Singularities in volt-ampere characteristics of superconductor-semiconductor-superconductor junctions

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It is shown that in superconductor-semiconductor-superconductor junctions the Riedel singularity in the amplitude of the superconducting current is considerably enhanced as a result of a resonance mechanism in the transmission of coherent electrons along special trajectories having a periodic arrangement of the impurities. In the asymmetrical case, this mechanism leads to the new singularities that appear when the voltage at each of the semiconductor-superconductor interfaces is equal to the corresponding value of the gap.

Josephson junctions in which the superconductors are separated by a semiconductor layer (*S-Sm-S*) have been investigated in a number of publications.[1–6] However, theoretical investigations have been carried out only for the static properties of the junctions: the critical current and its dependence on the temperature, thickness of the layer, and the parameters of the semiconductor have been calculated.[1–6] In this work we determine the superconducting current in the junction as a function of voltage for the case of a nondegenerate semiconductor containing impurity atoms.

This case is of particular interest, since under certain conditions resonance transmission of the electrons through the semiconducting layer becomes possible. In the semiconducting layer there are special trajectories along which there is a periodic distribution of impurity atoms and along which the coherent electrons, which carry the superconducting current, pass with little attenuation. Althoug the probability of formation of such trajectories is small, it increases with increasing impurity concentration, and well before the point of degeneracy the resonance mechanism of transport of the superconducting current is favored over the usual tunneling mechanism.[6–7]

It turns out that resonance tunneling has a substantial effect on the character of the singularities in the superconducting current. It is well known that in ordinary tunneling the amplitude of the superconducting current depends weakly on the applied voltage U_0, and only near the value $U_0 = 2\Delta_0$ (where Δ_0 is the modulus of the order parameter in the superconductor), does it have a Riedel logarithmic singularity.[8] At specified values of the total current this situation should lead to singularities in the volt-ampere (I–V) characteristic when the capacitance of the junction is small enough.[9] However, the fluctuations smear out the weak singularity and it has not proved possible to observe it.

Resonance tunneling leads to an enhancement of the singularity: there is a sharper rise in current over a wide range of voltage. At the same time, the capacitance of the superconductor-semiconductor-superconductor junctions is relatively low, because of the large thickness of the semiconductor layer. Therefore, in this case a singularity should appear in the I–V characteristic of the junction.

The band structure of the superconductor-semiconductor-superconductor junction is shown in Fig. 1. Schottky barriers are formed near the interfaces of the semiconductor with the superconductors. The resistance of the barriers is usually large and we can assume that all the voltage drop occurs at these barriers. The chemical potential μ in the semiconductor layer falls below V_0, the bottom of the conduction band (in the nondegenerate case) and near the impurity levels. The Fermi level in the superconductors is shifted relative to μ by an amount corresponding to the voltage drops U_1 and U_2 at the interfaces (the applied voltage is $U_0 = U_1 + U_2$, and the potential is referred to the level μ so that U_1 and U_2 have opposite signs).

The resonance trajectories are formed out of impurity levels located in a narrow energy range near the chemical potential. As a result, when the level μ passes through the bandgap (the crosshatched regions in Fig. 1), in the superconductors (as the voltage is varied), the superconducting current will show singularities. If the junction is symmetric, then at $U = 2\Delta_0$ (for a positive voltage) as in the usual tunnel junction there will be one singularity, to which, however, corresponds a hyperbolic dependence of the current on the voltage. In an asymmetric junction the order parameters Δ_1 and Δ_2 are different in the two superconductors and the voltage drops at the interfaces are also different (the ratio of the voltages is given by $\gamma = |U_2|/|U_1|$). In this case, for positive values of the total voltage U_0 there can be two square root singularities, when each of the voltages at the interfaces is equal in value to the respective energy gap.

Thus, by studying the singularities in the superconducting current it is possible to observe resonance tunneling in superconductor-semiconductor-superconductor junctions.

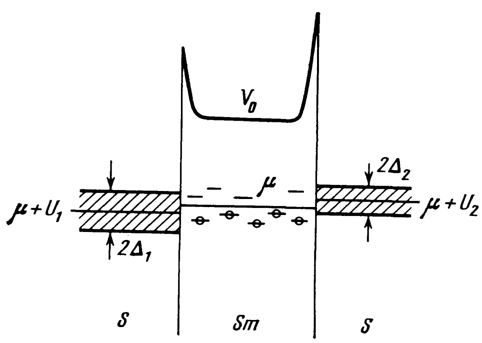

FIG. 1. Band structure of a *S–Sm–S* junction for voltage $U_0 = U_2 - U_1$.

