Bound states of solitons in inhomogeneous Josephson junctions

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We investigate theoretically static bound states of the magnetic flux in long Josephson junctions with local inhomogeneities. We show that for a qualitative and quantitative investigation of bound states it suffices to determine them at the bifurcation points that occur upon variation of the external parameters and are analogous to phase-transition points. Outside the bifurcation points, the bound states can be obtained with the aid of a special bifurcation perturbation theory. We analyze in detail the exactly solvable problems of physical interest, and propose to use for an approximate determination of the states and of their bifurcations an elementarily solvable piecewise-linear approximation of the sine-Gordon equation with inhomogeneities. We discuss the possibility of a direct experimental observation of the bound states and their bifurcations when the external magnetic field and the inhomogeneity locations are changed.

1. INTRODUCTION

Phenomena connected with the formation and propagation of solitons in unbounded homogeneous systems have been thoroughly investigated. The prospects of observing solitons and of their eventual applications will be determined by their interactions with inhomogeneities of the internal structure and with the boundaries of real physical systems. Of particular interest from this point of view are inhomogeneous long ("one-dimensional") Josephson junctions (LJJ, see e.g., Refs. 1–4). One reason is that the soliton propagation in such junctions, known also as fluxons or Josephson vortices, is described with sufficient accuracy by a relatively simple nonlinear equation. Another reason is that the LJJ are convenient for experimental observation of solitons.[1,5] Indeed,there is an ample choice of methods of observing physical soliton effects in Josephson junctions, viz., singularities produced in the current-voltage characteristics by fluxon production, radiation produced at the edge of the junction upon reflection of a moving fluxon, the unusual dependence of the maximum current flowing through the junction on the external magnetic field, and others (see, e.g., Refs. 6–10, with additional bibliography cited in Refs. 9 and 10). In addition, a new method was developed quite recently for a direct experimental study of the static distributions of a current in an LJJ by laser scanning.[11,12] This method permits direct observation of interactions between fluxons and inhomogeneities.

The possibilities of using solitons in LJJ with inhomogeneities as computer memory and switching devices were discussed many times, and the potential advantages of such elements are well known.[2,3] However, attempts at a concrete realization of these advantages will be possible only after a detailed theoretical and experimental study is made of the physical phenomena that take place in inhomogeneous LJJ. Until recently, the main obstacles to the experimental investigation of solitons in LJJ were the technological difficulties in the production of junctions having specified properties. It is now possible not only to manufacture homogeneous junctions of length $l=(10\text{–}20)\lambda_J$, where λ_J is the Josephson depth of penetration, but also to produce in them inhomogeneities with prescribed properties.[3] Thus an experimental investigation of solitons in inhomogeneous LJJ has become possible, and a theoretical analysis of the interaction of solitons with inhomogeneities is indeed essential.

Up to now, this interaction was studied using different variants of perturbation theory in terms of the inhomogeneity.[1,2,13] In this theory the unperturbed states are taken to be moving soliton or multisoliton states in a homogeneous infinite junction. The action of a perturbation on a single soliton causes the latter to behave as a weakly deformed particle acted upon by conservative forces that depend on the inhomogeneities, as well as by friction forces that describe the energy dissipation in the junction; the action of a perturbation on multisoliton states, sometimes called soliton "packets," is described in a similar context. Such a "soliton" perturbation theory (SPT) presents a very clear physical picture of the evolving processes and can explain lucidly some of the observed phenomena.[9,10] It becomes inapplicable, however, in those cases when the soliton or the multisoliton state is strongly deformed by the interaction with the inhomogeneity. Simplest examples of this detachment of a fluxon from the edge of an LJJ when the external magnetic field is increased, and localization of a fluxon on an inhomogeneity whose size is smaller than or of the order of the size of the fluxon, i.e., $\lesssim\lambda_J$. Solution of problems of this type calls for analytic and numerical methods capable in principle of yielding exact solutions. Thus, exact static distributions of the magnetic flux were obtained for a finite LJJ in an external magnetic field,[6–8] and exact solutions for the description of fluxon motion in a finite LJJ were obtained with very simple boundary conditions.[14] Approximate solutions were also obtained recently[4] to describe the entrance of a vortex lattice into a junction on the edge of which the maximum Josephson current decreases to zero smoothly, i.e., over a length $\gg\lambda_J$.

The problem of determining the static states of an LJJ with local inhomogeneities was posed in Refs. 15 and 16. The exact solution of this problem can be expressed in terms of elliptic functions, but the investigation of the properties of the exact formal solution is not a trivial task, since the un-

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known parameters that determine the state satisfy very complicated transcendental equations. An analysis of certain particular solutions has enabled us to reveal two new phenomena—formation of localized bound states on attracting microinhomogeneities, corresponding to a local decrease of the maximum Josephson current, and the existence of nontrivial bifurcations of these states when the external magnetic field is varied[15,16] (we recall that a bifurcation is defined as the generation or vanishing of solutions at certain critical values of the external parameters; if the parameters do not go through critical values, the number of solutions is conserved).

The present paper is devoted to a systematic study of static bound states in LJJ with local inhomogeneities, and particularly to their bifurcations. We show that for a qualitative and quantitative description of all the bound states it suffices to find the solutions at the bifurcation points, i.e., the critical states. This makes it possible to expand any state in powers of the deviation of the parameters from the bifurcation values; the terms of this series are determined recurrently and are explicitly expressed in terms of functions that characterize the critical states. This theory obviously differs in principle from the soliton perturbation theory, and we shall name it bifurcation perturbation theory (BPT). For a detailed quantitative study of the bound states and their bifurcations we propose a simple piecewise-linear model that can be solved in terms of elementary functions and provides good approximations for the exact solutions. We consider in detail the exact and approximate solutions of the simplest nontrivial problems, viz., an infinite junction with one and two inhomogeneities, and a semi-infinite junction with an inhomogeneity and a magnetic field at the edge. We discuss the possibilities of direct experimental observation of our predictions of the bound states of the fluxons and their bifurcations.

2. STATIC STATES AND THEIR BIFURCATIONS. BIFURCATION PERTURBATION THEORY

We consider the junction shown schematically in Fig. 1. In the general case the width w of the junction and the thickness d of the dielectric layer between the upper and lower superconductors depend on x. We regard the junction as one-dimensional, i.e., its dimensions must satisfy the conditions $w \ll \lambda_J \ll 1$. All the quantities depend here only on x, and the magnetic and electric fields are directed along the x and z axes, respectively. We know[1,3,5] that the propagation of electromagnetic waves, and particularly of solitons, in such an

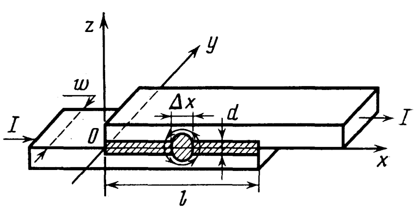

FIG. 1.

LJJ can be described by the equation ($\varphi_x = \partial\varphi/\partial x$, $\varphi_t = \partial\varphi/\partial t$).

$$L(L^{-1}\varphi_x)_x - LC\varphi_{tt} - \lambda_J^{-2}(x)\sin\varphi(x,\ t) = \bar{\alpha}\varphi_t, \qquad 0 \leqslant x \leqslant l. \tag{2.1}$$

