[\frac{W(k)}{n_0T}\right]^2 \frac{g(k)}{(kr_D)^2},$$
$$\gamma_c(k) \sim \frac{D}{\psi^2}, \quad D \sim \omega_p \left(\frac{W_c}{n_0T}\right)^2 \frac{1}{(kr_D)^2} \frac{k_c}{k}. \tag{31}$$

Here

$$g(k) \equiv \frac{kv_{Ti}}{\omega_k} \sim g\frac{k_0}{k} \sim \frac{k_*}{k},$$

$\psi(k)$ is the angular scale of the changes in the spectral density N_k of the waves. As the latter two processes do not change the frequency of the Langmuir waves, the energy flux with respect to frequency is generated by the induced scattering on ions. The energy density $W(k)$ of waves with wavelengths of order k^{-1} is determined from the condition that this flux be constant:

$$W(k) \sim n_0T \left(\frac{\Pi}{\omega_p n_0 T}\right)^{1/2} \frac{1}{g(k)}. \tag{32}$$

The spectrum (32) extends to the region of small wavenumbers up to the scale $\tilde{k}$ at which damping of the waves due to their induced scattering in the condensate is equal to the time for spectral transfer and the spectrum breaks off. To estimate $\tilde{k}$ we cannot turn to a consideration of the condensate but must use the connection obtained above between the parameters γ_* and Π external to the condensate. Using (8), (21), (32) we get the following equation for $\tilde{k}$:

$$U_{\tilde{k}\omega_{\tilde{k}}}/g(\tilde{k}) \sim (\Pi/\Pi_*)^{1/14}. \tag{33}$$

Even without assuming a Maxwellian form of the ion distribution function and exponential damping of the kernel $U_{k\omega_k}$ with respect to the parameter $(k/k_*)^2$ it is clear from this that $\tilde{k}$ depends very weakly on the flux Π and is practically the same as k_*.

Substituting the spectrum (32) into the estimate (31) we can express the non-linear growth rates in terms of the flux Π:

$$\gamma_i(k) \sim \omega_p \left(\frac{\Pi}{\omega_p n_0 T}\right)^{1/2} \frac{k_*}{k}, \quad \gamma_f(k) \sim \omega_p \frac{\Pi}{\omega_p n_0 T} \frac{1}{(k_* r_D)^2} \frac{k_*}{k};$$
$$\gamma_c(k)\psi^2(k) \sim D \sim \omega_p \frac{\Pi}{\omega_p n_0 T} \frac{1}{(k_* r_D)^2} \left(\frac{k_*}{k}\right)^3. \tag{34}$$

The ratio of the times of the first two processes is independent of k:

$$\gamma_f(k)/\gamma_i(k) \sim (\Pi/\Pi_*)^{1/2}. \tag{35}$$

This ratio is small because of the assumption (1′) that the four-plasmon process is slow when[8] $k \sim k_0$. The inequality $\Pi < \Pi_*$ is therefore in fact satisfied in the case of interest to us.

The ratio $D/\gamma_f \propto (k_*/k)^2$ increases with decreasing k but still remains small in the whole region $k > k_*$. Scattering on the condensate can therefore only have some value for strongly anisotropic spectra.

If it is unimportant in the resonance region, i.e.,

$$\gamma_c(k_0) < \gamma_f(k_0), \tag{36}$$

the angular width of the spectrum in that region is given by the estimate

$$\psi \sim \Delta\theta [\gamma_f(k_0)/\gamma_i(k_0)]^{1/2}. \tag{37}$$

Substituting (37) in the last of estimates (34) we establish easily that (36) is satisfied when $\Pi > (g/\Delta\Theta)^4 \Pi_*$.

If $\Pi < (g/\Delta\Theta)^4 \Pi_*$, we have $\gamma_f(k_0) < \gamma_c(k_0)$ and the width of the jet of Langmuir waves lying close to the maximum growth rate of the beam instability is determined from the condition

$$\gamma_i(k_0)(\psi/\Delta\theta)^2 \sim \gamma_c(k_0), \tag{38}$$

which gives

$$\psi \sim (g\Delta\theta)^{1/2} (\Pi/\Pi_*)^{1/8}.$$

The estimate of the quantity γ_c used above refers to the case

$$\psi > k_c/k_0 \sim g(\Pi/\Pi_*)^{1/7},$$

i.e.,

$$\Delta\theta > g(\Pi/\Pi_*)^{1/28}. \tag{39}$$

In that range of values of $\Delta\Theta$ the assumption $\psi < \Delta\Theta$ is satisfied automatically. It is also satisfied for smaller $\Delta\Theta$, since the angular scattering of the waves of the jet caused by the condensate is for $\psi < k_c/k_0$ not diffusive and γ_c no longer increases with decreasing ψ. The condensate therefore does not affect the stabilization of the beam instability.