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2. GENERAL EXPRESSION FOR SUPERCONDUCTING CURRENT IN A SUPERCONDUCTOR-SEMICONDUCTOR-SUPERCONDUCTOR JUNCTION

The current density is expressed by means of the matrix Green's function of the system $\hat{G}(\mathbf{r},\mathbf{r}';t,t')$, which we find by using the Gor'kov equation, written down by the use of the Keldysh method[10]:

$$\left[i\hat{\tau}_z\frac{\partial}{\partial t}+\frac{1}{2m}\frac{\partial^2}{\partial \mathbf{r}^2}+\mu-V(\mathbf{r})-U(z)+\hat{\Delta}\right]$$
$$\times\hat{G}(\mathbf{r},\mathbf{r}';t,t')=\delta(t-t')\delta(\mathbf{r}-\mathbf{r}'), \quad (1)$$

where $\hat{\tau}_z$ is the Pauli matrix, $V(\mathbf{r})$ is the scalar potential in the system without an external field (it is comprised of the bottom of the conduction band $V(z)$ and the impurity potential $V_{\rm imp}(\mathbf{r})$), and $V(z)$ is the potential distribution created by the applied voltage (z is the coordinate perpendicular to the plane of the junction). The matrix $\hat{\Delta}$ has the form

$$\hat{\Delta}=\begin{pmatrix}0 & \Delta(\mathbf{r},t)\\ -\Delta^*(\mathbf{r},t) & 0\end{pmatrix}. \quad (2)$$

The matrix Green's function $\hat{G}$ is given by the expression

$$\hat{G}=\begin{pmatrix}G^R & G\\ 0 & G^A\end{pmatrix}, \quad (3)$$

where G and $G^{B(A)}$ are, in turn, matrices consisting of the usual Green's function g and an anomalous function F:

$$G=\begin{pmatrix}g_1 & F_1\\ -F_2 & g_2\end{pmatrix}. \quad (4)$$

To solve equation (1) we perform a Fourier transformation with respect to the time difference:

$$\hat{G}(\mathbf{r},\mathbf{r}';t,t')=\int\frac{d\omega}{2\pi}\hat{G}_\omega(\mathbf{r},\mathbf{r}';t')e^{-i\omega(t-t')}. \quad (5)$$

From this we obtain

$$\left\{\tau_z[\omega-U(z)]+\frac{1}{2m}\frac{\partial^2}{\partial \mathbf{r}^2}+\mu-V(\mathbf{r})+\Delta(\mathbf{r},t')\right\}$$
$$\times\hat{G}_\omega(\mathbf{r},\mathbf{r}';t')=\delta(\mathbf{r}-\mathbf{r}'). \quad (6)$$

Here, we have used the explicit time dependence of the order parameter:

$$\Delta(\mathbf{r},t)=\Delta(z)e^{-2iU(z)t}, \quad (7)$$

where the order parameter in the semiconductor layer is taken to be zero because of the smallness of the electron-phonon coupling constant. The current density is expressed in terms of the Green's function by the formula[11]

$$j(t')=\frac{e}{4m}\int\frac{d\omega}{2\pi}\mathrm{Sp}\left[(1+\hat{\tau}_z)\left(\frac{\partial}{\partial z'}-\frac{\partial}{\partial z}\right)G_\omega(\mathbf{r},\mathbf{r}';t')|_{\mathbf{r}=\mathbf{r}'}\right]. \quad (8)$$

To find the Green's function G_ω it is convenient to write it in the form

$$G_\omega(\mathbf{r},\mathbf{r}';t')=\mathrm{th}\left[\frac{\omega-U(z)}{2T}\right](G_\omega^R-G_\omega^A)+G_\omega^{(1)}, \quad (9)$$

where the first term, in the absence of an external field, $[U(z)=0]$ goes over into the well-known solution[10], and for $G_\omega^{(1)}$ we obtain from Eq. (6)

$$\left\{\hat{\tau}_z[\omega-U(z)]+\frac{1}{2m}\frac{\partial^2}{\partial \mathbf{r}^2}+\mu-V(\mathbf{r})+\hat{\Delta}(z,t')\right\}G_\omega^{(1)}(\mathbf{r},\mathbf{r}';t')$$
$$=-\frac{1}{2m}\left[\frac{d}{dz}(G_\omega^R-G_\omega^A)\right]\frac{d}{dz}\mathrm{th}\left[\frac{\omega-U(z)}{2T}\right]$$
$$-\frac{1}{2m}\frac{d}{dz}\left\{(G_\omega^R-G_\omega^A)\frac{d}{dz}\mathrm{th}\left[\frac{\omega-U(z)}{2T}\right]\right\} \quad (10)$$

with the condition $G_\omega^{(1)}=0$ for $U(z)=0$. Using formulas (9) and (10) we can write the expression for the current density (8) in the form

$$j(t)=\frac{e}{4m}\int\frac{d\omega}{2\pi}\mathrm{Sp}(1+\hat{\tau}_z)\left(\frac{\partial}{\partial z'}-\frac{\partial}{\partial z}\right)\Big|_{\mathbf{r}'\to\mathbf{r}}$$
$$\times\left\{\mathrm{th}\left[\frac{\omega-U(z)}{2T}\right][G_\omega^R(\mathbf{r},\mathbf{r}';t')-G_\omega^A(\mathbf{r},\mathbf{r}';t')]\right.$$
$$-\frac{1}{2m}\int d^3\mathbf{r}_1\left[\frac{d}{dz_1}\mathrm{th}\frac{\omega-U(z_1)}{2T}\right]\left(\frac{\partial}{\partial z_1'}-\frac{\partial}{\partial z_1}\right)\Big|_{\mathbf{r}_1'\to\mathbf{r}_1}$$
$$\left.\times G_\omega^R(\mathbf{r},\mathbf{r}_1)[G_\omega^R(\mathbf{r}_1',\mathbf{r}')-G_\omega^A(\mathbf{r}_1',\mathbf{r}')]\right\}. \quad (11)$$

