Substrate-induced delocalization of electrons in a two-dimensional disordered system

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The interaction with a metal substrate is an important delocalizing factor in quasi-2D systems. Under certain conditions, a zero-charge situation can arise: As the dimension of the system, L, is increased the initial increase in the resistance t gives way to a decrease. This behavior results from an instability of the zero-component tensor σ model with respect to the inclusion of the symmetry-breaking gradient terms which result from tunneling into the substrate. The new "charges" associated with these terms increase under transformations of the renormalization group, changing the sign of the Gell-Mann–Low function $dt/d\xi$ at large values of L.

I. INTRODUCTION

Noninteracting electrons in a disordered two-dimensional (2D) system are localized if there is any, disorder, no matter how slight. This hypothesis, advanced by Abrahams *et al.*,[1] has been confirmed by a variety of approaches (see the review by Altshuler *et al.*[2]). The localization results from an interference upon the multiple scattering of a quantum-mechanical particle by the same center with a probability which tends toward unity in the limit $t\to\infty$ in systems of dimensionality $d\leqslant 2$. Any effects which cause a phase relaxation of the wave function will suppress the localization, since only repeated scattering events in a time t shorter than the phase relaxation time τ_φ will contribute to the interference. The time τ_φ is usually determined by inelastic processes and is proportional to T^{-p}. At a sufficiently low temperature T, the finite dimensions of the system are known to be a factor that limits τ_φ.

In this paper we show that under certain conditions the interaction with the metal substrate can be an important delocalizing factor in real quasi-2D systems. In this case, tunneling between the 2D and 3D bands gives rise to an additional contribution to τ_φ^{-1}. The effect of a metal substrate, however, is not simply one of reducing the phase relaxation time τ_φ. There is a completely different delocalization mechanism, which is unrelated to changes in this time. It turns out that interference effects accompanying tunneling into the substrate do not cause a delocalization even in the limit $\tau_\varphi\to\infty$. On the contrary, the initial decrease in the 2D quantum diffusion gives way to a logarithmic increase at a sufficiently large value of τ_φ.

From the mathematical standpoint, this effect stems from an instability of the zero-component tensor σ model with respect to the inclusion of symmetry-breaking gradient terms. A nonlinear tensor σ model[3] was proposed by Wegner[4] in an effort to describe quantum diffusion. In this model, the diffusion modes are described by a functional of the $(2N\times 2N)$ tensor field Q; the interaction of these modes is taken into account by the restriction $Q^2=1$. Following the replica method,[5] we should set $N=0$ in the final results. The breaking of the symmetry between replicas which was proposed by Wegner[4] is important for the description of diffusion.

The functional analyzed in Ref. 4 was derived rigorously in Refs. 6–8 with a different symmetry of the tensor field Q. This approach has the advantage over the standard crossing technique in that both the average over impurities and the integration over the electron variables which vary rpaidly over space are carried out in that stage of the derivation of the functional, in which the lower-order terms in the expansion in the hydrodynamic parameters $kl_0\ll 1$ and $\omega\tau\ll 1$ are taken into account ($l_0=v_F\tau$ is the mean free path).

We will taken the approach developed by Efetov *et al.*[6] to derive a functional which contains, along with the ordinary terms, some symmetry-breaking gradient terms which arise in higher orders of the expansion in $\omega\tau$. This functional is renormalizable in lowest order in the dimensionless resistance t and in the new charges Γ associated with these terms.

When tunneling into the volume occurs, small nucleating charges Γ grow in the renormalization process, changing the sign of the Gell-Mann–Low function $dt/d\xi$ at a large value of the logarithmic variable $\xi=\ln(\tau_\varphi/\tau)$. The initial increase in the resistance t thus gives way to a decrease to the zero-charge asymptote $t=(2\xi)^{-1}$. In this asympotic limit, Γ increases to ~ 1.

We derive the equations of the renormalization group for two cases. The first corresponds to scattering by a random potential relief. In the second case, the potential scattering is supplemented by a weak scattering by magnetic impurities or an interaction with a weak magnetic field.

The qualitative picture drawn above is common to these two cases. In the case of potential scattering, the dimensionless resistance reaches its maximum $t_{\max}\ll 1$ at values $\Gamma\sim 1$, and Γ remains ~ 1 even in the limit $\xi\to\infty$. This circumstance implies the qualitative applicability of the equations of the renormalization group derived in first order in the expansion in Γ. The results on potential scattering have been reported in part in a previous brief communication.[9]

In the magnetic case the delocalization effects are much stronger, since the localization in the magnetic case is weaker if there is no interaction with the volume. The resistance t may reach its maximum at values $\Gamma\sim t_{\max}\ll 1$, so that it is legitimate to restrict the analysis to the lowest order in t and Γ in the derivation of the equation of the renormalization group.

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2. DESCRIPTION OF THE MODEL

We consider a $2D$ electron gas with a diagonal disorder which is interacting with an ideal $3D$ system. The Hamiltonian is

$$\mathcal{H}=\sum_{ij}\varepsilon_{ij}a_i^+a_j+gV^{-1/2}\sum_{i,\mathbf{p}}(a_i^+c_\mathbf{p}e^{i\mathbf{p}\mathbf{R}_i}+\text{H.a.})+\sum_{\mathbf{p}}E(\mathbf{p})c_\mathbf{p}^+c_\mathbf{p}. \tag{1}$$

The operator a_i^+ creates an electron at the site $\mathbf{R}_i$ of the $2D$ lattice; the operator $a_\mathbf{p}^+$ creates a volume electron with a momentum $\mathbf{p}$ and an energy $E(\mathbf{P})$; and g is the interband transition constant (the hybridization constant). We assume that the overlap integrals ε_{ij} fall off rapidly. The position of the levels ε_{ii} is assumed to be a random quantity with a Gaussian distribution:

$$P(\varepsilon_{ii})=\frac{1}{\Delta(2\pi)^{1/2}}\exp\left[-\frac{(\varepsilon_{ii}-\varepsilon_0)^2}{2\Delta^2}\right]. \tag{2}$$

The restriction to a diagonal disorder in (1) is not important. Incorporating a nondiagonal disorder would lead to simply a redefinition of the constants, as in the case of a purely $2D$ disordered system.[7]

The only formally important restriction in (1) is the assumption that the volume band is ideal. It is this assumption which will later allow us to integrate over the volume variables and to derive an effectively $2D$ functional. It is physically justified to ignore the disorder in the volume since, in constrast with diffusion in $2D$ systems, the quantum diffusion in a slightly nonideal $3D$ system can be arbitrarily intense.

We wish to emphasize that we are interested in the diffusion of electrons through a $2D$ lattice. The coupling with the ideal volume in (1) does not lead to a "shunting" of the $2D$ system, as it might appear. Even at observation times much longer than the characteristic time for transitions from the surface band to the volume band the diffusion through the lattice is finite.

The interaction with the volume in (1) has been written in the simple single-particle approximation. We will show that even if we ignore correlation effects in the electron-electron interaction the coupling with the volume can cause qualitative changes in the kinetic properties of the $2D$ disordered system.