The stabilization condition $\gamma_i(k_0) \sim \gamma$ enables us to connect the flux Π with the growth rate γ of the beam instability:

$$\Pi \sim \omega_p n_0 T(\gamma/g\omega_p)^2. \qquad (40)$$

As all parameters of the spectrum have already been expressed in terms of Π, it is easy to express them also in terms of γ. We have already mentioned earlier that the restriction $\Pi < \Pi_*$ then reduces to inequality (1'); collisions do not affect the condensate ($\Pi > \Pi_v$) in a wide range of parameters:

$$\gamma > g^{10}\omega_0(\nu_e/\gamma)^7, \qquad (41)$$

and inequality (30) assumed to study the case $\Pi < \Pi_v$ is automatically satisfied with a margin of $(\gamma/g\nu_e)^2$ times.

We do not give formulae for the angular spread of the beam, the energy release, relaxation lengths, and so on as they are all determined by the spectrum in the resonance region and remain, according to what we said earlier, the same as when there is no condensate.

4. CONCLUSION

The main results of the present paper are the following.

The occurrence of a condensate does not change the mechanism of the stabilization of the beam instability. The absorption of waves brought out of resonance with the beam by induced scattering on ions occurs in the long-wavelength part of the condensate through Langmuir collapse. The collapsing cavitons hardly affect the spectrum of most of the condensed waves. It is possible to find that spectrum without going beyond the framework of weak turbulence theory. It depends on the energy flux entering the condensate from the short-wavelength region. When the flux increases the condensate "inflates." If the flux is not too small (so that collisions are unimportant) scaling occurs: the spectrum of the "inflating" condensate remains unchanged in shape and this is determined only by the symmetry of the excitation.

APPENDIX

Short-wavelength asymptotic behavior of the condensate

As is clear from the estimates of section 2 the spectrum in the region $k_c \ll k \ll k_*$ satisfies with reasonable accuracy the equation

$$\gamma_i(\mathbf{k})N_\mathbf{k} + I_\mathbf{k} = 0. \qquad (A1)$$

We now obtain the Boltzmann asymptotic behavior guessed in the same section, as well as the pre-exponential factor, directly from the kinetic Eq. (A1).

To simplify the formulae we introduce dimensionless variables

$$\mathbf{x} = \mathbf{k}/k_c, \quad w_\mathbf{x} = (k_c^3\omega_p/W_c)N_\mathbf{k}. \qquad (A2)$$

Substituting (A2) into (6), (7), (A1) and putting

$$\frac{W_c}{n_0T} = \frac{18}{(2\pi)^{1/2}}(k_cr_D)^2\frac{k_c}{k_*}, \qquad (A3)$$

we get for w_x the following equation

$$w_\mathbf{x}\int d^3\mathbf{x}_1\left(\frac{\mathbf{x}\mathbf{x}_1}{xx_1}\right)^2\frac{x^2-x_1^2}{|\mathbf{x}-\mathbf{x}_1|}w_{\mathbf{x}_1}$$
$$= \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 d^3\mathbf{x}_3\delta(\mathbf{x}-\mathbf{x}_1+\mathbf{x}_2-\mathbf{x}_3)\delta(x^2-x_1^2+x_2^2-x_3^2)$$
$$\times\left[\frac{(\mathbf{x}\mathbf{x}_1)(\mathbf{x}_2\mathbf{x}_3)+(\mathbf{x}\mathbf{x}_3)(\mathbf{x}_1\mathbf{x}_2)}{xx_1x_2x_3}\right]^2$$
$$\times(w_{\mathbf{x}_1}w_{\mathbf{x}_2}w_{\mathbf{x}_3}-w_\mathbf{x}w_{\mathbf{x}_1}w_{\mathbf{x}_2}+w_\mathbf{x}w_{\mathbf{x}_1}w_{\mathbf{x}_3}-w_\mathbf{x}w_{\mathbf{x}_2}w_{\mathbf{x}_3}). \qquad (A4)$$