Here $\varphi = 2\pi\Phi(x,t)/\Phi_0$, $\Phi(x,t)$ is the magnetic flux through the segment $(0,\ x)$, Φ_0 is the magnetic flux quantum, $\lambda_J^{-2}(x) = 2e\hbar^{-1}LI_c$; L, C, I_c are respectively the inductance, capacitance, and critical Josephson current per unit junction length. The right-hand side of the equation describes dissipative effects connected with the tunneling of normal electrons through the insulating layer. The junction geometry can be chosen such that the inductance is constant and $\lambda_J(x)$ changes noticeably only over small sections of size $\Delta x < \lambda_J$, where λ_J is the constant value of the Josephson length on the homogeneous segments. In fact, the tunnel current I_c decreases exponentially with increasing barrier thickness d, and $L = \mu_0(2\lambda_L + d)w^{-1}$, where λ_L is the London depth of penetration. Since $d \ll \lambda_L \ll \lambda_J$, we can, by increasing d on the segment Δx, decrease I_c to zero while maintaining the inductance practically constant (see Fig. 1). We choose the units of length and time to be the quantities λ_J and $\omega_J^{-1} = \lambda_J(L_0C_0)^{1/2}$, where L_0, C_0, and I_0 are the values of L, C, and I_c on the homogeneous intervals, and represent approximately the changes of the critical Josephson current at the inhomogeneities with the aid of δ functions. Equation (2.1) can then be written in the form ($x_0 = 0 < \ldots < x_i < x_{i+1} < \ldots < x_{n+1} = l$)

$$\varphi_{xx} - \varphi_{tt} - \left[1 - \sum_{i=1}^{n} \mu_i \delta(x - x_i)\right] \sin\varphi(x,t) = \alpha\varphi_t, \quad 0 \leqslant x \leqslant l. \tag{2.2}$$

Since we shall be interested in the static distributions of $\varphi(x)$, we neglect the fact that the damping $\alpha = \bar{\alpha}\lambda_J^2\omega_J$ and the signal propagation velocity $(LC)^{-1/2}$ can also depend on x. In addition, to describe a real junction it would be necessary to introduce in Eqs. (2.1) and (2.2) terms proportional to φ_{xxt} and to $\varphi_t \cos\varphi$ (Ref. 1). We disregard these terms, too, since they do not influence the static state and do not alter the qualitative picture of the temporal evolution of the states. At any rate, (2.2) can be regarded as a zeroth-approximation equation, and all the corrections, including those for the finite width of the inhomogeneity, can be regarded as a perturbation.

The energy concentrated inside the junction in a layer of thickness $2\lambda_L + d$ is the sum of the energies of the electromagnetic field and of the Josephson currents. The units for the electric and magnetic fields can be chosen such that $e(x,t) = \varphi_t$ and $h(x,t) = \varphi_x$. Then, expressing the energy in units of $L_0I_0^2$, we represent in standard fashion

$$\mathscr{E} = \int_0^l dx \left\{ \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\varphi'^2 + \left[1 - \sum \mu_i\delta(x - x_i)\right](1 - \cos\varphi) \right\}, \tag{2.3}$$

where $\dot{\varphi} = \varphi_t$, $\varphi' = \varphi_x$. From (2.2) it follows that

$$\dot{\mathscr{E}} = \dot{\varphi}_l\varphi_l' - \dot{\varphi}_0\varphi_0' - \int dx\,\alpha\dot{\varphi}^2. \tag{2.4}$$

Here and elsewhere we use the notation $\varphi i = \varphi(x_t,\ t)$,

$\varphi_i' = \varphi'(x_i, t)$. Equation (2.2) must be supplemented with boundary conditions at $x = 0$ and $x = l$. For an LJJ it is natural to specify the boundary values of the magnetic field

$$\varphi'(0, t) \equiv \varphi_0' = h_0, \quad \varphi'(l, t) \equiv \varphi_l' = h_l, \tag{2.5}$$

and we shall regard h_0 and h_l as external parameters (see Refs. 1, 5–8, and 10 concerning various realizations of the boundary conditions). Using the expression for the energy we can easily construct, in standard fashion, the Lagrangian and the dissipative function that lead to Eqs. (2.2). However, to obtain from the variational principle also the boundary conditions, it is necessary to choose another expression for the energy (Hamiltonian), in which account is taken of the energy of the external sources needed to maintain the field at the edges of the junction. In accord with the general prescription[17] we choose as the generalized Hamiltonian

$$\mathcal{J} = \mathcal{E} + h_0\varphi_0 - h_l\varphi_l. \tag{2.6}$$

It is easy to verify that

$$\dot{\mathcal{J}} = \dot{h}_0\varphi_0 - \dot{h}_l\varphi_l - \int dx\alpha\dot{\varphi}^2, \tag{2.7}$$

and at constant values of h_0 and h_l the change of $\mathcal{J}$ is determined only by the dissipative terms. If $h_0 = h_l = h_{ex}$, the increment to the energy in (2.6) is of the form $-(\varphi_l - \varphi_0)h_{ex}$, where $\varphi_l - \varphi_0 = 2\pi\Phi/\Phi_0$ is proportional to the total magnetization of the junction. Thus, the relation between $\mathcal{J}$ and $\mathcal{E}$ is similar to the relation between the thermodynamic potential and the free energy of the system in a magnetic field. Even though at $h_0 = h_l$ the generalized Hamiltonian $\mathcal{J}$ has a different physical meaning from the "genuine" thermodynamic potential used to describe the thermodynamics of a homogeneous LJJ in a magnetic field,[5] the analogy employed is useful.

In particular, the equation for the static states of the LJJ

$$\varphi''(x) = \left[1 - \sum \mu_i\delta(x - x_i)\right]\sin\varphi(x), \quad 0 \leqslant x \leqslant l, \tag{2.8}$$

and the boundary conditions (2.5) follows from the variational principle $\delta\mathcal{J}\{\varphi\} = 0$, and the requirement that the second variation be positive-definite, $\delta^2\mathcal{J}\{\varphi\} > 0$, is known[18,19] to be equivalent to the requirement that the boundary-value problem

$$-\psi''(x) + \left[1 - \sum \mu_i\delta(x - x_i)\right]\psi(x)\cos\varphi(x) = \omega^2\psi(x),$$
$$\psi'(0) = \psi'(l) = 0, \tag{2.9}$$

be positive definite, where $\varphi(x)$ is the solution of the problem $\delta\mathcal{J} = 0$. The same condition can be obtained by substituting in (2.2) and in the boundary condition (2.5) the perturbed solution

$$\varphi(x, t) = \varphi(x) + e^{-\alpha t/2}\sum_n \psi_n(x)\left(a_n e^{-i\bar{\omega}_n t} + a_n^* e^{i\bar{\omega}_n t}\right) + \ldots, \tag{2.10}$$

where $|a_n|, |a_n^*| \ll 1$ and $\delta\mathcal{J}\{\varphi\} = 0$. In the linear approximation we find that ψ_n are the eigenfunctions of the boundary-value problem (2.9) with eigenvalues $\omega_n^2 = \bar{\omega}_n^2 + \alpha^2/4$. It can be seen from (2.10) that the perturbed solution will not increase, i.e., will be stable, only if the lowest eigenvalue ω_0^2 is positive. In addition, the expansion (2.10) describes the time evolution of any small perturbation; it suffices to express a_n and a_n^* in terms of $\varphi(x, 0)$, $\varphi_t(x, 0)$, and the known eigenfunctions ψ_n. If $\omega_0^2 > 0$ and $4\omega_0^2\alpha^{-2} \gg 1$, then ω_0 determines the frequency of the response of the system in the state $\varphi(x)$ to an arbitrary small perturbation. If $\omega_0^2 < 0$, the state is unstable, but at $|\bar{\omega}_0^2| = |\omega_0^2| + 1/4\alpha^2 \ll 1$ one can speak of its "lifetime" $\Gamma^{-1} \sim (|\bar{\omega}_0| - \alpha/2)^{-1}$. Thus, for a complete description of the static state it is necessary to find also its natural frequencies ω_n.