The diffusion of electrons through the $2D$ lattice is described by the correlation function

$$\mathcal{K}_{ij}(\omega)=\langle\!\langle G_{ij}^R(\varepsilon+\omega/2)G_{ji}^A(\varepsilon-\omega/2)\rangle\!\rangle. \tag{3}$$

Here $\langle\!\langle\ldots\rangle\!\rangle$ means an average over random distribution (2). The retarded (or advanced) Green's function $G_{ij}^{R(A)}$ is conveniently written as a path integral over classical anticommuting fields, as in the procedure of Efetov *et al.*[6]:

$$G_{ij}^{R(A)}(\varepsilon_\pm)=i\int\varkappa(\mathbf{R}_i)\chi(\mathbf{R}_j)\exp(iS_\pm)D\chi(\mathbf{R})D\varkappa(\mathbf{R})D\chi(\mathbf{p})D\varkappa(\mathbf{p})\times\left\{\int\exp(iS_\pm)D\chi(\mathbf{R})D\varkappa(\mathbf{R})D\chi(\mathbf{p})D\varkappa(\mathbf{p})\right\}^{-1}. \tag{4}$$

Here $S_\pm$ is found from $\mathcal{H}-\varepsilon_\pm\mathcal{N}$ (the operator $\mathcal{N}$ represnets the total number of particles, and $\varepsilon_\pm=\varepsilon\pm\omega/2\pm i0$) by replacing the operators by the corresponding classical Fermi fields:

$$a_i^+\to\chi(\mathbf{R}_i),\quad a_i\to\varkappa(\mathbf{R}_i),\quad c_\mathbf{p}^+\to\chi(\mathbf{p}),\quad c_\mathbf{p}\to\varkappa(\mathbf{p}).$$

Since the action functional $S_\pm$ is quadratic in the volume variables $\chi(\mathbf{p})$ and $\varkappa(\mathbf{p})$, we can carry out the integration over these variables in expression (4). As a result we find an effectively $2D$ model with the action

$$S_\pm=\sum_{ij}\{\chi(\mathbf{R}_i)[\varepsilon_{ij}+U_{ij}(\varepsilon_\pm)-\varepsilon_\pm\delta_{ij}]\varkappa(\mathbf{R}_j)\}. \tag{5}$$

The action functional (5) differs from the action in a purely $2D$ disordered model by the presence of the additional term

$$U_{ij}(\varepsilon_\pm)=\frac{g^2}{V}\sum_{\mathbf{p}}\frac{e^{-i\mathbf{k}(\mathbf{R}_i-\mathbf{R}_j)}}{\varepsilon\pm\omega/2\pm i0-E(\mathbf{p})}=\sum_{\mathbf{k}}U(\varepsilon_\pm,\mathbf{k})e^{-i\mathbf{k}(\mathbf{R}_i-\mathbf{R}_j)}. \tag{6}$$

Here $\mathbf{k}$ is the $2D$ momentum, and $\mathbf{p}\equiv(\mathbf{k},p_z)$. We emphasize that the inclusion of this term by no means reduces to a renormalization of ε_{ij} in (5), since its odd part,

$$U^-(\mathbf{k})=\tfrac{1}{2}[U(\varepsilon_+,\mathbf{k})-U(\varepsilon_-,\mathbf{k})],$$

contributes to the difference S_+-S_-. We will need the value of $U^-(\mathbf{k})$ at $|k-k_F|\ll l_0^{-1}$, where l_0 is the unrenormalized mean free path, and k_F is the Fermi momentum of the electrons in the $2D$ band. Under the condition $k_Fl_0^{-1}(p_F^2-k_F^2)^{-1}\ll1$ (p_F is the Fermi momentum of the electrons in the $3D$ band) we can restrict the analysis to a calculation of $U^-(k_F)$. For a quadratic dispersion law for the electrons in the volume, $E(\mathbf{p})=p^2/2m$, we easily find from (6)

$$U^-(k_F)\equiv\Omega=i\frac{g^2}{2\pi}k_F^2m(p_F^2-k_F^2)^{-1/2},\quad p_F>k_F, \tag{7a}$$

$$U^-(k_F)\equiv\Omega=\omega\frac{g^2}{\pi}k_F^2m^2(k_F^2-p_F^2)^{-1/2},\quad p_F<k_F. \tag{7b}$$

The condition $p_F>k_F$ in case (7a) means that the Fermi surfaces of the $2D$ and $3D$ bands, which would be a cylinder and a sphere in the isotropic case, intersect. In this case [in contrast with case (7b)] there can be an interband tunneling of electrons with the Fermi energy ε_F. It was assumed in the derivation of (7) that the Fermi energies in the $2D$ and $3D$ bands are the same, since the system is near equilibrium in the low-frequency limit of interest here.

3. DERIVATION OF THE FUNCTIONAL OF THE ZERO-COMPONENT TENSOR σ MODEL WITH THE SYMMETRY-BREAKING GRADIENT TERMS

To average correlation function (3) over a random distribution of levels of the $2D$ electrons, we use the replica method of Refs. 5 and 6. As a result of this averaging and a transformation from the site description to a continuous description, the correlation function becomes

$$\mathcal{K}(\mathbf{r},\mathbf{r}';\omega) = -\frac{\int \varkappa_\alpha{}^1(\mathbf{r})\chi_\alpha{}^1(\mathbf{r}')\varkappa_\beta{}^2(\mathbf{r}')\chi_\beta{}^2(\mathbf{r})e^{iS}D\chi D\varkappa}{N^2\int e^{iS}D\chi D\varkappa}\Bigg|_{N=0}. \quad (8)$$

The replica indices written as subscripts here (Greek letters) vary from 1 to N (we are to set $N=0$ in the final results); the (block) superscripts $a, b = 1, 2$ specify the orgins of the fields from G^R and G^A, respectively; and any repeated tensor index implies a summation. The action functional is given by

$$S[\psi] = S_0[\psi] + S_i[\psi], \quad (9)$$

$$S_0[\psi] = -\int d^2r\bar{\psi}(\mathbf{r})\left\{[\varepsilon - \varepsilon(\hat{k})] + \frac{1}{2}\tilde{\omega}\Lambda\right\}\psi(\mathbf{r}), \quad (10)$$

$$S_i[\psi] = \frac{i}{2\pi\nu\tau}\int d^2r[\bar{\psi}(\mathbf{r})\psi(\mathbf{r})]^2.$$

Here $\varepsilon(k)$ aqnd k_F are the dispersion law and Fermi momentum of the electrons in the $2D$ band (all energies are reckoned from the bottom of the volume band), and $\tau = k_F^2/4\pi^2\nu\Delta^2$ is the unrenormalized mean free time, where ν is the electron state density in the $2D$ band, and Δ is the width of distribution (2). We can then write

$$\Lambda^{ab} = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix}, \quad \tilde{\omega} \equiv \omega + \Omega$$

[see (7)]. To simplify the notation, we have converted from the double set of fields $\chi, \varkappa$ to the spinor field ψ in Eqs. (9) and (10) (Ref. 6):

$$\psi_\alpha{}^a = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_\alpha{}^a \\ \varkappa_\alpha{}^a \end{pmatrix}, \quad \bar{\psi}_\alpha{}^a = (\psi_\alpha{}^a)^+ = \frac{1}{\sqrt{2}}(-\varkappa_\alpha{}^a, \chi_\alpha{}^a). \quad (11)$$

The fields ψ and $\bar{\psi}$ contain rapid spatial oscillations. The diffusion modes correspond to slow spatial variations. It is thus convenient to transform [6-8] in Eqs. (8)–(10) to the tensor field Q which is the adjoint of the bilinear combination $\psi\otimes\bar{\psi}$.