With the use of Eq. (6) the Green's function of the system in the superconducting state can be expressed in terms of the normal Green's functions

$$G_\omega^{R(A)}(\mathbf{r},\mathbf{r}';t')=G_{\omega n}^{R(A)}(\mathbf{r},\mathbf{r}')$$
$$-\int d^3\mathbf{r}_1G_{\omega n}^{R(A)}(\mathbf{r},\mathbf{r}_1)\hat{\Delta}(z_1,t')G_\omega^{R(A)}(\mathbf{r}_1,\mathbf{r}';t'). \quad (12)$$

For the derivatives of the normal Green's functions that appear in (11) we have the simple identity derived in Ref. 4:

$$\int\frac{d^2\rho}{m}\left(\frac{\partial}{\partial z'}-\frac{\partial}{\partial z}\right)\Big|_{\mathbf{r}\to\mathbf{r}'}G_{\omega n}^{R(A)}(\mathbf{r},\mathbf{r}_1)G_{\omega n}^{R(A)}(\mathbf{r}_2,\mathbf{r}')$$
$$=G_{\omega n}^{R(A)}(\mathbf{r}_2,\mathbf{r}_1)[\mathrm{sign}\,z_1-\mathrm{sign}\,z_2]. \quad (13)$$

Here, the points $\mathbf{r}_1$ and $\mathbf{r}_2$ are located in the superconducting regions and the point $\mathbf{r}$ is in the semiconducting layer, $\boldsymbol{\rho}$ is a vector which lies in the plane of the junction and in terms of which it is subsequently convenient to transform to the momentum representation, with the corresponding momentum $\mathbf{p}$. We note that the origin of the coordinate system is taken in the middle of the semiconductor layer, so that expression (13) vanishes if both coordinates are in the same superconductor region.

Using formulas (12) and (13) and expanding the matrix derivatives in (11), we obtain for the superconducting current a final expression that contains two terms having different analytic properties:

$$J_S(t)=J_S^{(1)}+J_S^{(2)}, \quad (14)$$

$$J_S^{(1)}=e\int\frac{d\omega}{4\pi}dz_1\,dz_2\,d^2\mathbf{p}_1\,d^2\mathbf{p}_2\left\{\mathrm{th}\left[\frac{\omega-U_2}{2T}\right]\theta(z_2-z_1)\right.$$
$$\left.-\mathrm{th}\left[\frac{\omega-U_1}{2T}\right]\theta(z_1-z_2)\right\}|\mathrm{sign}\,z_1-\mathrm{sign}\,z_2|\Delta(z_1,t)\Delta^*(z_2,t)$$
$$\times[g_{1\omega n}^R(\mathbf{p}_1,\mathbf{p}_2;z_1,z_2)g_{2\omega}^R(\mathbf{p}_2,\mathbf{p}_1;z_2,z_1)$$
$$-g_{1\omega n}^A(\mathbf{p}_1,\mathbf{p}_2;z_1,z_2)g_{2\omega}^A(\mathbf{p}_2,\mathbf{p}_1;z_2,z_1)], \quad (15)$$

$$J_S^{(2)}=e\int\frac{d\omega}{4\pi}\left[\mathrm{th}\left(\frac{\omega-U_1}{2T}\right)-\mathrm{th}\left(\frac{\omega-U_2}{2T}\right)\right]\int dz_1\,dz_2$$

$$\times d^2\mathbf{p}_1\, d^2\mathbf{p}_2 |\operatorname{sign} z_1 - \operatorname{sign} z_2| \Delta(z_1, t)\Delta^*(z_2, t)$$
$$\times [g^R_{1\omega n}(\mathbf{p}_2, \mathbf{p}_1; z_2, z_1) g_{2\omega}{}^R(\mathbf{p}_1, \mathbf{p}_2; z_1, z_2)$$
$$-g^{RA}_{2\omega}(\mathbf{p}_1, \mathbf{p}_2; z_1, z_2) g^{AR}_{1\omega n}(\mathbf{p}_2, \mathbf{p}_1; z_2, z_1)]. \quad (16)$$

In these formulas under the integral sign there are always products of two Green's functions with coordinates z_1 and z_2 which lie on different sides of the semiconductor barrier (although the integration is taken over all z_1 and z_2, when this condition is satisfied, then Δ in the semiconductor layer and the factor $|\operatorname{sign} z_1 - \operatorname{sign} z_2|$ are both zero). The integrals are taken over transverse momenta $\mathbf{p}_1$ and $\mathbf{p}_2$.

The integration over ω was carried out along the real axis, and the rule for going around the singularities was determined by the analytic properties of the Green's functions. Here, in addition to the functions $g^{B(A)}$ that are analytic in the upper and lower half-planes, it is appropriate to introduce the functions g^{RA} and g^{AR} which have more complicated rules for integrating around the poles; we shall formulate these rules after we write down explicit expressions for the Green's functions. We note only that as a consequence, in the term $J_s^{(2)}$ there is a contribution to the current not only due to the pole of the hyperbolic tangent, as in the term $J_s^{(1)}$, but also due to the poles of the Green's functions.