When the parameters $p = (h_0, h_l, x_i, \mu_i)$ are varied, the states $\varphi(x) \equiv \varphi(x; p)$ themselves and the frequencies $\omega_n \equiv \omega_n(p)$ vary continuously, but their number changes jumpwise when one of the frequencies goes through zero.[1)] The points $p = p_c$ in the space of the parameters in which one of the eigenvalues $n(p)$ vanishes will henceforth be identified with the bifurcation points, and the surface defined by the equation $\omega_n(p) = 0$ will be called the bifurcation surface of the catastrophe surface of the family of states $\varphi(x; p)$. Knowledge of the bifurcation surfaces makes it possible to obtain a qualitative idea of the LJJ states, and near these surfaces we can construct simple expansions of the solution $\varphi(x; p)$ itself, of its energy $\mathcal{J}(p)$, and of its natural frequencies $\omega_n(p)$ in powers of $p - p_c$. Although such a bifurcation perturbation theory is quite general, we shall not dwell on the general procedure, and construct the needed expansion in terms of the deviation of the boundary magnetic field h_0 from the critical value h_c at which a certain natural frequency vanishes, e.g., $\omega_0(h_c) = 0$. In this case $\varphi(x; h_0)$, $\mathcal{J}(h_0)$ and $\omega^2(h_0)$ can be expanded in powers of the parameter $\varepsilon = (h_0 - h_c)^{1/2}$, and this corresponds to creation or annihilation of two solutions at the bifurcation point (cf. Refs. 20–23). Representing $\varphi(x; h_0)$ in the form

$$\varphi(x; h_0) = \varphi_c(x) + \sum_{n=1}^{\infty} \varepsilon^n\chi_n(x), \tag{2.11}$$

where $\varphi_c(x) = \varphi(x; h_c)$, we can find the functions $\chi_n(x)$ by substituting (2.11) in (2.8) and (2.5). To simplify the notation we put $h_l = 0$ and introduce for the energy of the Josephson currents the notation

$$\left[1 - \sum \mu_i\delta(x - x_i)\right](1 - \cos\varphi) = V(\varphi; x), \tag{2.12}$$

which emphasizes the general character of the formulas that follow. Equation (2.8) can then be represented in the form $\varphi'' = V^{(1)}$, where $V^{(n)} = \partial^n V/\partial\varphi^n$, and (2.9) takes the form

$$\hat{K}\psi \equiv -\psi'' + V^{(2)}\psi = \omega^2\psi. \tag{2.9a}$$

It is easy to verify that the functions χ_n satisfy the equations

$$\hat{K}\chi_1 = 0, \quad \hat{K}\chi_n = f_n(V_c^{(3)}, \ldots, V_c^{(n+1)}; \chi_1, \ldots, \chi_{n-1}), \quad n \geqslant 2, \tag{2.13}$$

where $V_c^{(n)} = V^{(n)}(\varphi_c; x)$, as well as the boundary conditions

$$\chi_n'(0) = 0, \quad n \neq 2; \quad \chi_2'(0) = 1; \quad \chi_n'(l) = 0. \tag{2.14}$$

We note that the functions f_n can be easily calculated, e.g.,

$$f_2 = -{}^1/_2 V_c^{(3)}\chi_1^2, \quad f_3 = -(V_c^{(3)}\chi_1\chi_2 + {}^1/_6 V_c^{(4)}\chi_1^3).$$

Since (2.9) has a solution at $\omega^2 = 0$, there exists for the equation $\hat{K}\chi_1 = 0$ a solution that satisfies the conditions

$\chi_1'(0)=\chi_1'(l)=0$ and is definite apart from the normalization. Choosing the second solution $\bar{\chi}_1$ of this equation with normalization $\chi_1'(0)=1$, we can write the solutions of Eqs. (2.13) at $n\geqslant 2$ in the form

$$\chi_n(x)=c_n\chi_1(x)-\chi_1^{-1}(0)\times\int_x^l dy[\chi_1(x)\bar{\chi}_1(y)-\bar{\chi}_1(x)\chi_1(y)]f_n(y). \qquad (2.15)$$

The condition $\chi_n'(l)=0$ holds for this equation, and the conditions $\chi_2'(0)=1$, $\chi_n'(0)=0$, $n>2$, enable us to fix the normalization of χ_1 and obtain in succession the unknown constants c_n. From the condition $\chi_2'(0)=1$ follows the relation

$$\chi_1(0)=\int_0^l dx f_2(x)\chi_1(x)\equiv\langle\chi_1 f_2\rangle=-\frac{1}{2}\langle V_c^{(3)}\chi_1^3\rangle, \qquad (2.16)$$

which fixes the real functions $\chi_1(x)$ apart from the sign. The two choices of the sign of $\chi_1(0)$ correspond to the two solutions that appear at the bifurcation point and merge as $\varepsilon\to 0$.

It is now easy to get, with the aid of the obtained expansion of φ, the series for ω^2 and $\mathscr{J}$ in terms of ε. To obtain the expansion of ω^2 we represent ψ, $V^{(2)}$ and ω^2 in (2.9a) in the form of series in powers of ε:

$$\varphi=\sum_{n=0}^{\infty}\varepsilon^n\psi_{(n)},\quad \omega^2=\sum_{n=1}^{\infty}\varepsilon^n\omega_{(n)}^2,$$

$$V^{(2)}(\varphi;x)=V_c+\varepsilon V_c^{(3)}\chi_1+\dots.$$

It is easy to obtain for $\psi_{(n)}$ integral representations that are perfectly analogous to (2.15), and from the conditions $\psi_{(n)}'(0)=0$ we can find in succession $\omega_{(n)}^2$. In first-order approximation we get

$$\omega^2=-2\varepsilon\chi_1(0)\langle\chi_1^2\rangle^{-1}, \qquad (2.17)$$

so that the solution, for which $\chi_1(0)>0$, is always unstable.

An expansion for $\mathscr{J}$ can be obtained by substituting (2.11) in (2.6) and using (2.13) and (2.14). It is simpler, however, to use the general properties of the function $\mathscr{J}(h_0, h_l)$, which we rewrite, taking the notation (2.12) into account, in the form

$$\mathscr{J}=\int_0^l dx\left[\frac{1}{2}\varphi'^2+V(\varphi;x)\right]+h_0\varphi_0-h_l\varphi_l. \qquad (2.18)$$

Differentiating this expression with respect to h_0 (or h_l), with account taken of the equation $\varphi''=V^{(1)}$ and of the boundary conditions, we easily find that

$$\partial\mathscr{J}/\partial h_0=\varphi_0,\quad \partial\mathscr{J}/\partial h_l=-\varphi_l. \qquad (2.19)$$

If $h_0=h_l=h_{ex}$, then $\partial\mathscr{J}/\partial h_{ex}=-(\varphi_l-\varphi_0)$, in full accord with the thermodynamic analogy. With the aid of (2.19) and (2.11) we now obtain

$$\mathscr{J}=\mathscr{J}_c+\int_{h_0}^{h_c} dh_0\varphi_0=\mathscr{J}_c+\varepsilon^2\varphi_c(0)+\frac{2}{3}\varepsilon^{3/2}\chi_1(0). \qquad (2.20)$$

We see hence that the derivative $\partial\mathscr{J}/\partial h_0$ is continuous, while $\partial^2\mathscr{J}/\partial h_0^2$ becomes infinite as $h_0\to h_c$, so that this bifurcation point is analogous to a second-order phase-transition point. Comparison of (2.20) and (2.17) shows that the state with higher energy ($\chi_1(0)>0$) is always unstable, inasmuch as $\omega_2<0$ for this state. Moreover, as $h_0\to h_c$ we have $\partial^2\mathscr{J}/\partial h_0^2<0$ for the state with lower energy and $\partial^2\mathscr{J}/\partial h_0^2>0$ for the state with higher energy.

For a qualitative analysis of the energy spectrum of the bound states and for an approximate estimate of their energies, a more useful property is that the function $\mathscr{J}(h_0, h_l)$ is convex in all of its branches corresponding to stable states. To prove this it suffices to recall that $\delta\mathscr{J}=0$, $\delta^2\mathscr{J}>0$ for any variation of the function φ, and in particular for $\delta\varphi=\rho_0\partial\varphi/\partial h_0+\rho_l\partial\varphi/\partial h_l$, where ρ_0 and ρ_1 are independent infinitely small parameters. With the aid of the equation and the boundary conditions for φ we easily find that

$$\mathscr{J}\{\varphi+\delta\varphi\}-\mathscr{J}\{\varphi\}=-\frac{\rho_0^2}{2}\frac{\partial\varphi_0}{\partial h_0}+\frac{\rho_l^2}{2}\frac{\partial\varphi_l}{\partial h_l}+\frac{\rho_0\rho_l}{2}\left(\frac{\partial\varphi_0}{\partial h_l}-\frac{\partial\varphi_l}{\partial h_0}\right)+\dots. \qquad (2.21)$$

Since this expression is positive, we have[2)]

$$\partial^2\mathscr{J}/\partial h_0^2=\partial\varphi_0/\partial h_0<0,\quad \partial^2\mathscr{J}/\partial h_l^2=\partial\varphi_l/\partial h_l>0. \qquad (2.22)$$

According to (2.19),

$$\partial\varphi_0/\partial h_l=\partial^2\mathscr{J}/\partial h_0\partial h_l=-\partial\varphi_l/\partial h_0. \qquad (2.23)$$

Using this condition, we can easily show that (2.21) and (2.22) lead to the inequality

$$(\partial\varphi_0/\partial h_l)^2=(\partial\varphi_l/\partial h_0)^2<-(\partial\varphi_0/\partial h_0)(\partial\varphi_l/\partial h_l). \qquad (2.24)$$

When (2.22) and (2.23) are taken into account, this demonstrates in fact that $\mathscr{J}(h_0, h_l)$ is convex.