The functional of the tensor field $Q(\mathbf{r})$ was derived in Refs. 6, 8, and 10. We offer an alternative derivation here, which we belive to be more natural; this alternative derivation has the methodological advantage that the slow variation of the field $Q(\mathbf{r})$ is exploited only in the last step of the derivation. This derivation is in the spirit of that of Ref. 7, but there several difficulties arose in the functional intergration over intermediate Bose fields.

We write correlation function (8) in the form

$$\mathcal{K}(\mathbf{r},\mathbf{r}';\omega) = \frac{1}{4N^2}\mathrm{Sp}\left\{\frac{\delta^2}{\delta h(\mathbf{r})\delta h(\mathbf{r}')}\left[\frac{Z(h)}{Z(0)}\right]_{h=0,N=0}\right\}, \quad (12)$$

where the generating functional is

$$Z(h) = \int \exp\{iS[\psi] + iS[h,\psi]\}D\chi D\varkappa. \quad (13)$$

Here

$$S[h,\psi] = -\frac{i}{2}\int d^2r\bar{\psi}h\psi = \frac{i}{2}\int d^2r\,\mathrm{Sp}\,h\psi\otimes\bar{\psi}. \quad (14)$$

It follows from (11) that $\psi\otimes\bar{\psi}$ is a Hermitian quaternion-real $(2N\times 2N)$ matrix. The field $h(\mathbf{r})$ is conveniently chosen to be also Hermitian and quaternion-real. The functional derivative in (12) is evaluated in terms of independent quaternion components. If the correct coefficient of the exponential function in (8) is to be obtained, the matrix structure of $h(\mathbf{r})$ must be

$$h = \begin{pmatrix} 0 & h_0 \\ h_0{}^T & 0 \end{pmatrix}\otimes\tau_0 + \begin{pmatrix} 0 & h_3 \\ -h_3{}^T & 0 \end{pmatrix}\otimes\tau_3,$$

where τ are quaternions, τ_0 is the (2×2) unit matrix, $\tau_j = -i\sigma_j$ (σ_j are tha Pauli matrices, $j = 1,2,3$), and h_0 and h_3 are real $N\times N$ matrices.

We multiply the numerator asnd denominator in (12) by

$$\int \exp\left[-\frac{\pi\nu}{8\tau}\int d^2r\,\mathrm{Sp}\,Q^2(\mathbf{r})\right]DQ, \quad (15)$$

where $Q(\mathbf{r})$ is a $(4N\times 4N)$ Hermitian field, and we use the linear substitution

$$Q \to Q - (2/\pi\nu)\psi\otimes\bar{\psi} + \tau h. \quad (16)$$

The small displacement (16) causes the contributions of (10) and (14) to functional (13) to cancel out, so that the action becomes bilinear in ψ and $\bar{\psi}$, and the field h is the adjoint of the field Q:

$$S[Q,h] = \frac{i\pi\nu}{4}\int d^2r\left\{\mathrm{Sp}\,hQ + \frac{\tau}{2}\mathrm{Sp}\,h^2\right\}. \quad (17)$$

After calculation the functional derivative in (12) we find the following expression for the correlation function:

$$\mathcal{K}(\mathbf{r},\mathbf{r}';\omega) = \left(\frac{\pi\nu}{2N}\right)^2 \times \frac{\int \mathrm{Sp}[Q^{12}(\mathbf{r})(\tau_0 + i\tau_3)Q^{21}(\mathbf{r}')(\tau_0 + i\tau_3)]e^{-F[Q]}DQ}{\int e^{-F[Q]}DQ}\Bigg|_{N=0} - 2\pi\nu\tau\delta(\mathbf{r}-\mathbf{r}'). \quad (18)$$

The "free-energy" functional in (18) can be written

$$F[Q] = \frac{\pi\nu}{8\tau}\int d^2r\,\mathrm{Sp}\,Q^2(\mathbf{r}) - \ln\int \exp\{iS[Q,\psi]\}D\chi D\varkappa, \quad (19)$$

where

$$S[Q,\psi] = S_0[\psi] + \frac{i}{2\tau}\int d^2r\bar{\psi}Q\psi, \quad (20)$$

and $S_0[\psi]$ is given by (9). Since $\psi\otimes\psi$ and h are quaternion-real Hermitian matrices, only the quanternion-real part of the Hermitian field Q contributes to the action in (17), (20). The field Q will thus be assumed to be a Hermitian quaternion-real field:

$$Q = Q_j\tau_j = Q_j{}^T\tau_j{}^+, \quad Q_j \equiv \{Q^{ab}_{\alpha\beta,j}\}, \quad j = 0,1,2,3. \quad (21)$$

Correlation function (18) is essentially the same as that derived in Refs. 6 and 8. We should emphasize, however, that our derivation did not require that the field $Q(\mathbf{r})$ vary slowly. Another important point for the discussion below is the presence of a δ-function in (18); this function was not considered in Refs. 6–8.

It was shown in Refs. 6–8 that in the long-wave limit and under the condition[1)]

$$\varepsilon_F\tau\gg 1 \tag{22}$$

the only important contribution to correlation function (18) is made by the gap-free modes which satisfy the transversality condition

$$Q^2(\mathbf{r})=1, \quad \mathrm{Sp}\, Q=0. \tag{23}$$

The minimum of the free energy in (19) is reached on a class of fields Q_0 which are constant over space and which satisfy condition (23) [if the symmetry-breaking[2)] Λ term in (9) is ignored]. We write the free-energy functional (19) as an expansion in deviations from the minimum, $\delta Q(\mathbf{r}) = Q(\mathbf{r}) - Q_0$, where $Q(\mathbf{r})$, like Q_0, satisfies condition (23):

$$F-F_0=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2\tau)^n n}\left\langle\left(\int d^2r\bar{\psi}(\mathbf{r})\,\delta Q(\mathbf{r})\,\psi(\mathbf{r})\right)^n\right\rangle. \tag{24}$$

Here $\langle\ldots\rangle$ means average with functional (20), in which we set $Q=Q_0$.

Expectation values of the products of the ψ fields in (24) are calculated from Wick's theorem; only coupled diagrams are taken into account. We recall that in the calculation of Gaussian integrals over the fields $\bar{\psi}$ and ψ the integration should actually be carried out over the adjoint Fermi fields χ and $\varkappa$ (these fields were originally written as expansions in anticommuting Grassmann variables, for which integration operations have been defined[6,8,11]). Expressing ψ and $\bar{\psi}$ in terms of the original fields χ and $\varkappa$, and carrying out a Gaussian integration, we find

$$-i\langle\psi(\mathbf{r})\otimes\bar{\psi}(\mathbf{r}')\rangle={}^1/_2 G(\mathbf{r}-\mathbf{r}'). \tag{25}$$

In addition to (25), we have some other nonvanishing quadratic expection values:

$$-i\langle\psi\otimes\psi^T\rangle={}^1/_2 G\tau_2, \quad -i\langle\bar{\psi}^T\otimes\bar{\psi}\rangle={}^1/_2\tau_2 G. \tag{26}$$

In (26) each element of the $(2N\times 2N)$ quaternion matrix G is multiplied (on the left or right) by the quaternion $\tau_2=-i\sigma_0$. Here G is the Green's function of Eq. (20) with $Q=Q_0$:

$$G(k)=\left(\xi-\frac{1}{2}\tilde{\omega}\Lambda+\frac{i}{2\tau}Q_0\right)^{-1}, \tag{27}$$

$$\xi=\varepsilon(k)-\varepsilon_F=v_F(k-k_F).$$

In a description of quantum diffusion it is sufficient to derive the explicit form of the functional (24) for fields $Q(\mathbf{r})$ which vary slowly over space. For this purpose, we should carry out a hydrodynamic expansion in the parameters kl_0 and $\omega\tau$ in (24), where k^{-1} is the characteristic dimension for the spatial variations of the field $Q(\mathbf{r})$, and $l_0=v_F\tau$ is the unrenormalized mean free path. In lowest order in these parameters, the first two terms of the expansion in (24) contribute to the free energy.