Expressions (15) and (16), obtained by the method discussed here, are very convenient for finding the superconducting current in the superconductor-semiconductor-superconductor junction. The dependence of the order parameter on the z-coordinate in these formulas can be taken as

$$\Delta(z) = \Delta_1 \exp\left[-2i\left(U_1 t + \frac{\chi}{2}\right)\right], \quad z < -a,$$
$$\Delta(z) = \Delta_2 \exp\left[-2i\left(U_2 t - \frac{\chi}{2}\right)\right], \quad z > a, \quad (17)$$

since the smallness of the derivatives of the Green's functions under the integral sign automatically causes the correction to the current in the superconductors due to the z dependence $\Delta(z)$ to be small. With formulas (15) and (16) one can also easily take into account the explicit z-dependence of the Green's functions (because of the complex shape of the potential barrier) and average over the transverse coordinates, that is, over the different resonance trajectories. For the present problem, this is how our method is better than the method of integrating the Green's functions over the energy variable.[11]

In the absence of an external field, formulas (14)–(16) go over into the expression derived in Ref. 14 for the superconducting current of a superconductor-semiconductor-superconductor junction. The well known results for the nonstationary Josephson effect in a superconductor-insulator-superconductor junction[8,12] also follow from these formulas.

2. CONTRIBUTION OF RESONANCE TRAJECTORIES TO THE SUPERCONDUCTING CURRENT

It is necessary to average formulas (15) and (16) over the positions and energies of the impurities in the semiconductor (the Green's functions depend on these parameters). In doing so, it is found that the major contribution to the current is due to resonance trajectories with a periodic distribution of the impurity atoms. Such a trajectory is characterized by a certain definite distance $2y$ between the impurity atoms (the important values of y are such that $y \ll a$, so that there are many impurity atoms on a trajectory) and by an angle θ, which defines the crookedness of the trajectory (the important values of θ are $\theta \ll 1$). If it is assumed that the trajectory begins and ends at a Schottky barrier, then in order to obtain cyclic boundary conditions, it is necessary that the first and last impurities be located at the same distance y from the ends of the trajectory (the impurity levels within the Schottky barrier can be neglected, since they lie considerably above μ.

The spread E_D in energy of the impurity levels causes large-scale fluctuations in the potential, and the correlation length usually exceeds the thickness of the semiconductor layer.[13] Therefore the value of $E_D - \mu$, as well as the height of the barrier $V_0 - \mu$ can be considered constant in each trajectory. We shall now determine the contribution to the superconducting current from the resonance trajectories and then average over the parameters y, θ, and E_D (taking into account the probability of formation of a trajectory). In the case of not too small impurity concentrations, this contribution is found to be larger than the ordinary tunneling superconducting current.

For periodic trajectories and for a δ-function potential for the interaction with the impurities, Eq. (6) reduces to a system of difference equations solved by the method developed in Ref. 6 and 7. As a result we obtain

$$g^{R(A)}_{1\omega n}(\mathbf{p}_1, \mathbf{p}_2; z, z') = I^{R(A)}_{\omega\, imp}(y, \theta, E_D)$$
$$\times \exp(i\mathbf{p}_1\boldsymbol{\rho}_0 - i\mathbf{p}_2\boldsymbol{\rho}_N) g^{R(A)}_{1\omega n}(\mathbf{p}_1; z, z_0) g^{R(A)}_{1\omega n}(\mathbf{p}_2; z_N, z'), \quad (18)$$

$$I^{R(A)}_{\omega\, imp}(y, \theta, E_D) = \frac{V_0 - \mu}{\pi m \alpha B} \frac{\lambda_- - \lambda_+}{(1+\lambda_+)(1+\lambda_-)(\lambda_+{}^N - \lambda_-{}^N)},$$

$$\lambda_\pm = \frac{\omega + \mu - E_D}{2B} \pm i\left[1 - \left(\frac{\omega + \mu - E_D}{2B}\right)^2\right]^{1/2},$$

$$B = \frac{V_0 - \mu}{\alpha y} e^{-2\alpha y}, \quad (19)$$

$$\alpha = (2m)^{1/2}(V_0 - \mu)^{1/2}, \quad N = \frac{a}{y}\left(1 + \frac{\theta^2}{2}\right).$$

In (18) the factor $I_{\omega\,\mathrm{imp}}$ describes the resonance tunneling, while the Green's functions on the right hand side of the formula describe the system without impurities. The coordinates z and z' lie in the superconducting regions, and z_0 and z_N are the coordinates of the first and last impurities in a trajectory (the transverse radius vectors are denoted, respectively, $\boldsymbol{\rho}_0$ and $\boldsymbol{\rho}_N$). The quantity B has the meaning of the width of the band formed by the periodic distribution of the impurities; the parameter α characterizes the decay of B within the barrier, and N is the number of impurities in a trajectory.