As noted above, the bifurcation perturbation theory has an entirely different character. In Ref. 24, for example (see also Ref. 19), analogous expansions were used to analyze Abrikosov vortices. It can be stated that the BPT is a generalization of expansions in the order parameter near phase transition points. The analysis performed shows that for a qualitative understanding of the arrangement of the bound states in an inhomogeneous system it suffices to determine the bifurcation points and the corresponding state φ_c. We can then use the BPT for the quantitative calculations.

All the foregoing is obviously valid not only for inhomogeneous LJJ, but also for any system described by a scalar field $\varphi(x, t)$ with a potential $V(\varphi; x)$ satisfying natural physical restrictions (cf. Refs. 18, 19, 22, and 23). The remaining analysis pertains to LJJ with inhomogeneities, i.e., the potential is defined by relation (2.12).

3. BOUND STATES AND BIFURCATIONS IN LJJ WITH MICROINHOMOGENEITIES

It is clear from the foregoing that the qualitative structure of the static states is the following. Given the values of the magnetic field at the edges, there exist usually several states with different values and distributions of the magnetic flux and with different energies $\mathscr{J}$. Some of these states may turn out to be locally stable, and in principle more than one stable state can exist for given values of the edge fields. It will be shown below that this situaton, which is of general-physics and applied interest, is realized already for a semi-infinite LJJ with one microinhomogeneity and lends itself to a complete investigation.

The solutions of Eq. (2.8) with boundary conditions (2.5) are easiest to obtain using the first integrals on the homogeneous sections

$$\varphi'^2/4=k_i^2-\cos^2(\varphi/2), \quad x_i\leqslant x\leqslant x_{i+1}, \tag{3.1}$$

where k_i are integration constants. It follows from (3.1) that the general solution on any homogeneous interval can be written in the form

$$\cos(\varphi/2)=-k_i \operatorname{sn}(x+l_i|k_i), \tag{3.2}$$

where sn is the Jacobi elliptic sine[25] and l_i are constants. The unknowns k_i, l_i ($i=0, 1, \ldots, n$) are determined from the n conditions for the continuity of $\varphi(x)$ at the points x_1, from the n conditions for the jump of the magnetic field

$$\varphi'(x_i+0)-\varphi'(x_i-0)=-\mu_i \sin\varphi(x_i) \tag{3.3}$$

and from the two boundary conditions (2.5).

An equation for the states of the LJJ can be obtianed in the following manner. At any given φ_0 the value of k_0^2 is determined from (3.1):

$$k_0^2=c_0^2+h_0^2/4, \tag{3.4}$$

where we use the notation $c_i=\cos(\varphi_i/2)$, $s_i=\sin(\varphi_i/2)$, $i=0, 1, \ldots$, and $n+1$. From the known φ_i and k_i^2 we can determine φ_{i+1} and k_{i+l}^2. Indeed, it follows from (3.2) that[25]

$$x_{i+1}-x_i=F(\operatorname{Arcsin}(c_i/k_i)|k_i)-F(\operatorname{Arcsin}(c_{i+1}/k_i)|k_i), \tag{3.5}$$

where F is an elliptic integral of the first kind. From (3.5) we can find c_{i+1} from the known c_i and k_i. At $k_i>1$ it is more convenient to use the relation

$$x_{i+1}-x_i=k_i^{-1}[F((\pi-\varphi_i)/2|k_i^{-1})-F((\pi-\varphi_{i+1})/2|k_i^{-1})], \tag{3.5a}$$

which is obtained from (3.5) by the known[25] transformation of the function F. Equation (3.5a) can be obtained also directly from (3.1):

$$x_{i+1}-x_i=\int_{x_i}^{x_{i+1}} dx=\int_{\varphi_i}^{\varphi_{i+1}} d\varphi[\varphi'(\varphi)]^{-1}$$

$$=k_i^{-1}\int_{c_i}^{c_{i+1}} dc[(1-c^2)(1-k_i^{-2}c^2)]^{-1/2}, \tag{3.6}$$

which is equal to (3.5a) in accord with the definition of F (Ref. 25). Representation of $x_{i+1}-x_i$ in the form of an integral with respect to c is usually more convenient for numerical calculations and for a qualitative analysis, and will be frequently used below. The value of k_{i+1}^2 is now determined from the conditions (3.1) and (3.3)

$$(k_{i+1}^2-c_{i+1}^2)^{1/2}=(k_i^2-c_{i+1}^2)^{1/2}-\mu_i s_{i+1}c_{i+1}, \tag{3.7}$$

where the square root in the right-hand side must be understood in its algebraic sense. It is easy in practice to track the sign reversals.

Using relations (3.4), (3.5), and (3.7) we can ultimately express k_n^2 and φ_{n+1} in terms of the free parameter φ_0. Substituting these values in the boundary condition

$$k_n^2-c_{n+1}^2=h_l^2/4, \tag{3.8}$$

we obtain for φ_0 an equation that yields all the possible states. It is convenient to plot them on a diagram in the (φ, φ') plane, Fig. 2. Corresponding to each state is a piecewise-continuous curve that starts at (φ_0, h_0), ends at (φ_l, h_c) and is made up of pieces described by Eqs. (3.1). The values of φ_i should satisfy relations (3.5), and the values of k_i are determined from (3.7)

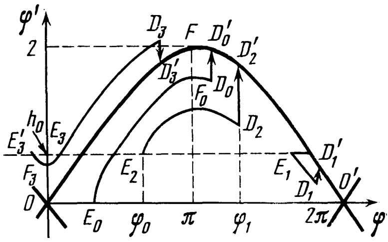

FIG. 2.

Let us investigate the simplest bound states. For an infinite junction with one microinhomogeneity ($x_0=-\infty$, $x_1=0$, $x_2=+\infty$) there exist only two static solutions corresponding to a fluxon or an antifluxon localized in the inhomogeneity. In fact, the energy can be finite only under the conditions $\cos\varphi(\pm\infty)=1$, $\varphi'(\pm\infty)=0$ i.e., $(\varphi(\pm\infty), \varphi'(\pm\infty))=(2\pi N_\pm, 0)$, where $N_\pm$ are integers. Since only lines corresponding to $k_0=k_1=1$ enter these points, we should have $N_+-N_-=\pm1$, where we can choose $N_-=0$ and $N_+=1$ or $N_-=1$ and $N_+=0$; the solutions obtained by making the shift $\varphi\to\varphi+2\pi N$ are physically identical. The first solution corresponds in Fig. 2 to the curve OFO', and the second to its mirror image relative to the φ axis. It follows from the condition $k_0=k_1=1$ that the jump of φ' at the point $x_1=0$ is equal to zero, and according to (3.3) we have $\varphi_1=\pi$. From (3.2) we easily find that

$$\varphi(x)=2\arccos(\mp \operatorname{th} x)=4\operatorname{arc\,tg} e^{\pm x}, \tag{3.9}$$

where the upper sign pertains to the bound fluxon. The beam distribution (3.9) coincides with the known distribution for a free fluxon at rest in a homogeneous junction,[1] but its energy, as can be easily found from (2.3), is equal to $\mathscr{E}=8-2\mu_1$. At $\mu_1>0$ this is less than the minimum energy $\mathscr{E}_0=8$ of a free fluxon in a homogeneous junction, and it is this which allows us to speak of a bound state and of attraction of a fluxon by its inhomogeneity. The state (3.9) exists at any value of μ_1, including also at $\mu_1<0$, but in the latter case it is unstable. Indeed, for the distribution (3.9) we have $\cos\varphi=1-2\operatorname{sech}^2 x$ and Eq. (2.9) can be solved exactly by reducing it, using the substitution $\psi=\{[\exp(-p|x|)]v\}\times(\tanh|x|)$ to the Jacobi equation for the function v. A simple investigation yields the single eigenvalue belonging to the discrete spectrum and corresponding to the eigenfunction

$$\omega_0^2=\frac{1}{2}\mu_1[(1+(\mu_1/4)^2)^{1/2}-(\mu_1/4)],$$

$$\psi_0=e^{-p_0|x|}(p_0+\operatorname{th}|x|), \tag{3.10}$$

where $p_0^2=1-\omega_0^2$. At $\omega^2\geqslant1$ the spectrum is continuous. It can be seen from (3.0) that the state (3.9) is stable at $\mu_1>0$ and unstable at $\mu_1<0$, i.e., $\mu_1=0$ is the bifurcation point of the bound states.