Equations (25) and (26) can be used to express the second-order expectation value as the difference between two terms with the following structure:

$$\mathrm{Sp}(G\delta Q G\delta Q)-\mathrm{Sp}(G\tau_2\delta Q^T\tau_2 G\delta Q). \tag{28}$$

Making use of the known properties of quaternions and the circumstance that δQ is a Hermitian, quaternion-real matrix, we easily conclude that the two terms make identical contributions to the expectation value. Since the condition for a minimum, (23), was derived by ignoring the symmetry-breaking term in (9), a term linear in δQ also appears in expansion (24). As a result we find the following expression for the free-energy functional:

$$F=\frac{\pi\nu D}{8}\int d^2r\{\mathrm{Sp}(\nabla Q)^2+\gamma_0\,\mathrm{Sp}\,\Lambda Q\}. \tag{29}$$

Here $D=v_F^2\tau/2$ is the unrenormalized diffusion coefficient, and $\gamma_0=2i\tilde{\omega}D^{-1}$. Functional (29) (with $\tilde{\omega}\equiv\omega$) has been suggested in a paper by Wegner[4] for describing $2D$ quantum diffusion; it was derived in Refs. 6–8.

To describe the effects of the interaction with the volume, we will also calculate the contribution to the free-energy functional which is quadratic in $\tilde{\omega}\tau$. For this purpose we need to consider, in addition to the expansion of the Green's functions in (28) in $\tilde{\omega}\tau$, the variations of third and fourth orders in $\delta Q=Q-Q_0$. The corresponding calculations show that the cross terms QQ_0 cancel out. The following symmetry-breaking terms are added to functional (29) [these terms are also calculated under condition (23)]:

$$\Phi={}^1/_8\pi\nu D\int d^2r\{\gamma_1\,\mathrm{Sp}\,\Lambda Q(\nabla Q)^2+\beta l_0^{-2}\,\mathrm{Sp}(\Lambda Q)^2$$
$$-\Gamma_1\,\mathrm{Sp}(\Lambda\nabla Q)^2+\Gamma_2\,\mathrm{Sp}(\Lambda Q)^2(\nabla Q)^2+\Gamma_3\,\mathrm{Sp}(\Lambda Q\nabla Q)^2\}. \tag{30}$$

The unrenormalized values of the coefficients in (30) are

$$\gamma_1=3i\tilde{\omega}\tau, \quad \beta=\beta_0\equiv-(\tilde{\omega}\tau)^2,$$
$$\Gamma_1=7\beta_0/4, \quad \Gamma_2=5\beta_0/2, \quad \Gamma_3=5\beta_0/4. \tag{31}$$

The hydrodynamic limit of model (1) is thus described by free-energy functional (29), (30), in which the field satisfies conditions (21) and (23).

If the system also contains magnetic impurities, or if the system is in a weak magnetic field, then fluctuations of the quaternion components Q_1 and Q_2 are suppressed in the limit of sufficiently small momenta, as was shown in Refs. 6 and 8. The modes corresponding to field fluctuations $Q=Q_0\tau_0+Q_3\tau_3$ remain Goldstone modes. The model is thus described by functional (29), (30) under condition (23) in the magnetic case; the Hermitian matrix Q in condition (23) consists of $4N^2$ complex numbers.[3)]

4. APPROXIMATION OF FREE DIFFUSIONS

In calculating the correlation function (18) with the functional (29), (30), we need to choose a parametrization of the field Q, which belongs to the manifold (21), (23). Following Ref. 6, we write Q as

$$Q(\mathbf{r})=U^{-1}(\mathbf{r})\Lambda U(\mathbf{r}). \tag{32}$$

Condition (21) holds for the field Q if $U(\mathbf{r})$ is an arbitrary quaternion-real unitary matrix, i.e., if $U\in\mathrm{Sp}(2N)$, where $\mathrm{Sp}(N)$ is a symplectic group. The block-diagonal matrices $(U\in\mathrm{Sp}(N)\times\mathrm{Sp}(N))$ commute with the matrix Λ and thus do not convert it[4)] into another matrix Q. Such matrices are eliminated from (32) by the following choice of parametrization[6]:

$$U=\exp\left(\frac{W}{2}\right), \quad W=\begin{pmatrix}0 & B\\ -B^+ & 0\end{pmatrix}, \tag{33}$$

where B is an arbitrary $(N\times N)$ quaternion-real matrix. Rep-

resentation (32), (33) is also valid in the magnetic case. Here U is a unitary matrix, while B is an arbitrary field of complex numbers.

The approximation of free diffusions corresponds to an expansion [with the help of (32), (33)] of all the matrices Q in correlation function (18) out to terms quadratic in W. After the corresponding Gaussian integrals have been calculated, the correlation function can be written as follows (for $N=0$):

$$\mathcal{K}(\mathbf{k},\omega)=\frac{2\pi\nu}{\tilde{D}k^2-i\omega-i\Omega-\beta\tau^{-1}}-2\pi\nu\tau. \tag{34}$$

The effective diffusion coefficient $\tilde{D}$ in (34) is

$$\tilde{D}=D(1+\gamma_1+\Gamma_1+\Gamma_2+\Gamma_3+\beta_0/2). \tag{35}$$

The term $\beta_0/2$ results from the change in the unrenormalized diffusion coefficient in (29) in second order in $\tilde{\omega}\tau$.

When (31) and (35) are used, the correlation function of the zeroth approximation, (34), is the same as the correlation function derived by summing the ladder diagrams in the crossing technique with the same accuracy in $\tilde{\omega}\tau$. Conservation of the number of particles requires that the exact correlation function by proportional to ω^{-1} at $k=0$. Because of the constant term in (34) [i.e., the δ-function in (18)], the correction $\beta\propto(\omega\tau)^2$ cancels out in the denominator in (34). The number of particles is thus conserved at $\Omega=0$, with the same accuracy with which the correlation function is derived. When there is tunneling into the volume [case (7a)], the number of particles in the $2D$ band is not conserved; this situation is described by the Ω term in correlation function (34).

We will be interested in the renormalized values of the coefficients in (34). As will be shown below, the renormalization of the coefficient β is determined to a large extent by the shortwave ($k\sim l_0^{-1}$) contributions to functional (29), (30), so that it is not possible to determine the behavior of β upon a change in the characteristic dimension by the renormalization-group approach. In contrast, the renormalization of all the gradient terms is determined exclusively by long-wave fluctuations of the field $Q(\mathbf{r})$.

5. DERIVATION OF THE EQUATIONS OF THE RENORMALIZATION GROUP

How do the symmetry-breaking gradient terms in (30) affect the properties of the model in the renormalization-group approach? To single out the "fast" diffusion modes, we use the method proposed by Polyakov,[13] which was used in Refs. 6 and 8 to analyze a tensor σ model with functional (29).