The expression of the function g_2 is obtained from formulas (18) and (19) by substitution of the frequency ω by $-\omega$ in the expression for $I_{\omega\,\mathrm{imp}}$.

Let us now discuss the analytic properties of the Green's functions. The expression for $I_{\omega\,\mathrm{imp}}$ has N poles for real values of ω, which can be easily found by rewriting the expression in the form

$$I^{R(A)}_{\omega\,imp}=\frac{V_0-\mu}{2\pi\alpha mB}\frac{\mathrm{tg}\,(\varphi/2)}{\sin\varphi N},\quad \varphi=\arccos\frac{\omega+\mu-E_D}{2B}. \quad (20)$$

The "pure" Green's functions $g_1^{R(A)}(\mathbf{p}_1;z_1,z_0)$ and $g_1^{R(A)}(\mathbf{p}_2;z_N,z')$ have, as usual, a branch point at $\omega=\pm\Delta_{1,2}+U_{1,2}$, and the contour passes above them for the retarded Green's functions and below them for the advanced Green's functions.[8] The contours around the poles of the function $I^{R(A)}_{\mathrm{imp}}$ must be the same. For the functions g^{RA} and g^{AR} the rules for integrating around the poles are more complicated. For instance, for the function g^{RA} we have

$$g^{RA}_{1\omega n}(\mathbf{p}_1,\mathbf{p}_2;z,z')=I^{RA}_{\omega\,imp}(y,\theta,E_D)$$
$$\times\exp[i\mathbf{p}_1\boldsymbol{\rho}_0-i\mathbf{p}_2\boldsymbol{\rho}_N]g^{R}_{1\omega n}(\mathbf{p}_1;z,z_0)g^{A}_{1\omega n}(\mathbf{p}_2;z_N,z'), \quad (21)$$

where the poles of $I^{RA}_{\omega\,\mathrm{imp}}$ are encircled from above when the difference $U(z)-U(a')>0$ and from below when $U(Z)-U(z')<0$. For the function $I^{AR}_{\omega\,\mathrm{imp}}$ the rules are the opposite. Thus, for any sign of the applied voltage U_0 the functions g^{RA} and g^{AR} are not analytic in the upper half-plane or in the lower half-plane of ω.

Using the formulas for the Green's functions of ordinary superconductors,[4,5] we bring expressions (14)–(16) for the superconducting current to the form

$$J_S=J_S^{(a)}\sin 2(U_0t+\chi)+J_S^{(b)}\cos 2U_0t,$$
$$J_S^{(a)}=eD(\mu)\Delta_1\Delta_2\,\mathrm{Im}\left\{\int\frac{d\omega}{2\pi}\left(\mathrm{th}\frac{\omega-U_2}{2T}+\mathrm{th}\frac{\omega-U_1}{2T}\right)\right.$$
$$\times f_1(\omega)([\Delta_1^2-(\omega-U_1+i\delta)^2][\Delta_2^2-(\omega-U_2+i\delta)^2])^{-1/2}$$
$$+\left(\mathrm{th}\frac{\omega-U_2}{2T}-\mathrm{th}\frac{\omega-U_1}{2T}\right)f_2(\omega)$$
$$\left.\times([\Delta_1^2-(\omega-U_1+i\delta)^2][\Delta_2^2-(\omega-U_2-i\delta)^2])^{-1/2}\right\}, \quad (22)$$
$$J_S^{(b)}=eD(\mu)\Delta_1\Delta_2\,\mathrm{Re}\int\frac{d\omega}{2\pi}\left(\mathrm{th}\frac{\omega-U_2}{2T}-\mathrm{th}\frac{\omega-U_1}{2T}\right)$$
$$\times\{f_1(\omega)([\Delta_1^2-(\omega-U_1+i\delta)^2][\Delta_2^2-(\omega-U_2+i\delta)^2])^{-1/2}$$
$$-f_2(\omega)([\Delta_1^2-(\omega-U_1+i\delta)^2][\Delta_2^2-(\omega-U_2-i\delta)^2])^{-1/2}\}, \quad (23)$$

where $D(\mu)$ are the derivatives of the transmission factors of the Schottky barriers, as were calculated in Refs. 1 and 4.

If we take into account only ordinary tunneling, then $f_1=f_2=S\exp(-4\alpha a)$, where S is the area of the junction, and from these formulas we obtain the well-known singularities.[8,12] In the resonance case, using formula (20) we obtain for the functions f_1 and f_2

$$f_1=\left\langle\left(\frac{y^3}{a}\right)\left\{\left(\mathrm{ctg}\frac{\varphi_+}{2}\sin\varphi_+N+i\delta\cos\varphi_+N\right)\right.\right.$$
$$\left.\left.\times\left(\mathrm{ctg}\frac{\varphi_-}{2}\sin\varphi_-N-i\delta\cos\varphi_-N\right)\right\}^{-1}\right\rangle_{E_D,y,\theta}, \quad (24)$$