We note that new bound states can set in at large values of $|\mu_i|$. For example, if $|\mu_1|\geqslant 2$, a state of the "fluxon-antifluxon" type can occur, at which the sign of the magnetic field is jumpwise reversed at the point x_1. We consider hereafter only attracting inhomogeneities, for which $0<\mu_i<1$. The restriction $\mu_i<1$ stems from the fact that a real inhomogeneity is produced by a decrease of the critical Josephson current on a segment of finite length Δx (see Fig. 1). Approximation of this current by the expression $(1-\mu_i\delta(x-x_i))$ calls for satisfaction of the inequality $\Delta x-\mu_i\geqslant 0$. The δ-function approximation is meaningful only at sufficiently small Δx, smaller than the fluxon dimension, and hence the limit on μ_i. A large inhomogeneity can, of course be approximated by several closely lying local inhomogeneities.

A more abundant set of bound states is produced in an infinite junction with two inhomogeneities. We consider for simplicity the case $\mu_1=\mu_2=\mu$ and put $x_0=-\infty$, $x_1=-\Delta$, $x_2=\Delta$, $x_3=+\infty$. Just as in the preceding case $(\varphi(\pm\infty), \varphi'(\pm\infty))=(2\pi N_\pm, 0)$, but now the possible values of $|N_+-N_-|$ depend on Δ. Using the symmetry of the problem under the reflection $x\to -x$, we can reduce its solution to an analysis of four cases: 1) $\varphi(0)=\pi$, $\varphi(+\infty)=2\pi$, $\varphi(-\infty)=0$, i.e., $\Phi=\Phi_0$; 2) $\varphi(0)=0$, $\varphi(\pm\infty)=\pm 2\pi N$, $N=1, 2,\ldots$, i. e., $\Phi=2N\Phi_0$; 3) $\varphi(0)=-\pi$, $\varphi(+\infty)=2\pi N$, $\varphi(-\infty)=-2\pi(N+1)$, i.e., $\Phi=(2N+1)\Phi_0$; 4) $\varphi(0)=2\arccos k_1$, $\varphi(\pm\infty)=2\pi$, $\Phi=0$. The remaining solutions can be obtained by making the substitution $\varphi'\to-\varphi'$ and by a physically immaterial shift of φ. It suffices to obtain the solution on the semi-infiite interval $0\leqslant x<+\infty$ at a known value of $\varphi(0)$. In Fig. 2 the solution of the first type corresponds to the curve $F_0D_0D_0'O'$, the solution of the second type at $N=1$ to the curve $F_3D_3D_3'O'$, the solution of the fourth type to $E_0D_0D_0'O'$, etc. The value of k_1^2 is determined from (3.7), i.e.,

$$k_1^2-1=\mu_2c_2(1-c_2^2)(2+\mu_2c_2). \tag{3.11}$$

When c_2 varies in the range $-1\leqslant c_2\leqslant +1$, the parameter k_1^2 takes on values in the interval $k_-^2\leqslant k_1^2\leqslant k_+^2$, where k_-^2 and k_+^2 are respectively the minimum and the maximum of the function $k_1^2(c_2)$. At $\mu_2<1$ we can use the approximation

$$k_\pm^2=k_1^2(c_\pm)=1\pm\frac{4\mu_2}{3\sqrt{3}}\left(1\pm\frac{\mu_2}{2\sqrt{3}}+\ldots\right),$$
$$c_\pm=\pm\left(\frac{1}{\sqrt{3}}\pm\frac{\mu_2}{18}+\ldots\right). \tag{3.12}$$

Instead of solving (3.5), it is convenient to simply specify a value of c_2, determine from (3.11) the value of k_1^2, and use (3.5) or (3.6) to calculate $\Delta=(x_2-x_1)/2$. The extremal values of Δ correspond to the bifurcation points. It is thus easy to verify that states of the first type and states of the second type with a flux $2\Phi_0$ exist at all Δ, whereas the other states are produced only at a sufficiently large value[3] of Δ. For example, the bound state of a fluxon and antifluxon, i.e., solutions of the fourth type (see curve $E_0D_0D_0'O'$ in Fig. 2) cannot exist if $\Delta<K(k_-)\equiv F(\pi/2|k_-)$. In fact,

$$\Delta=F(\pi/2|k_1)-F(\arcsin(c_2/k_1)|k_1)>K(k_1)\geqslant K(k_-), \tag{3.13}$$

and the exact value of the minimum of Δ, which depends on μ, can be easily obtained by numerical means.

The natural frequencies of the investigated states are easiest to calculate in the approximation described in the next section, although at $\varphi(\Delta)\sim\pi N$ it is possible to solve also the exact equations. The general structure of the bound states on two inhomogeneities is simple enough, and we leave out the details. The problem of three and more inhomogeneities is substantially more complicated and is worthy of a special investigation.

For a semi-infinite junction $(x_0=0, x_1>0, x_2=+\infty)$ with a given value of h_0 it is also possible to obtain a complete solution of the problem of bound static states on the basis of exact equations. Such a junction is physically realized if $x_2=l\gg 1$, $l-x_1\gg x_1$ and $h_l=0$. On the right end we put $\varphi(+\infty)=2\pi$. Although Eq. (2.8) and the boundary condition (2.5) are invariant to the shift $\varphi\to\varphi+2\pi N$, the expression (2.6) for the energy is not invariant, $\mathscr{J}\to\mathscr{J}+2\pi Nh_0$. This is why it is necessary to fix the value $\varphi(+\infty)=2\pi N_+$. The observed quantities (energy differences at a given h_0, magnetic fields, currents) do not depend on the choice of N_+, and are continuous functions of h_0 at a given N_+.

The problem posed can be solved in the same manner as the problem of two inhomogeneities in an infinite junction. Obviously $k_1=1$ and k_0 is determined from the specified value of c_1 by Eq. (3.11), in which we must replace, k_1, c_2, and μ_2 by k_0, c_1, and μ_1. The values of c_0 are obtained from (3.4), after which we calculate x_1 from (3.5) or (3.6). Typical states are shown in Fig. 2, viz., the curves $E_1D_1D_1'O'$, $E_2D_2D_2'O'$, $E_3D_3D_3'O'$, $E_3'D_3D_3'O'$. At $h_0=0$ and at small values of x_1 there exists only one stable state $\varphi(x)=2\pi$. It follows from (3.13) in which $\Delta, k_1, c_2\to x_1, k_0, c_1$, that at $x_1>K(k_-)$ there can appear in a zero field one more stable bound state corresponding to the curve $E_0D_0D_0'O'$.