The only possible solution of (A4) in the region $x \gg 1$ has the form

$$w_\mathbf{x} = \Phi(\mathbf{x}/x)x^a\exp(-x^2). \qquad (A5)$$

Equality of the coefficient in front of $-x^2$ in the exponential to unity must be considered to be the quantitative definition of the quantity k_c. For such a definition of k_c the quantity W_c evaluated using Eq. (A3) differs from the total energy density of the waves in the condensate by a factor

$$\int d^3\mathbf{x}w_\mathbf{x} \sim 1.$$

We substitute (A5) into (A4) and retain the main terms in the parameter $x^{-1} \ll 1$.[9] For instance, on the right-hand side of (A4) we retain only the first of all the combinations of the spectral functions w_x. Cancelling identical exponentials we get

$$x^{a+1}\Phi\left(\frac{\mathbf{x}}{x}\right)\int d^3\mathbf{x}_1\left(\frac{\mathbf{x}\mathbf{x}_1}{xx_1}\right)^2 w_{\mathbf{x}_1}$$
$$= \int d^3\mathbf{x}_1 d^3\mathbf{x}_3\delta(\mathbf{x}-\mathbf{x}_1-\mathbf{x}_3)\delta(x^2-x_1^2-x_3^2)x_1^a x_3^a$$
$$\times\Phi(\mathbf{x}_1/x_1)\Phi(\mathbf{x}_3/x_3)\int d^3\mathbf{x}_2 e^{-x_2^2}w_{\mathbf{x}_2}\left[\frac{(\mathbf{x}\mathbf{x}_1)(\mathbf{x}_2\mathbf{x}_3)+(\mathbf{x}\mathbf{x}_3)(\mathbf{x}_1\mathbf{x}_2)}{xx_1x_2x_3}\right]^2. \qquad (A6)$$

The right-hand side of (A6) behaves x as x^{2a+1} as function of. Comparing the exponents of x in (A6) we find

$$a = 0. \qquad (A7)$$

The condition that the coefficients of the powers of x are the same gives an equation for the function $\Phi(n)$:

$$A_{\alpha\beta}n_\alpha n_\beta\Phi(\mathbf{n}) = \frac{B_{\alpha\beta}}{8}\int d^2\mathbf{n}'\Phi\left(\frac{\mathbf{n}+\mathbf{n}'}{|\mathbf{n}+\mathbf{n}'|}\right)\Phi\left(\frac{\mathbf{n}-\mathbf{n}'}{|\mathbf{n}-\mathbf{n}'|}\right)$$
$$\times\frac{n_\alpha n_\beta-(n_\alpha n_\beta'+n_\alpha' n_\beta)(\mathbf{n}\mathbf{n}')+n_\alpha' n_\beta'(\mathbf{n}\mathbf{n}')^2}{1-(\mathbf{n}\mathbf{n}')^2}. \qquad (A8)$$

Here $\mathbf{n} = \mathbf{x}/x$; $\int d^2\mathbf{n}'$ indicates integration over all directions of the unit vector $\mathbf{n}'$; the coefficients

$$A_{\alpha\beta} = \int d^3\mathbf{x}n_\alpha n_\beta w_\mathbf{x}, \quad B_{\alpha\beta} = \int d^3\mathbf{x}n_\alpha n_\beta e^{-x^2}w_\mathbf{x} \qquad (A9)$$

are determined by the spectrum in the region $x \sim 1$, and their symmetry depends on the symmetry of γ_*. Solving Eq. (A8) one can express the pre-exponential coefficient $\Phi(\mathbf{n})$ in the symptotic expression (A5) in terms of the integral characteristics (A9) of the condensate. For instance, for isotropic γ_* and, hence, isotropic condensate, we find

$$\Phi(\mathbf{n}) = \frac{2}{\pi}\int_0^\infty dxx^2w_x\left[\int_0^\infty dxx^2w_xe^{-x^2}\right]^{-1}. \qquad (A10)$$

[1]To fix the ideas we assumed in deriving Eq. (6) that the function f is Maxwellian.

[2]These considerations are not connected with a specific interaction and therefore applicable in all cases when the quasi-particles in any part of

the spectrum interact faster than the spectrum as a whole evolves. For instance, one can, without doing calculations, state that the short-wavelength asymptotic form of the self-similar spectrum of capillary waves, considered in Ref. 10, must be Boltzmannian. The difference between the result obtained in Ref. 10 and the Boltzmannian one is connected with the incorrectness of the diffusion approximation used by the authors.

[3)]This conclusion was not formulated explicitly although it followed directly from the results obtained.

[4)]The spectrum (19) was first obtained in Ref. 9 as an exact solution of the kinetic equation describing the four-plasmon interaction of waves with $k \ll k_*$.

[5)]We note that automatically $\Pi_\nu \ll \Pi_*$ by virtue of (1),(2).

[6)]We note that the main part, in the parameter $\ln\psi^{-1}$, of the four-plasmon interaction does not change the wavenumber k, it merely "regularizes" the width ψ of the jet. This fact strongly simplifies the problem of the analytical search for the spectrum.

[7)]It is understood that the electron non-linearities are unimportant.

[8)]We recall that the opposite case has already been studied.[1]

[9)]Taking corrections into account goes beyond the accuracy of (A1).

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