$$f_2=\left\langle\left(\frac{y^3}{a}\right)\left\{\left(\mathrm{ctg}\frac{\varphi_+}{2}\sin\varphi_+N+i\delta\,\mathrm{sign}\,U_0\cos\varphi_+N\right)\right.\right.$$
$$\left.\left.\times\left(\mathrm{ctg}\frac{\varphi_-}{2}\sin\varphi_-N+i\delta\,\mathrm{sign}\,U_0\cos\varphi_-N\right)\right\}^{-1}\right\rangle_{E_D,y,\theta}, \quad (25)$$

where the angles $\varphi_\pm$ are defined by formula (20) when $\pm\omega$ are substituted into it. The factors $\cos\varphi N$ and $\mathrm{sign}\,U_0$ for an infinitely small positive δ give the rules described above for integrating around the poles. The brackets $\langle...\rangle$ denote an average over the corresponding quantities. We note that this method that we have developed enables one to calculate in addition the superconducting current in other systems of weakly coupled superconductors with low-transmittance interfaces, e.g., in superconductor-insulator-normal metal-superconductor (S–I–N–S) junctions.[14] In this case, only the functions f_1 and f_2 are different, while the form of the equation for the current is the same.

Now it is necessary to perform the average and find the optimum trajectories. In accordance with Ref. 13, and with the use of the distribution function for the random potential $\mathscr{F}(E)$, the average over the values of E_D was taken according to the formula

$$\langle f(\omega)\rangle_{E_D}=\sqrt{2\pi}\mathscr{F}(\mu)Bf(\omega). \quad (26)$$

In this formula the factor B (the width of the impurity band) appears for the following reason: f as a function of $E_D-\mu$ falls off already for values of $E_D-\mu\sim B$ (this can be seen from (25)) whereas the scale of the fluctuations $\langle(E_D-\mu)^2\rangle^{1/2}$ is large in comparison to B (see Ref. 12).

The average over y and θ are taken according to the formula

$$\langle f(\omega)\rangle_{E_D,y,\theta}=\sqrt{2\pi}\mathscr{F}(\mu)S\int\frac{dy\,d\theta}{y^3}W(y,\theta)Bf(\omega). \quad (27)$$

For the probability of formation of a trajectory we have, with exponential accuracy[6,7]

$$W(y,\theta)=\exp[N\ln(y^2\theta^2n/\alpha)-\pi nNy^3], \quad (28)$$

where n is the concentration of majority impurities in the semiconductor. Ordinarily, only the first term appears in this formula. The integrals in (27) are calculated by the method of steepest descent. Here it is convenient to substitute (27) into formulas (22) and (23) for the current and first calculate the integral over ω.

In the static case ($U_0=0$) the imaginary Matsubara frequencies (the poles of the hyperbolic tangents) at which the function $f(\omega)$ is exponentially small are important. The saddle point is determined by just this exponential and it turns out to be larger than the ordinary tunneling transmittance of the entire barrier.[6] In the nonstationary case the situation is different. We shall consider first for simplicity the symmetrical case ($-U_1=U_2=U_0/2$; $\Delta_1=\Delta_2=\Delta_0$). When the voltage U_0 is not too small and not too close to $\pm 2\Delta_0$, the main contribution to the integral over ω is given by the poles of the function I_{imp} (the important frequencies are $\omega\sim B$). Therefore the integral over ω proves to be proportional to the range of integration B and the saddle point is determined by the behavior of the function B^2W. For the saddle point in this case we have

$$\frac{\theta_0}{x_0}\ln(x_0^2\theta_0^2c)+\frac{2}{x_0\theta_0}=0,\quad 2+\frac{L}{x_0^2}\ln(x_0^2\theta_0^2c)=0, \quad (29)$$

where we introduce the dimensionless quantities: the thickness of the semiconductor layer $L=2a\alpha$, the concentration of impurities $c = n/\alpha^3$, and length $x = 2\alpha y$. By solving the system (29) it is easy to find the optimal values of x_0 and θ_0 (the resonance percolation trajectories of Lifshitz[7]). The value of cL is usually small and accordingly $\theta_0 \ll 1$, while the number of impurities $N \sim L/x$ in a trajectory is large. For the width of the impurity band we obtain

$$B=(V_0-\mu)\beta^{-1}e^{-\beta/\sqrt{2}}, \quad \beta=(L|\ln cL|)^{1/2}. \tag{30}$$

The magnitude of B increases with increasing impurity concentration, but up to the point of degeneracy B is usually small relative to Δ_0

From formula (22) (in which the f_2 term is the important one) we obtain for the superconducting current, with an accuracy up to a numerical coefficient,

$$J_S^{(a)} = eSD(\mu)\mathcal{F}(\mu)\frac{\Delta_0^2(V_0-\mu)^2}{L\beta^2}\frac{e^{-2\sqrt{2}\beta}}{|U_0^2-4\Delta_0^2|}\operatorname{th}\frac{|U_0|}{4T}, \quad |U_0\pm 2\Delta_0| \gg B. \tag{31}$$