The unitary matrix $U(\mathbf{r})$ in (32) breaks up into the product of a "fast" part $U_0(\mathbf{r})$ and a "slow" part $\tilde{U}(\mathbf{r})$: $U=U_0\tilde{U}$, where the matrix U_0 can be written in the form in (33). Using $[\Lambda W]_+\equiv\Lambda W+W\Lambda=0$, we then find from (32), (33)

$$Q=\tilde{U}^+Q_0\tilde{U}, \tag{36}$$

where the fast matrix is

$$Q_0=\Lambda\exp(W_0). \tag{37}$$

From (36) we find the following convenient representation of ∇Q:

$$\nabla Q=\tilde{U}^+([Q_0,\ \mathbf{A}]_-+\nabla Q_0)\tilde{U}, \tag{38}$$

where

$$\mathbf{A}=\nabla\tilde{U}\tilde{U}^+=-\tilde{U}\nabla\tilde{U}^+. \tag{39}$$

The functional $\tilde{\mathcal{F}}$, which describes the slow fluctuations, can be found by integrating over the fast variable $Q_0(\mathbf{r})$:

$$\tilde{\mathcal{F}}[\tilde{Q}]=-\ln\int\exp\{-\mathcal{F}[Q]\}DQ_0. \tag{40}$$

Here $\mathcal{F}=F+\Phi$ [see (29) and (30)], while the slow field $\tilde{Q}(\mathbf{r})$ is

$$\tilde{Q}=\tilde{U}^+\Lambda\tilde{U}.$$

To simplify the calculations we can, following Ref. 8, immediately choose the gauge of the matrix $\mathbf{A}$, setting $\mathbf{A}^{11}=\mathbf{A}^{22}=0$. We might note that the expression for $\nabla\tilde{Q}$ [see (38) with $Q_0=\Lambda$] does not contain $\mathbf{A}^{11}$ or $\mathbf{A}^{22}$.

The functional integral in (40) can be evaluated through an expansion in the dimensionless resistance $\tilde{t}$, where

$$16\pi\tilde{t}\equiv t\equiv(\pi\nu D/8)^{-1}. \tag{41}$$

This expansion corresponds to an expansion in W_0 of the matrix Q_0 in (37) in the functional $\mathcal{F}[Q]$. To calculate $\tilde{\mathcal{F}}$ in first order in t it is sufficient to use the approximation quadratic in W_0 (the terms linear in W_0 do not contribute to the functional, since W_0 varies rapidly). Expressions of three types which are quadratic in W_0 arise in each of the gradient terms in the functional $\mathcal{F}[Q]$. The three types of expressions ($\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$) contain zero, one, and two matrices $\mathbf{A}$, respectively. A special role is played by the functional F_0 which is obtained in this manner from the functional (29):

$$F_0=-\frac{1}{t}\int d^2r\,\mathrm{Sp}(\nabla W_0)^2=\frac{2}{t}\int d^2r\,\mathrm{Sp}\,\nabla B\nabla B^+. \tag{42}$$

This functional does not contain the slow field at all, and it is adopted as the zeroth-approximation functional.

As in our original functional, (29), (30), we consider in $\tilde{\mathcal{F}}[\tilde{Q}]$ terms no higher than quadratic in $\nabla\tilde{Q}$. Here it is sufficient to retain in the expansion of the exponential function in (40) in $\mathcal{F}-F_0$ only the terms which are quadratic in $\mathbf{A}$, which are then averaged with a weight factor $\exp(-F_0)$. This average will be denoted below as $\langle\ldots\rangle_0$.

In contrast with F_0 in (42), the terms Φ_0 found from functional (30) contain both the fast variables W_0 and the slow variables $\tilde{Q}$, so they cannot be taken into account in the zeroth approximation. These terms, which do not depend on gradients of the slow variable $\tilde{Q}$, give rise to terms which are quadratic in $\nabla\tilde{Q}$ during the expansion of the expontential function in (40) in an arbitrary nth) order in γ_1 and Γ (and in the lowest order in t), because of expectation values of the types

$$\langle(\Phi_0)^nF_2\rangle_0,\quad\langle(\Phi_0)^{n-1}\Phi_2\rangle_0,\quad\langle(\Phi_0)^{n-2}(\Phi_1)^2\rangle_0.$$

In zeroth order in γ_1 and Γ [i.e., when only functional (29) is taken into account], the contribution to $\tilde{\mathcal{F}}$ is

$$\langle F_2\rangle_0+\tilde{F}_0, \tag{43}$$

where $\tilde{\mathcal{F}}_0$ stems from a term in expression (37) which does not contain W_0, and it reproduces in the slow variable $\tilde{Q}$ the gradient term in functional (29). In (43) we have used the circumstance that we have $F_1=0$ when $\mathbf{A}^{11}=\mathbf{A}^{22}=0$. Here

$$F_2=\frac{1}{t}\int d^2r\,\mathrm{Sp}[\mathbf{A},W_0]_-{}^2. \tag{44}$$

In first order in the small parameters γ_1 and Γ, the contribution to $\tilde{\mathcal{F}}$ which is quadratic in $\nabla\tilde{Q}$ is given by

$$\langle\Phi_2\rangle_0-\langle\Phi_0F_2\rangle_0+\tilde{\Phi}_0. \tag{45}$$

The terms $\tilde{\Phi}_0$ reproduce in the slow variables the form of functional (30).

As an example, we describe a method for renormalizing the Γ_1 term in functional (30). In the approximation quadratic in W_0, $\Phi(\Gamma_1)$ is

$$\Phi_0(\Gamma_1)=-\frac{\Gamma_1}{t}\int d^2r\,\mathrm{Sp}(\Lambda\tilde{Q}\nabla W_0)^2,$$

$$\Phi_2(\Gamma_1)$$

$$=-\frac{\Gamma_1}{t}\int d^2r\,\mathrm{Sp}\{(\Lambda\tilde{Q}[W_0,\mathbf{A}]_+)^2+2\Lambda\tilde{Q}\mathbf{A}\Lambda\tilde{Q}[W_0^2,\mathbf{A}]_+\}. \tag{46}$$

We have not written out the terms of $\Phi_1(\Gamma_1)$ which are linear in $\mathbf{A}$, since [see (45)] they do not contribute to the functional in first order in Γ_1.

To evaluate the expection values in (45) we should express the matrices W_0 in (46) explicitly in terms of B and B^+, working from (33). For potential scattering the matrices B consist of N^2 real quaternions:

$$B=B_j\tau_j,\qquad B^+=B_j^T\tau_j^+.$$

In this case the expectation value which are quadratic in B are, in terms of the parameter $\tilde{t}$ in (41),

$$\langle B_{\alpha\beta,i}(\mathbf{k}_1)B_{\mu\nu,j}(-\mathbf{k}_2)\rangle_0=\frac{2\pi\tilde{t}}{k_1^2}\delta_{\mathbf{k}_1\mathbf{k}_2}\delta_{\alpha\mu}\delta_{\beta\nu}\delta_{ij}. \tag{47}$$

In the magnetic case, in which the matrices B consist of N^2 complex numbers, the only nonvanishing expectation values are

$$\langle B_{\alpha\beta}{}^+(\mathbf{k}_1)B_{\mu\nu}(-\mathbf{k}_2)\rangle_0=\frac{8\pi\tilde{t}}{k_1^2}\delta_{\mathbf{k}_1\mathbf{k}_2}\delta_{\alpha\nu}\delta_{\beta\mu}. \tag{48}$$

Here we are assuming that the fast field $B(\mathbf{k})$ is nonzero only in the region $\lambda k_0<k<k_0$, where $0<\lambda<1$, and $k_0\sim l_0^{-1}$ is a cutoff parameter. Expection values of the products of the matrices B can be calculated with the help of expression (47) or (48) and Wick's theorem. To take the logarithm in (40) is, as usual, equivalent to considering only the coupled diagrams in expression (45).