The solutions with $|k_0^2-1|\ll 1$, particularly all solutions at $\mu_1\ll 1$, can be expressed in terms of elementary functions by expanding the integral in (3.6) in powers of (k_0^2-1). In the general case they are represented by rather cumbersome formulas and we present the results only for states with small $|s_0|$ and $|s_1|$ as $h_0\to 0$. If $|2\pi-\varphi_1|<|2\pi-\varphi_0|$, we have

$$s_0={}^1/_2h_0\,\mathrm{cth}\,(x_1+\nu_-)+\ldots,$$
$$s_1={}^1/_2h_0\,\mathrm{ch}\,(\nu_-)/\mathrm{sh}\,(x_1+\nu_-)+\ldots; \tag{3.14}$$
$$\mathscr{J}=2\pi h_0-{}^1/_2h_0^2\,\mathrm{cth}\,(x_1+\nu_-)+o(h_0^2),$$

and at $|\varphi_0|$, $|\varphi_1|\ll 1$

$$s_0=-{}^1/_2h_0\,\mathrm{th}\,(x_1-\nu_+)+\ldots,$$
$$s_1={}^1/_2h_0\,\mathrm{sh}\,(\nu_+)/\mathrm{ch}\,(x_1-\nu_+)+\ldots; \tag{3.15}$$
$$\mathscr{J}=8-{}^1/_2h_0^2\,\mathrm{th}\,(x_1-\nu_+)+o(h_0^2),$$

where $\nu_\pm=1/2\ln(\pm 1+2/\mu)$. We recall that to calculate $\mathscr{J}$ we need not know $\varphi(x)$, and it suffices to replace with the aid of (3.1) the integration with respect to x by integration with respect to $\cos(\varphi/2)$ or $\sin(\varphi/2)$ [cf. (3.6)], after which (3.14) and (3.15) are easily obtained.

Simple and useful analytic expressions exist also for states similar to a bound fluxon in an infinite junction, i.e., at $|c_1| \ll 1$, $|k_0^2 - 1| \ll 1$.. With the aid of (3.4), (3.6), and (3.11) it is easy to calculate expansions of c_0 and h_0 in powers of c_1. If we stipulate $dh_0/dc_1 = 0$ at $c_1 = 0$, we can obtain a bifurcation point whose all characteristics can be expressed in terms of elementary functions. The condition $dh_0/dc_1 = 0$ at $c_1 = 0$ yields the relation between μ_1 and x_1:

$$\tfrac{1}{2}\mu_1 = [(1+\mathrm{ch}^2 x_1)\,\mathrm{cth}\, x_1 - 1]^{-1}. \qquad (3.16)$$

In this case $h_c = 2\,\mathrm{sech}\, x_1$ and

$$h_0^2 - h_c^2 = \mu_1 c_1^2 [3\,\mathrm{sh}\, x_1\, \mathrm{ch}^{-3} x_1 + \mu_1 (5 + 3\,\mathrm{sh}^2 x_1 - 4\,\mathrm{sh}^{-2} x_1)] + o(c_1^2). \qquad (3.17)$$

With the aid of (3.14) and (3.11) it is easy now to obtain the first terms of the expansions of $\varphi(0)$ and $\mathscr{F}$ in powers of $\varepsilon = (h_0 - h_c)^{1/2}$ [see (2.11) and (2.20] and verify that the solution at $c_1 > 0$ has the lower energy. Since $\varphi_c(x) = 2$ arccos$(-\tanh(x - x_1))$, Eq. (2.9) reduces to a Jacobi equation, and it can be shown that the smallest eigenvalue ω_0^2 at $\varphi = \varphi_c$ is zero. Thus, the solution with the lower energy at $\varepsilon \neq 0$ is stable. If $\varepsilon = 0$ (i.e., $c_1 = 0$), the energy of the bound state is equal to

$$\mathscr{F}_c = 4[1 + \mathrm{th}\, x_1 + \mathrm{sech}\,(x_1)\arccos(\mathrm{th}\, x_1) - \tfrac{1}{2}\mu_1], \qquad (3.18)$$

where μ_1 is determined from (3.16). When μ_1 and x_1 do not satisfy this relation, the bifurcation takes place at $c_1 \neq 0$. Numerical calculation with the aid of (3.4), (3.6), and (3.11) shows that for the typical value $\mu_1 \sim 1/2$ at $0.5 \lesssim x_1 \lesssim 1.5$ the bifurcation does indeed take place at small c_1 and at $h_c \approx 2\,\mathrm{sech}\, x_1$.

We examine now the general picture of the evolution of the static states when c_1 is varied from -1 to $+1$, without fixing the value of x_1 but calculating it from (3.5) or (3.6). It is convenient to represent the states on the (x_1, h_0) plane by curves corresponding to a constant c_1, Fig. 3. The shapes of these curves can be easily interpreted with the aid of Eq. (3.6), which in this case takes the form

$$x_1 = \int_{c_1}^{c_0} dc[(1-c^2)(k_0^2 - c^2)]^{-1/2}. \qquad (3.6a)$$

Given c_1, the parameter k_0^2 is determined from (3.11), where k_1, c_2, $\mu_2 \to k_0$, c_1, μ_1, with $c_0 = \pm(k_0^2 - r_0^2)^{1/2}$, $r_0 = h_0/2 \equiv h_0/h_m$ [see (3.4)]. Differentiating (3.6a) with respect to r_0 we easily verify that at a fixed c_1

$$dr_0/dx_1 = \mp[(k_0^2 - r_0^2)(r_0^2 - k_0^2 + 1)]^{1/2},$$
$$d^2 r_0/dx_1^2 = -r_0(2r_0^2 - 2k_0^2 + 1), \qquad (3.19)$$

i.e., these curves have a maximum at $r_0^2 = k_0^2$ and a minimum at $r_0^2 = k_0^2 - 1$. Putting in (3.6a) $c_0 = 0$ and $c_0 = \pm 1$, we determine the trajectories of the maxima and of the minima. They are shown dashed in Fig. 3, and the direction of motion with increasing c_1 is indicated by arrows. The thin solid lines are typical plots of constant c_1, wherein the curves with the maxima M_0 and M_1 correspond to $c_1 < 0$ (states $E_1D_1D_1'O'$, $E_2D_2D_2'O'$ in Fig. 2), while the curves with the minima M_2, M_3', and M_4 correspond to $c_1 > 0$($E_3D_3D_3'O'$ or $E_3'D_3D_3'O'$). It is easy to show that the curve $c_1 = c_-$ has the smallest maximum and the curve $c_1 = c_+$ has the largest

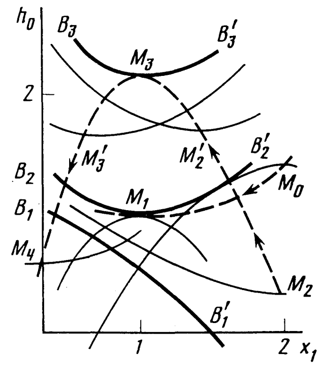

FIG. 3.

minimum. It can now be readily understood that the family of the $c_+ =$ const curves has three envelopes shown by the thick lines. The envelope B_1B_1' corresponds to small c_1, while the value of c_1 is close to c_- for B_2B_2' and to c_+ for B_3B_3'. The curves B_2B_2' and B_3B_3' have minima M_1 and M_3 at $x_1 \sim 1$. The envelopes described are in fact the bifurcation curves. At fixed x_1 there are two states below B_1B_1' (intersection of the curves M_4 and M_1, two more states appear between B_1B_1' and B_2B_2' (intersection of the curves M_0 and M_1), and two states remain between B_2B_2' and B_3B_3'. It can be easily discerned which of the states are stable, As $h_0 \to 0$ the state described by the equations in (3.14) has the lowest possible energy and is obviously stable. Therefore the states on the left-hand branch of the M_0 curve are stable up to the point of tangency with the envelope, while the states to the right of the tangency point, in accord with the general theory, are unstable. It was shown above that the states on the curves of type M_2 and M_2' are stable up to the points of tangency with B_1B_1' or B_2B_2'. It follows from the general theory that the states on the shown sections of the curves, of type M_3' and M_4, are unstable. To show the dependence of $\mathscr{F}$ on h_0 at a given x_1 it suffices now to find the values of $h_0 = h_{ci}$, calculate $\mathscr{F}_{ci} = \mathscr{F}(h_{ci})$, and use Eqs. (3.14) and (3.15) as well as the general conditions (2.19), (2.20), and (2.22). As a result we obtain the typical ($\mu_1 \sim 1/2$) curve shown in Fig. 4, where a deformation was introduced to preserve the signs of $\mathscr{F}'(h_0)$ and $\mathscr{F}''(h_0)$ and the sequence of the points h_{ci}. As shown above, the curves OB_2 and B_3B_1 in Fig. 4 correspond to locally stable states, and B_2B_1 and B_3O' to unstable ones. The energy of the states on OC and CB_3 has an