As can be seen, the singularity at $U_0 = \pm 2\Delta_0$ arises as a result of the merging of two square roots in the symmetrical case for small ω. Formula (31) does not take into account the contribution from nonresonance electrons, which is proportional to the usual tunneling transmission e^{-2L} and is relatively small even for small impurity concentrations

$$|\ln cL| \ll L. \tag{32}$$

Near a singularity $U_0 = \pm 2\Delta_0$ a different formula for the superconducting current is valid

$$J_S^{(a)}=eSD(\mu)\mathcal{F}(\mu)\frac{\Delta_0(V_0-\mu)}{L\beta}\ln\left(\frac{B}{|U_0\pm 2\Delta_0|}\right)e^{-2\beta}\operatorname{th}\frac{\Delta_0}{2T}, \quad \ln\left(\frac{B}{|U_0\pm 2\Delta_0|}\right)\gg\beta. \tag{33}$$

This result is obtained from formula (22), where now with logarithmic accuracy the values $\omega \sim U_0 \pm 2\Delta_0$ are important in the integral over ω, and the saddle point in θ and y is found in the expression BW. As can be seen, near a singularity the Riedel logarithmic peak is preserved. Formulas (32) and (33) match up in the region $1 \ll \ln(B/|U_0 \pm 2\Delta_0|) \ll \beta$. For exponentially small voltages (32) also does not determine the value of the superconducting current, since in this case the principal contribution to the current is due to the saddle-point current, which corresponds to the critical current of the junction.

For $U_0 < 2\Delta_0$ the value of $J_s^{(b)}$ vanishes, while for $U_0 > 2\Delta_0$ it is determined by the same formulas as (32) and (33) (in the latter formula one should replace the logarithm by a constant $-\pi$).

We shall also study the singularities in the I–V characteristics for the asymmetric case, going back to the general formulas (22) and (23). In the nonresonance case (for ordinary tunneling) the frequencies $\omega \sim U_{1,2} \pm \Delta_{1,2}$ are important. This corresponds[8] to the known values of the total voltage $U_0 = \pm|\Delta_1 \pm \Delta_2|$. In the resonance case, small values of ω, $\omega \sim B$ are important, and for the appearance of singularities it is sufficient that one of the voltages be close to the corresponding value of the gap [as can be seen from (22), the singularity in this case is a square root singularity]. As a result we obtain two singularities in the current (for a positive voltage), which correspond to the chemical potential level passing through the edges of each of the band gaps in the superconductors.

$$J_S^{(a)} = SeD(\mu)\mathcal{F}(\mu)\frac{\Delta_1\Delta_2(V_0-\mu)^2}{L\beta^2} \times |(\Delta_1^2-U_1^2)(\Delta_2^2-U_2^2)|^{-1/2} \times e^{-2\sqrt{2}\beta}\left(\operatorname{th}\frac{|U_1|}{2T}+\operatorname{th}\frac{|U_2|}{2T}\right), \quad |U_{1,2}\pm\Delta_{1,2}|\gg B. \tag{34}$$

In the immediate vicinity of the singularities the superconducting current becomes constant (the singularity is "clipped"). For example, for the singularities $U_1 = -\Delta_1$ and $U_2 = \gamma\Delta_1$, we have

$$J_S^{(a)} = eSD(\mu)\mathcal{F}(\mu)\frac{\Delta_1^{1/2}\Delta_2(V_0-\mu)^{3/2}}{L\beta^{3/2}}|\Delta_2^2 - \gamma^2\Delta_1^2|^{-1/2}e^{-\sqrt{6}\beta}\left(\operatorname{th}\frac{\Delta_1}{2T}+\operatorname{th}\frac{\gamma\Delta_1}{2T}\right). \tag{35}$$

An analogous expression is obtained for the singularities $U_2 = \Delta_2$ and $U_1 = -\Delta_2/\gamma$. As can be seen, in this case the singularity is distinct from the Riedel singularity over the entire voltage range, since it is caused by a different mechanism. Formula (35) is valid when the parameter γ, which defines the voltage ratio, is not too close to Δ_2/Δ_1, specifically, $|\gamma - (\Delta_2/\Delta_1)| \gg B/(\Delta_1 + \Delta_2)$. In the contrary case the singularities begin to overlap and for $\gamma = \Delta_2/\Delta_1$ there is one singularity for a total voltage $U_0 = \Delta_1 + \Delta_2$, and which is described by the formulas (31) and (33). Thus, these formulas correspond to a more general case than the fully symmetric junction, where $\Delta_2 = \Delta_1$ and $\gamma = 1$.

4. DISCUSSION OF RESULTS

The results of this investigation show that resonance tunneling of electrons appears most strongly in the singularities of the superconducting current in superconductor-semiconductor-superconductor junctions. In the symmetric case of identical superconductors and identical Schottky barriers, resonance tunneling leads to an enhancement of the Riedel logarithmic singularity in the amplitude of the superconducting current. The superconducting current increases according to the much faster hyperbolic law [formula (31)] as the singularity is approached (Fig. 2), and the logarithmic

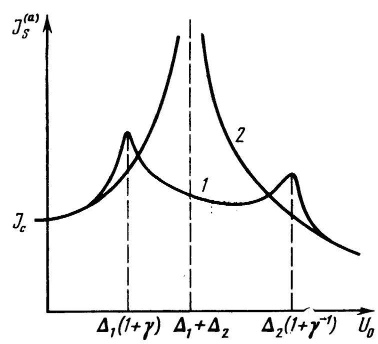

FIG. 2. Amplitude of resonance superconducting current as a function of applied voltage. 1) asymmetric case, 2) for $\gamma = \Delta_2/\Delta_1$ the singularities merge.

dependence (formula (33)) is recovered only in a narrow voltage range near the singularity.