As a result we find the following expressions:

$$\langle\Phi_2(\Gamma_1)\rangle_0-\langle\Phi_0(\Gamma_1)F_2\rangle_0=\xi\Gamma_1\int d^2r\{-\alpha\,\mathrm{Sp}(\nabla\tilde{Q})^2$$

$$+(\alpha+4N)\,\mathrm{Sp}(\Lambda\nabla\tilde{Q})^2+(2+\alpha)\,\mathrm{Sp}\,\Lambda\nabla\tilde{Q}\,\mathrm{Sp}\,\Lambda\nabla\tilde{Q}\}. \tag{49}$$

Here $\xi=\ln\lambda^{-1}$. The coefficient α in (49) depends on the symmetry of the matrix Q: For quanternion-real matrices (for the case of potential scattering) we would have $\alpha=-1$, while for complex matrices (the magnetic case) we would have $\alpha=0$.

It can be seen from (49) that the integration over the fast variables not only leads to a renormalization of the coefficients in functional (29), (30) but also generates a new contribution to the functional [the last term in (49)]. Corresponding additional contributions are generated upon the renormalization of $\Phi(\Gamma_2)$ and $\Phi(\Gamma_3)$ in functional (30).

The following functional is thus added to functional (29), (30):

$$\Phi=\frac{1}{t}\int d^2r\{\Gamma_1\,\mathrm{Sp}\,\Lambda\nabla Q\,\mathrm{Sp}\,\Lambda\nabla Q-\Gamma_2\,\mathrm{Sp}\,\Lambda Q\,\mathrm{Sp}\,\Lambda Q(\nabla Q)^2$$

$$-\Gamma_3\,\mathrm{Sp}\,\Lambda Q\nabla Q\,\mathrm{Sp}\,\Lambda Q\nabla Q\}. \tag{50}$$

The last term in (50) is nonzero only in the magnetic case, in which the Hermitian matrix Q consists of $4N^2$ complex numbers. In the case of potential scattering, in which the matrix Q consists of $4N^2$ real quaternions, this term is identically zero.

The theory described by functional (29), (30), (50) is renormalizable. Each $\Gamma-\overline{\Gamma}$ pair is renormalized independently (without reference to the other pairs). From expressions of the type in (19) we find the equations of the renormalization group, which turn out to be identical for the $\Gamma_1-\overline{\Gamma}_1$ and $\Gamma_2-\overline{\Gamma}_2$ pairs:

$$\frac{d}{d\xi}(\Gamma\tilde{t}^{-1})=-(\alpha+4N)\Gamma+2\overline{\Gamma}, \tag{51}$$

$$\frac{d}{d\xi}(\overline{\Gamma}\tilde{t}^{-1})=(2+\alpha)\Gamma-4N\overline{\Gamma}. \tag{52}$$

The vertex $\overline{\Gamma}_3$ in functional (50) is nonzero only in the magnetic case. In this case, in which we have $\alpha=0$, the equations of the renormalization group for the pair $\Gamma_3-\overline{\Gamma}_3$ are also the same as (51) and (52). In the case of potential scattering, on the other hand, we are left with a single independent equation for the renormalization of Γ_3:

$$\frac{d}{d\xi}(\Gamma_3\tilde{t}^{-1})=-(1+4N)\Gamma_3. \tag{53}$$

Renormalization of the vertex γ_1 in functional (30) leads to the equation

$$\frac{d}{d\xi}(\gamma_1\tilde{t}^{-1})=-2N\gamma_1. \tag{54}$$

This equation differs only in the sign of its right side from the equation from the renormalization of the vertex γ_0 of functional (29) obtained in Refs. 3 and 6. In the limit of interest here ($N=0$), the vertex $\gamma_1\tilde{t}^{-1}$ is not renormalized. Also in this limit, the charges Γ grow in magnitude, telling us that the σ model is unstable with respect to the inclusion of symmetry-breaking gradient terms. For any natural number N, all the charges in Eqs. (51)–(54) vanish, so that there is no instability with respect to their inclusion.

Since the unrenormalized values of the coefficients γ_1 and Γ are such that the condition $|\gamma_1^2\sim|\Gamma|$ holds, we also need to take into account the contribution of second order in the coefficient γ_1 to the functional (40). This contribution is given by the expectation values

$$\frac{1}{2}\langle\Phi_0{}^2(\gamma_1)F_2\rangle_0-\frac{1}{2}\langle\Phi_1{}^2(\gamma_1)\rangle_0-\langle\Phi_0(\gamma_1)\Phi_2(\gamma_1)\rangle_0. \tag{55}$$

At $N=0$, the expectation values (55) contribute only to equations of the type (52) for the vertices $\overline{\Gamma}_2$ abnd $\overline{\Gamma}_3$: We need to add $-(2+\alpha)\gamma_1^2/4$ to the right side of the equation

for $\overline{\Gamma}_2$ and $-(1+\alpha)\gamma_1^2/4$ to the right side of the equation for $\overline{\Gamma}_3$.

Renormalization-group equations were derived in Refs. 3, 6, and 8 for the gradient term of functional (29). In the magnetic case, the Gell-Mann–Low function vanishes in lowest order in $\tilde{t}$, so that we need to consider the next term in the expansion in $\tilde{t}$ (Refs. 2, 3, and 8). For the system under consideration here, the vertices Γ and $\overline{\Gamma}$ also contribute to the renormalization of functional (29), as can be seen from expressions of the type in (49). As a result, the equation of the renormalization group for the dimensionless resistance $\tilde{t}$ becomes ($N=0$)

$$d\tilde{t}^{-1}/d\xi=\alpha-{}^1/_2\tilde{t}(1+\alpha)-\alpha(\Gamma_1-\Gamma_2-\Gamma_3)+2(\overline{\Gamma}_1-\overline{\Gamma}_2)+2\overline{\Gamma}_3(1+\alpha). \quad (56)$$

Unfortunately, it is not possible to derive a renormalization-group equation for the gradient-free β term in functional (30). The reason is that expectation values of the types $\langle\Phi_0\rangle_0$ and $\langle\overline{\Phi}_0\rangle_0$ contribute to the renormalization of the β vertex. Expectation values of this type diverge quadratically at the upper limit; i.e., they are proportional to the square of the cutoff parameter, $k_0^2\sim l_0^{-2}$. The order of magnitude of the contribution of these terms of Γl_0^{-2} (or $\gamma_1^2 l_0^{-2}$), i.e., on the same order as the original coefficient of the gradient-free term, βl_0^{-2} [see (30) and (31)]. The renormalization of the β vertex thus depends strongly on the cutoff parameter k_0; i.e., it is determined by the contribution of shortwave fluctuations, which cannot be described by the renormalization-group method.