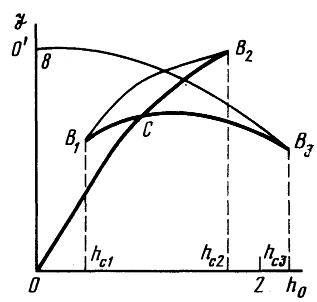

FIG. 4.

absolute minimum, so that they are absoutely (globally) stable. The states on B_1C and CB_2 correspond only to a local minimum of the energy, so that generally speaking they are metastable. Transitions from the states B_1C and B_2C into the states OC and B_3C, at which the total flux changes by an amount $\sim(1/2)\Phi_0$, can result from large quantum-mechanical or temperature fluctuations. Although processes of this type were investigated in homogeneous systems, the elucidation of all the transition mechanisms and the determination of the corresponding lifetimes in an inhomogeneous system constitute a separate and still unsolved problem. We note in conclusion that the peculiar dependence of $\mathcal{J}$ on h_0 is similar to the pressure dependence of the thermodynamic potential of a Van der Waals gas at $T<T_c$, with the states corresponding to stable phases on the sections OC and B_3C and to metastable ones on CB_2 and CB_1.

4. PIECEWISE-LINEAR APPROXIMATION

To calculate the natural frequencies of the bound states obtained above and to solve more complicated problems, we propose here a simple approximation in which all the equations contain only elementary functions. We assume for the energy of the Josephson currents the continuously differentiable approximation

$$1-\cos\varphi\approx{}^1/_2(-1)^N(\varphi-\pi N)^2+{}^1/_8\pi^2[1-(-1)^N] \qquad (4.1)$$

on the intervals I_N: $(N-1/2)\pi\leqslant\varphi\leqslant(N+1/2)\pi$, with N an integer. We then obtain for the Josephson current, by differentiating with respect to φ, a piecewise-linear ("sawtooth") approximation, and for $\cos\varphi$ in (2.9) we obtain after one more differentiation the piecewise continuous approximation $\cos\varphi\approx(-1)^N$. We assume for simplicity that the sought solution $\varphi(x)$ increases monotonically, and put $\varphi(\bar{x}_N)=(N-1/2)\pi$, $\varphi(\bar{x}_{N+1})=(N+1/2)\pi$. Denoting by $\bar{I}_N$ the interval of x in which $\varphi\in I_N$, i.e., $\bar{I}_N=(\bar{x}_N,\bar{x}_{N+1})$, we replace Eqs. (2.8) and (2.9) by the approximate ones

$$\varphi''=\Big[1-\sum\mu_i\delta(x-x_i)\Big](-1)^N(\varphi-\pi N),\quad x\in\bar{I}_N, \qquad (4.2)$$

$$-\psi''+(-1)^N\Big[1-\sum\mu_i\delta(x-x_i)\Big]\psi=\omega^2\psi,\quad x\in\bar{I}_N. \qquad (4.3)$$

The functions φ and ψ and their first derivatives are continuous at the points $\bar{x}_N$, while on the edges of an LJJ they satisfy the usual boundary conditions [see (2.5) and (2.9)]. The approximation (4.1) can be refined by multiplying the right-hand side by a certain number λ^2. An obvious renormalization of the values of x, μ_1, h_0, and h_1 allows us to preserve the forms of Eqs. (4.2) and (4.3) and of the boundary conditions; it suffices therefore to consider only the case $\lambda=1$.

The solutions of Eq. (4.2) on the intervals $\bar{I}_N$ can be represented in the form

$$(\varphi-\pi N)={}^1/_2\pi\{-\operatorname{ch}(x-\bar{x}_N)+a_N\operatorname{sh}(x-\bar{x}_N)\},$$
$${}^1/_2\pi\{\operatorname{ch}(x-\bar{x}_{N+1})+a_N'\operatorname{sh}(x-\bar{x}_{N+1})\}, \qquad (4.4)$$

$$(\varphi-\pi N)={}^1/_2\pi\{-\cos(x-\bar{x}_N)+a_N\sin(x-\bar{x}_N)\},$$
$${}^1/_2\pi\{\cos(x-\bar{x}_{N+1})+a_N'\sin(x-\bar{x}_{N+1})\}, \qquad (4.5)$$

where the parameters a_N and a'_N assume the role of the constants k_i^2 in (3.1); we can put on the corresponding intervals $k_1^2-1\approx(a_N^2-1)\pi^2/16$. If there are no inhomogeneities on $\bar{I}_N$, we have

$$a_N=a_N'=\operatorname{cth}[(\bar{x}_{N+1}-\bar{x}_N)/2],\ \operatorname{ctg}[(\bar{x}_{N+1}-\bar{x}_N)/2]. \qquad (4.6)$$

If $x_1\in\bar{I}_N$, then a'_N differs from a_N. All the parameters a_N and a'_N can be easily expressed in terms of h_0, h_l, and $\bar{x}_N$. For example, in an LJJ with one inhomogeneity the value of a_N to the left of the inhomogeneity can be expressed in terms of h_0 and $\bar{x}_N<x_1$, using the boundary condition and the continuity of $\varphi'(x)=\bar{x}_N$. We obtain similary a'_N to the right of the inhomogeneity. The equations for $\bar{x}_N$ are obtained from the condition that $\varphi(x)$ be continuous at $x=x_1$, from the discontinuity condition

$$\varphi'(x_i+0)-\varphi'(x_i-0)=\mu_i(-1)^N(\varphi(x_i)-\pi N),\quad x_i\in\bar{I}_N \qquad (4.7)$$

and from the relations (4.6) for the intervals that do not contain x_i. The equations for $\bar{x}_N$ are obviously nonlinear. Our approximation leads therefore to linear equations only for solutions that are all concentrated on one of the $\bar{I}_N$ intervals. The approximate solutions (4.4) and (4.5) describe well the dependence of the exact solutions on x. If no account is taken of the solutions that contain pieces with $k_i^2<1/2$ an accuracy to within $\sim10\%$ is guaranteed for all the observable quantities.

To determine ω^2 and $\psi(x)$ from (4.3) it is convenient to put $f(x)=\psi'(x)/\psi(x)$. We then have at $x\neq x_i$, if N is even

$$f=q\operatorname{th}(qx+\theta),\quad x\in\bar{I}_N, \qquad (4.8)$$

and if N is odd

$$f=-p\operatorname{tg}(px+\theta),\quad x\in\bar{I}_N, \qquad (4.9)$$

where $p^2=1-\omega^2$, $q^2=1+\omega^2$. If $x_i\in\bar{I}_N$, the θ has different values, which we designate θ_N and θ'_N, on the left and on the right of the homogeneity. The connection between them is determined by the condition for the discontinuity of ψ', i.e.,

$$f(x_i+0)-f(x_i-0)=(-1)^{N+1}\mu_i,\quad x_i\in\bar{I}_N. \qquad (4.10)$$

On an interval without inhomogeneities we have $\theta_N=\theta'_N$. At the boundary points $f(x_0)=f(x_{n+1})=0$; if $x_0=-\infty$, $x_{n+1}=+\infty$, then $f(x_0)=q$, $f(x_{n+1})=-q$. Using (4.8)–(4.10) we can, for given values of $\bar{x}_N$, eliminate the unknown parameters and obtain an equation for ω^2. It is then easy to obtain $\psi(x)$ by integrating (4.8) and (4.9).