For the resonance effect to contribute more to the superconducting current than the ordinary tunneling, a sufficiently high concentration of impurities is necessary. A comparison of the tunneling and resonance exponentials in the expression for the superconducting current shows that resonance tunneling becomes dominant for impurity concentrations defined by formula (32). Transforming to dimensional quantities, we obtain

$$n \gg a_B^{-3}(a/a_B)^{-1}\exp(-2a/a_B), \qquad a_B=(2m)^{-1/2}(V_0-\mu)^{-1/2}, \tag{36}$$

where a_B is the Bohr radius of the impurity. It can be seen that resonance transmission is preferred even at exponentially low impurity concentrations (the thickness 2a of the semiconducting layer is large compared to the Bohr radius).

Resonance tunneling is due to the presence in the semiconductor layer of special trajectories with a periodic distribution of impurities connecting the superconductors. The result is the formation of a narrow impurity band [its width B for optimum trajectories is given by formula (30)] along which coherent electrons can pass easily. Such trajectories are exponentially few in number, but the ordinary Josephson current too is proportional to a small tunneling exponential function. Therefore, in the range of concentrations defined by (36), the resonance current is greater. We note that the limitation on the concentration (36) is much weaker than in the stationary case. The stationary superconducting current always transports electrons of energy the order of T, and these electrons lose their coherence at spacings the order of the pair distance. When a voltage is applied in the resonance case, electrons with low energies $\sim B$ become important, and these are transmitted along the impurity band without attenuation. In order for the resonance effect to enhance the singularity, the impurity concentration cannot be too large. The superconducting current senses the edge of the band gap only when the width B of the impurity band is small compared to Δ_0. However, because of the exponential smallness of B, this limitation is weak. For temperatures not too close to the critical temperature and for a sufficiently thick semiconductor layer, where

$$a > a_B \ln^2[(V_0-\mu)/\Delta_0],$$

the condition on the impurity concentration is satisfied up to the point of degeneracy. Finally, we note that the calculations were carried out in the approximation $cL \gg 1$, and this condition puts a significant upper limit on the concentration:

$$n \ll a_B^{-3}(a/a_B)^{-1}. \tag{37}$$

The study of the superconducting current in the asymmetric case is of particular interest, since in this case new singularities appear. These are due to the passage of the narrow impurity band through the edges of the band gaps in the superconductors at characteristic voltages at each of the interfaces: $U_1 = \pm\Delta_1$ and $U_2 = \pm\Delta_2$. In this case the current increase first follows a square root law [formula (34)] and in the immediate vicinity of the singularity becomes constant [formula (35)]. Such singularities are produced only in superconductor-semiconductor-superconductor junctions during resonance transmission of electrons, whereas, in other weakly bound systems the singularities occur at characteristic values of the total voltage (the voltage distribution in this case is unimportant).

The values of the total voltage corresponding to the singularities are given by the expression $\Delta_{1,2}(1+\gamma^{\pm 1})$, where the quantity γ, equal to the ratio of the voltages at the interfaces, is determined by the parameters of the Schottky barriers. These values coincide with the voltage $\Delta_1 + \Delta_2$, at which the ordinary Josephson current shows a singularity only for a specific value $\gamma = \Delta_2/\Delta_1$. Consequently, in the asymmetric case, in addition to the usual singularities, there are additional resonance singularities which become the stronger when the condition (36) is satisifed. We note that the quasistatic current in this case is small, as it is proportional to the smaller (with respect to barrier transmittance) exponential.

In the calculation of the superconducting current we did not take into account the interaction among the electrons. As is known, when this interaction is taken into account, a Coulomb gap can be formed.[13] The magnitude of the gap is determined by the interaction energy of electrons separated by a distance the order of the correlation length. Therefore, in the case of weakly and strongly compensated semiconductors, as well as amorphous semiconductors (when the correlation length is large), the gap will be small. The formula for the superconducting current derived in this paper is valid when the gap is small compared to the band width B. However, in the opposite case the position of the resonance singularities is the same, while only the magnitude of the superconducting current changes. To calculate the latter it is necessary to use a more exact form of the distribution function of the random potential $\mathscr{F}(E)$.

In conclusion, let us formulate the optimum conditions for the observation of resonance singularities in superconductor-semiconductor-superconductor junctions. The thickness of the semiconductor layer should be as large as possible, but such that one can reliably measure the superconducting current through the junction. The temperature must be low, so that the order parameter Δ is of the order T. Under these conditions there is, whithin the limits of the inequalities (36) and (37), a broad range of impurity concentration for which singularities, characteristic of the resonance mechanism of electron transmission, should appear in the amplitude of the superconducting current. These singularities are perceptible directly in the I–V characteristics under conditions of a given total current if the capacitance of the junction is small,[9] or can be observed by *uhf* irradiation by the same means as the Ridel singularities.

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