6. ANALYSIS OF THE RENORMALIZATION-GROUP EQUATIONS

We begin the analysis with the magnetic case. Equation (56) with $\alpha=0$ is conveniently rewritten

$$d\tilde{t}^{-1}/d\xi=-\frac{\tilde{t}}{2}+2\Gamma, \quad \Gamma=\Gamma_1-\overline{\Gamma}_2+\overline{\Gamma}_3. \quad (57)$$

Introducing the corresponding combination $\Gamma=\Gamma_1-\Gamma_2+\Gamma_3$, we find the following equations (for $N=0$) from Eqs. (51) and (52), taking the contribution of the γ_1 vertex into account:

$$\frac{d}{d\xi}(\Gamma\tilde{t}^{-1})=2\overline{\Gamma}, \quad (58)$$

$$\frac{d}{d\xi}(\overline{\Gamma}\tilde{t}^{-1})=2\Gamma+\gamma_1^2(\tilde{t}/2\tilde{t}_0)^2. \quad (59)$$

Here we have taken it into account that by virtue of (54) we have $\gamma_1(\xi)=\gamma_1\tilde{t}/\tilde{t}_0$, where $\gamma_1\equiv\gamma_1(\xi=0)$. The unrenormalized value of the dimensionless resistance (41) is

$$\tilde{t}_0\equiv\tilde{t}(\xi=0)=(\pi\varepsilon_F\tau)^{-1}\ll 1. \quad (60)$$

The vertices $\overline{\Gamma}$ arise only in the course of the renormalization. The initial conditions for Eqs. (58) and (59) with (31) are thus

$$\overline{\Gamma}_0=0, \quad \Gamma_0=\beta_0/2, \quad \gamma_1^2=9\beta_0. \quad (61)$$

It is easy to see that even in the initial stages of the renormalization (while $\tilde{t}$ has essentially not yet changed) the equality $\Gamma=\overline{\Gamma}$ becomes established, and both charges increase in proportion to $\propto\beta_0\exp(2\tilde{t}_0\xi)$. We thus see that if

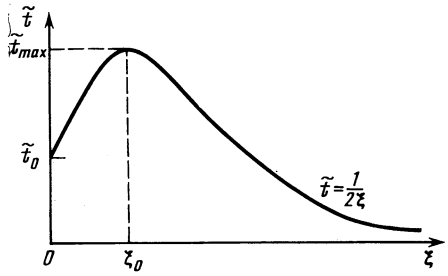

FIG. 1. The dimensionless resistance versus the logarithmic variable $\xi=\ln\lambda^{-1}$. In the case of potential scattering we would have $\xi_0\sim\tilde{t}_0^{-1}$, while in the magnetic case we would have $\xi_0\sim\min\{\tilde{t}_0^{-1}\ln|\tilde{\omega}\tau|^{-1}, \tilde{t}_0^{-2}\}$.

$\beta_0>0$ the sign of the Gell-Man–Low function (57) changes at sufficiently large values of ξ. As a result, the initial growth of $\tilde{t}(\xi)$ gives way to a zero-charge asymptotic behavior (Fig. 1):

$$\tilde{t}=(2\xi)^{-1}. \quad (62)$$

Under the condition

$$\Gamma_0\geqslant\exp(-\pi\varepsilon_F\tau) \quad (63)$$

we have $\tilde{t}_{max}\ll 1$. It can be seen from (57) that the maximum of $\tilde{t}$ corresponds to $\overline{\Gamma}=t_{max}/4$; i.e., condition (63) is also a sufficient condition for the use of the Γ expansion. In asymptotic expression (62), the quantity Γ (and $\overline{\Gamma}$) tends toward unity (Fig. 2).

If $\beta_0<0$, the sign of the Gell-Mann–Low function in (57) does not change, so that the incorporation of the Γ terms does not cause qualitative changes of any sort. For a purely $2D$ system ($\tilde{\omega}\equiv\omega$) we would be dealing with precisely this case. Incorporating the volume states in case (7b), in which there are no real transitions to the volume, leads to only a slight renormalization of the frequency ω, so that the diffusion falls off montonically (as it does in the purely $2D$ case). The zero-charge situation ($\beta_0>0$) described above prevails if the Fermi surfaces of the $2D$ and $3D$ bands intersect, and it becomes possible for electrons to tunnel into the volume [case (7a)].

We turn now to the case of potential scattering. The equations of the renormalization group are, with $\alpha=-1$ and $N=0$,

$$d\tilde{t}^{-1}/d\xi=-1+\Gamma+2\overline{\Gamma}, \quad (64)$$

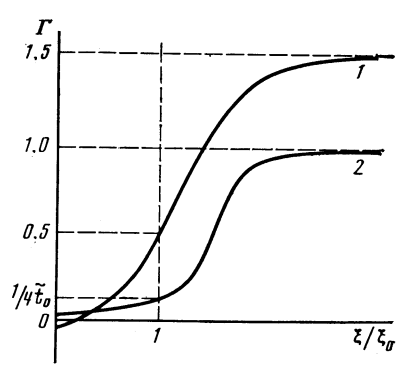

FIG. 2. The charges associated with the symmetry-breaking gradient terms versus the logarithmic variable $\xi=\ln\lambda^{-1}$. 1—Potential scattering; 2—magnetic case.

$$\frac{d}{d\xi}(\Gamma\tilde{t}^{-1})=\Gamma+2\overline{\Gamma}, \tag{65}$$

$$\frac{d}{d\xi}(\overline{\Gamma}\tilde{t}^{-1})=\overline{\Gamma}+\gamma_1^2\left(\frac{\tilde{t}}{2\tilde{t}_0}\right)^2. \tag{66}$$

Here $\Gamma=\Gamma_1-\Gamma_2$ and $\overline{\Gamma}=\overline{\Gamma}_1-\overline{\Gamma}_2$. We have ignored the term Γ_3 in (64), since in the case of potential scattering the magnitude $|\Gamma_3|$ decreases [see (53)]. The initial conditions for Eqs. (64)–(66) are given in (60) and (61), but we should substitute a different value for Γ_0, namely $-3/4\,\beta_0$, into these equations.

As in the magnetic case, Γ and $\overline{\Gamma}$ are coupled even in the initial stages of the renormalization: $\Gamma=2\overline{\Gamma}=C(1-\tilde{t}_0\xi)^{-3}$. The proportionality coefficient is $C=2/3(\Gamma_0+\gamma_1^2/4)=\beta_0$. Consequently, as in the magnetic case, $\Gamma(\xi)$ is positive if $\beta_0>0$ (despite the opposite sign of the unrenormalized value of Γ_0). The qualitative behavior of $\tilde{t}(\xi)$ and $\Gamma(\xi)$ is the same as in the magnetic case (Figs. 1 and 2).

Interestingly, the zero-charge asymptotic behavior which $\tilde{t}(\xi)$ reaches in the case $\beta_0>0$ is precisely the same as the zero-charge asymptotic behavior in the magnetic case, (62). The same asymptotic behavior in a magnetic field has been derived for a purely $2D$ disordered system with a Coulomb interaction.[10]

The primary distinction from the magnetic case is that the Gell-Mann–Low function in (64) vanishes at $\Gamma=1/2$, i.e., at the boundary of the range of applicability of the Γ expansion. At $\xi\gg\tilde{t}_0^{-1}$, the parameter Γ tends toward the asymptotic value $\Gamma=3/2$. This result indicates that the Γ expansion remains applicable at least qualitatively in the case of potential scattering.