Equations (4.4)–(4.10) together with the boundary conditions and the continuity requirements enable us thus to obtain the bound states, which are now determined by the parameters $\bar{x}_N$, and calculate their natural frequencies. We present simple nontrivial examples. Consider on a semi-infinite junction states that satisfy the condition $\varphi_0,\varphi_1\in I_1$. Eliminating all the unknown parameters except $\eta=\bar{x}_2-x_1+\pi/4$, we can obtain for η the equation

$$r_0\equiv h_0/h_m=\cos x_1[\sin\eta+(\operatorname{tg}x_1-\mu_1)\cos\eta], \qquad (4.11)$$

where $h_m=\pi/\sqrt{2}$ is the maximum value of h_0 at which static states can exist in a homogeneous junction ($h_m=2$ for the exact equations). The bifurcation points $\eta=\eta_c$ are defined by the condition $dr_0/d\eta=0$. Eliminating from this equation and from (4.11) the value $\eta=\eta_c$, we obtain the equation of the bifurcation curve:

$$r_c^2 \equiv (h_c/h_m)^2 = \cos^2 x_1 [1+(\mathrm{tg}\, x_1 - \mu_1)^2]. \qquad (4.12)$$

This equation describes approximately the curve B_2B_2' on Fig. 3 and yields an approximate value of h_{c2} for different x_1 and μ_1. From (4.12) we can also obtain the position of the point M_1 on the bifurcation curve:

$$\mathrm{tg}\, x_1 = {}^1/_2 [\mu_1 + (4+\mu_1^2)^{1/2}], \qquad r_c = \mathrm{ctg}\, x_1.$$

At $\mu_1 = 1/2$ and $\mu_1 = 1$ we obtain from this formula $(x_1; r_c) = (0.91; 0.78), (0.90; 0.66)$. The numerical calculation of the exact equations yields respectively (0.78; 0.82) and (1.02; 0.62). Thus, we obtain a fair approximation even for a very weak characteristic of the spectrum of the bound states. The method described above yields an equation for p:

$$\mu_1 = p\, \mathrm{tg}[p(\bar{x}_2 - x_1) - \arccos(p/\sqrt{2})] + p\, \mathrm{tg}(px_1). \qquad (4.13)$$

The state is stable if this equation has no roots $0 < p < 1$. If we put $p = 1$ in (4.13), we obtain the bifurcation curve obtained above from the condition $dr_0/d\eta = 0$.

We consider in conclusion a state with $x_0 \in \bar{I}_0$ and $x_1 \in \bar{I}_1$. For the two independent parameters $\eta_1 = x_1 - \bar{x}_1$ and $\eta_2 = \bar{x}_2 - x_1 + \pi/4$ we have the equations

$$\sin\eta_1 \sin\eta_2 + (\cos\eta_1 + \mu_1 \sin\eta_1)\cos\eta_2 = 1/\sqrt{2}, \qquad (4.14)$$

$$(r_0\sqrt{2} + \mathrm{sh}\, \bar{x}_1)\sin\eta_1 = (\sqrt{2}\cos\eta_2 + \cos\eta_1)\, \mathrm{ch}\, \bar{x}_1. \qquad (4.15)$$

The first equation permits $\cos\eta_2$ to be expressed in terms of η_1, and we are left with one transcendental equation that connects r_0, x_1, and η_1. The bifurcation curve B_1B_1' on Fig. 3 is easiest to find in this case from the condition $\omega^2 = 0$. Calculation in accord with the scheme described above leads to an equation for p:

$$\mu_1 = p\, \mathrm{tg}[p(x_1 - \bar{x}_1) - \mathrm{arctg}(qp^{-1}\, \mathrm{th}(q\bar{x}_1))]$$
$$+ p\, \mathrm{tg}[p(\bar{x}_2 - x_1) - \arccos(p/\sqrt{2})], \qquad (4.16)$$

where $q^2 = 1 + \omega^2 = 2 - p^2$, and the equation for the bifurcation curve is obtained at $q = p = 1$. In particular, for the state with $\varphi_1 = \pi$ we have $a_0' = \alpha_1 = a_1' = a_2 = 1$ [see (4.4) and (4.5)], i.e., $\bar{x}_2 - \bar{x}_1, = \pi/2$, $\eta_1 = \pi/4$, $\eta_2 = \pi/2$, and from (4.15) it follows that $\sqrt{2r_0} = \exp(-x_1 + \pi/4)$. For this state we find from (4.16) at $q = p = 1$ that $x_1 - \pi/4 = -1/2 \ln \mu_1$. The expressions obtained approximate well the exact relations $r_0 = \mathrm{sech}\, x_1$ and (3.16).

The same method can calculate also other states. At low values of x_1 and μ_1 there can exist only states with a total flux $\lesssim (3/2)\Phi_0$. It is easy to calculate that there exist only eight different types of such states. The equations that determine all the states and their natural frequencies are not more complicated that (4.13)–(4.16). The problem of a junction of finite length involves more complicated equations and calls for a special investigation.

5. CONCLUSION

It was shown above that the magnetic flux can be localized on inhomogeneities that attract fluxons (they were called in Ref. 15 microresistances, in contrast to microshorts). In this case bound states of deformed fluxons are produced and are stable to small fluctuations, or even absolutely stable. Such states can be produced in experiment by varying the magnetic field at the edge of the junction, and the corresponding distributions of the current and of the magnetic field inside the junction might be observable directly by the procedure, mentioned in the Introduction, of scanning the LJJ with a focused low-power laser beam.[11,12] The distributions of the current and field along the junctions are then directly determined by the changes, due to local heating, of the maximum total current at zero voltage. If the state is rigidly localized, i.e., its binding energy and natural frequency are high enough, a perturbation due to illumination of the junction will not distort noticeably the investigated current distribution, which can be predicted theoretically.

We note that one might attempt, by locally heating the junction with a second more powerful focused laser beam, to produce a microinhomogeneity whose position can be easily varied. This would uncover a possibility of directly observing the evolution of the bound states and of their bifurcations while varying simultaneously the position of the microinhomogeneity and of the external magnetic field.

Scanning of rigidly localized states at different values of the edge magnetic field can also be used for diagnostics of junctions with inhomogeneities. Indeed, our analysis shows that each LJJ has an individual "portrait" of bound states, and in principle it is easy to use the abundant set of experimental distributions to solve the inverse problem, i.e., to find the parameters that characterize the inhomogeneities. The position of the microinhomogeneities can be observed in this case literally "with unaided eye" (see Figs. 3 and 5 in Ref. 12), and a more detailed theoretical analysis is needed only to determine the values of μ_i. A more direct method of finding μ_i is to measure the natual frequencies of the bound states. It suffices for this purpose to excite natural oscillations and to observe them by one of the methods used to study the motion of fluxons in homogeneous LJJ (Refs. 9 and 10).

We note that bound states can exist also in junctions with repelling inhomogeneities (microshorts); for example, a soliton may fall into a trap between two microshorts that are far enough from each other.[1,2] Although such states can also be found by the methods of the present paper, soliton perturbation theory is usually sufficient for their description. The stable states in the traps are usually weakly localized and have low natural frequencies, so that their natural oscillations are easily excited and attenuate slowly. For this reason, the prospects of using fluxons localized on microresistances are more favorable. Furthermore, as shown above, even in a very simple system it is possible to realize two stable states (bistability). These states can be smoothly controlled by varying the magnetic field far from the bifurcations, and an abrupt switching (quenching) into a state with lower energy is possible at the bifurcation point. For a detailed quantitative description of these phenomena we must solve a nonstationary equation with allowance for the static solutions obtained above. An investigation of the nonstationary problem is needed also for the calculation of the probability of transiton from a metastable to a stable state. It is quite probable that the use of information on static states, including also on sufficiently long-lived unstable ones, can provide a simple description of the nonstationary processes, similar to the S-matrix scattering theory. Bound states and resonances of the S matrix will correspond in this case to the stable and unstable static distributions investigated in the present paper.

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[1)]This follows from the general theory of stability and bifurcations, originated in the papers of A. M. Lyapunov and E. Schmidt.[20,21] In problems similar to ours, this phenomenon was recently observed in Refs. 22 and 23.

[2)]At $h_0 = h_l = h_{ex}$ it follows hence that $\partial\Phi/\partial h_{ex} > 0$, i.e., the total flux increases with increasing external field. This condition is similar to the known thermodynamic relation $(\partial V/\partial p)_T < 0$, with $-h_{ex}$ the analog of the prsssure and Φ the analog of the volume.

[3)]Besides ground states of the first and fourth type, at sufficiently large values of Δ there exist unstable excited states in which φ' reverses sign many times in the interval $(-\Delta,\Delta)$; these states will not be considered here.

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