We recall that the effective diffusion coefficient $\tilde{D}$ in (35) consists of two parts:

$$\tilde{D}=(2\pi^2\nu\tilde{t})^{-1}(1+\Gamma_+), \tag{67}$$

where $\Gamma_+=\Gamma_1+\Gamma_2+\Gamma_3$ [and where we have discarded a term $\gamma_1\tilde{t}^{-1}$, which remains constant during the renormalization according to (54)]. In the nonmagnetic case, as in the magnetic case, the quantity $\Gamma_+(\xi)$ is proportional to $\Gamma(\xi)$ even in the initial stages of the renormalization. In the case of potential scattering, the two contributions to the diffusion in (67) are comparable in magnitude even near the minimum of D, and they remain comparable in the asymptotic limit $\xi>\tilde{t}_0^{-1}$. In the magnetic case, the additional contribution to the diffusion becomes important only in the asymptotic region (62).

What are the physical consequences of the results derived by the renormalization-group approach? The resistance $\tilde{t}(\xi_f)$, where ξ_f is that value of the parameter $\xi=\ln\lambda^{-1}$ at which the renormalization process is topped, has a physical meaning. This stopping occurs when the diffusion modes cease to be Goldstone modes (i.e., at $Dk_0^2\lambda^2\sim\tilde{\omega}$) or when the finite size of the system becomes important ($\lambda k_0\sim L^{-1}$). At $T=0$ we thus have

$$\xi_f=\min\{{}^1\!/_2\ln(\tilde{\omega}\tau)^{-1},\ \ln(L/l_0)\}. \tag{68}$$

At $T\neq0$, the quantity ξ_f is determined by temperature-dependent inelastic scattering[13]: $\xi_f(T)\sim|\ln(T\tau)|$. The $t(\xi)$ dependence (Fig. 1) is actually a dependence on $\xi_f(T)$ as long as $\xi_f(T)$ is smaller than ξ_f in (68). At lower temperatures, the T dependence of the resistance dissapears. With $\omega=0$ and $L=\infty$, the temperature dependence reaches saturation at $\xi_f(T)\sim\xi_f(\Omega)=1/2\ln(\Omega\tau)^{-1}$. If $\xi_f(\Omega)$ is greater than the value at which the resistance t reaches its maximum (Fig. 1), this maximum may also be manifested in the temperature dependence of the resistance.

7. CONCLUSION

Can the effects predicted here be observed experimentally? If so, it will be of fundamental importance to prevent a shunting of the resistance of the $2D$ system by a conducting substrate.

A shunting results from the penetration of the electric field into the volume even if there is no direct contact between the current-carrying electrodes and the substrate. The shunting effect is described formally by the correlation function $\ll\mathbf{j}_s\,\mathbf{j}_b\gg$, which, along with the correlation function $\ll\mathbf{j}_s\,\mathbf{j}_s\gg$, contributes to the surface conductivity in accordance with the Kubo formula (the operators $\mathbf{j}_b$ and $\mathbf{j}_s$ are the volume and surface current densities). This contribution is proportional to $|\Omega(\omega+i/\tau_b)^{-1}|$, where $|\Omega|$ is the tunneling frequency (7a), and τ_b is the mean free time of the electrons in the substrate. This contribution diverges in the limit $\omega\to0$, $\tau_b\to\infty$.

This formal divergence, however, has no bearing on the effects discussed above. The density correlation function $\ll\rho_s\,\rho_s\gg$ (which is coupled to only the single correlation function $\ll\mathbf{j}_s\,\mathbf{j}_s\gg$, by virtue of the Einstein relation) remains finite in the limit $\tau_b\to\infty$. The possible zero-charge decrease in the resistance with the temperature is thus totally unrelated to the shunting effect of the substrate.

In a real situation (with a nonideal substrate and a finite value of τ_b), the shunting contribution proportional to $|\Omega\tau_b|$ is finite and can be ignored if the tunneling frequency Ω is sufficiently low. In the calculation of the correlation function $\ll\rho_s\,\rho_s\gg$, on the other hand, the finite value of τ_b leads to only small corrections $\sim(\varepsilon_F\tau_b)^{-1}$. It is thus completely legitimate to ignore the fact that the volume is not ideal in a description of surface diffusion.

Experimentally, shunting can be suppressed by separating the conducting film and the conducting substrate by an insulating barrier (by depositing a film on an oxidized metal, for example). Let us calculate the tunneling frequency in a three-layer system of this sort. The hybridization constant [see (1)] is $g\sim\varepsilon_F\lambda^{3/2}\exp(-d/\lambda)$, where d is the thickness of the insulating layer, and λ is the characteristic tunneling length (on the order of 1 Å). The quantity Ω in (7a) is thus

$$|\hbar\Omega|\sim g^2n(v_F\hbar)^{-1}\sim\exp(-2d/\lambda)\cdot1\text{ eV}.$$

Here $n\sim10^{15}$ cm^{-2} is the electron density in the $2D$ band, and $v_F\sim10$cm/s. Consequently, with an insulating layer only a few angstroms thick, the parameter $\Omega\tau_b$ is small even for ultrapure metal substrates. On the other hand, the tunneling effects which depend logarithmically on $\Omega\tau$ may be important even at very small values of Ω.

Experimentally, tunneling effects may be manifest, in particular, as a saturation of the temperature dependence of the resistance. At sufficiently low temperatures (below 1K),

one ordinarily observes a logarithmic increase of the resistance with decreasing temperature. This temperature dependence disappears when the temperature decreases to the point that the phase relaxation time determined by temperature-dependent inelastic-scattering processes[14,2] becomes shorter than the phase relaxation time determined by the effects of the finite size of the system and the tunneling into the substrate. The tunneling effects outshadow the effects of the finite size of the system if $|\Omega| > v_F^2 \tau / L^2$. Adopting $\tau \sim 10^{-14}$ s and $L \sim 1$ mm for an estimate, we find that this inequality holds for an insulating layer with a thickness $d \lesssim 10\lambda \sim 10$–$20$ Å.

It would be particularly interesting to examine the conductivity of a rather dirty metal film (with $\varepsilon_F \tau$ not too large) in this three-layer system. In this case we may see at

$$\exp(-\pi\varepsilon_F\tau) > |\Omega\tau| > (l_0/L)^2$$

a maximum in the temperature dependence of the resistance associated with the maximum in the $t(\xi)$ dependence (Fig. 1).

The effects discussed here are single-particle effects and do not depend on the statistics, so that the results derived here may be qualitatively applicable to the quantum diffusion of arbitrary surface excitations. The most interesting systems (from the standpoint of a study of the effect of the volume) are those in which it is possible to study surface diffusion directly. The surface diffusion of excitons in molecular crystals, for example, can be detected by measuring the emission from these excitons at molecular traps (see the review by Agranovich[15]). In such systems, the band of surface excitations is not spatially distinct from the volume band, so that interband transitions should have an extremely important effect on surface diffusion.

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[1)]Condition (22) is equivalent to the condition $(\Delta/\varepsilon_F)^2 \ll 1$, where is the spread of the levels in distribution (2), corresponding to a slight diagonal disorder in (1).

[2)]The A terms lowers the symmetry group of the action functional (9), (10) from $Sp(2N)$ to $Sp(N) \times Sp(N)$ [$Sp(N)$ is the symplectic group].

[3)]Taking the effect of the magnetic field on the volume states into account leads to only some unimportant and small renormalizations in U_{ij} [See Eq. (6); these renormalizations are on the order of $(eH/mc)\varepsilon_F^{-1}$].

[4)]This means that $Sp(N) \times Sp(N)$ is an isotropy group of manifold (23); i.e., these conditions detrermine a quaternion Grassmann manifold $Sp(2N)/Sp(N) \times Sp(N)$ (Ref. 12).

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