Integral equations for resonance and virtual states

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(Submitted 19 October 1983)
Zh. Eksp. Teor. Fiz. **86**, 1600–1617 (May 1984)

Integral equations are derived for the resonance and virtual (antibound) states consisting of two or three bodies. The derivation is based on the analytic continuation of the integral equations of scattering theory to nonphysical energy sheets. The resulting equations can be used to exhibit the analytic properties of amplitudes that are necessary for practical calculations using the equations for the quasistationary levels and Gamov wave functions derived in this paper. The Fourier transformation and the normalization rule for the wave function are generalized to the case of nonstationary states. The energy of the antibound state of the tritium nucleus is calculated for a "realistic" local potential.

1. INTRODUCTION

Recent years have seen renewed interest in physical problems that can be solved by the methods of quantum mechanics. Studies of resonance and virtual states in strongly interacting systems occupy an important position among these problems. From the formal mathematical point of view, descriptions of unstable (Gamow) and bound states are almost equivalent, since they correspond to the same condition for the emission of an outgoing spherical wave in the radical part of the wave function, i.e., the poles of the S matrix are the zones of the Jost function. However, the fact that, for unstable states, the poles of the S matrix lie on the nonphysical sheet of the Riemann energy surface[1,2] ($\mathrm{Im}\sqrt{E} < 0$) leads to well-known differences. Hence, even in those cases where, for given interaction, the bound-state problem can be solved reasonably rigorously (for example, for systems consisting of two or three bodies), it is found that difficulties arise when resonance or virtual states are considered. For example, a serious obstacle to using the Schrödinger equation to find the poles of the S matrix is that the asymptotic form of the Gamow wave function increases exponentially. On the other hand, the integral equations for the t matrix, which are widely used for bound states, are formulated on the physical sheet of the energy E, and cannot therefore be used directly for unstable states. Having found the t matrix on the physical sheet, it is, in principle, possible to continue it analytically to the nonphysical sheet. If the scattering amplitude is known in an analytic form, this does not, of course, present any problems. However, in most physically interesting cases, only a numerical solution can be obtained, and its analytic continuation is not a simple matter. This is the reason why, until quite recently, rigorous results for even the two-body potential problem were known only for the square-well[2] and separable potentials of finite rank (see, for example, Ref. 3) for which the scattering amplitude has an analytic form.

We therefore have to face the problem of analytic continuation of the integral equations for the t matrix to nonphysical energy sheets in order to transform the resonance problem to a form analogous to the bound-state problem. This is, in fact, the aim of the present paper, in which we derive integral equations for the resonance and virtual states of two- and three-body systems. The resulting equations are analogs of the corresponding Schrödinger equations for the bound states in momentum representation, and can be used directly to find the poles of the S matrix on nonphysical sheets, and the corresponding Gamow wave functions. We note that the evaluation of the scattering amplitude in the physical energy region, i.e., the solution of the problem for the continuous spectrum, is known to be unrelated to the problems considered here. However, the fact that the cross section is a maximum necessarily signifies the presence of the corresponding pole in the S matrix, but the determination of this pole is a separate problem.

The approach developed here was formulated perviously by one of the present authors in a brief communication,[4] in which he examined the analytic continuation of the Lippmann-Schwinger equation for the two-body potential problem. The general character of this approach distinguishes it from other methods that have a limited range of application or are computationally laborious and frequently do not guarantee that correct results will be obtained. Let us enumerate briefly the other methods used in the literature without any pretence to completeness. These methods include the effective range approximation[5,6] for near-threshold poles, the evaluation of the Regge trajectory,[7] and analytic continuation in the coupling constant[8] that so far has been developed only for single-channel problems. A method of summing the divergent perturbation-theory series by the Pade-Borel method has recently been developed.[9] Complex poles have also been investigated by the N/D method in which the discontinuity in the scattering amplitude on the left cut is specified instead of the potential.[10,11] The availability of numerous methods for studying the resonances indicates not only the degree of interest in this particular field, but also the unsatisfactory situation in this branch of physics. An example of this is the recent paper by Kloet *et al.*,[12] who resorted to an analysis of the numerical convergence of Neumann iteration series for the t matrix in order to elucidate the existence of a resonance pole (it is known that the series diverges near this pole), and arrived at an incorrect conclusion (see Ref. 13).

Attempts to obtain the analytic continuation of the integral equations for the t matrix, in particular, for the three-body problem, had been undertaken previously in Refs. 13–

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15. A particular deformation of the contour of integration was used there to isolate singularities in the kernel of the integral equation. This exposes part of the nonphysical sheet on which the required poles of the S matrix may be located. This method is technically quite complicated and, most importantly, requires an *a priori* knowledge of the singularities of the t matrix. In the present paper, we consider a deformation of the contour of integration that arises when a point on the plane of the complex variable E passes from the physical to the nonphysical sheet, and is actually dictated by the conditions of the problem in its most general case. This not only enables us to write down the integral equation for the unstable state in a simple form, but also establishes the analytic properties of its solution, both in momentum and energy. The latter is fundamentally important for the practical implementation of the method of analytical continuation itself. Finally, we have been able to accomplish the complete analytic continuation in energy of the Faddeev equation for the three-body problem.[16] This problem has not been examined in detail in the literature.

The results obtained below remove the currently existing gap between descriptions of bound and unstable states, and enable us to consider poles of the S matrix on a unified basis, independently of their position, and to exploit the methods of solution developed for bound states. Our approach is universal in the sense that it is valid for any integral equation with a Cauchy kernel (i.e., a free Green function).

Our plan is as follows. Section 2 examines the two-body potential problem. Section 3 generalizes the Fourier transformation and the normalization of the wave function to the case of unstable states. Section 4 is devoted to the multichannel problem. Sections 5 and 6 discuss the method of calculating quasistationary and virtual states for the three-body system. Finally, Section 7 applies this method to the evaluation of the antibound state of the tritium nucleus.

2. INTEGRAL EQUATION FOR THE RESONANCE STATE OF A TWO-BODY SYSTEM WITH A SHORT-RANGE POTENTIAL

The Lippmann-Schwinger equation for the partial t matrix ouside the energy surface can be written in the following form (z is the complex energy on the physical sheet I_+, i.e., for $\mathrm{Im}\sqrt{z}>0$):

$$t_l(q,q';z)=V_l(q,q')+4\pi\int_0^\infty \frac{V_l(q,k)t_l(k,q';z)}{z-k^2/2\mu}k^2\,dk, \qquad (1)$$

where

$$V_l(q,q')=(2\pi^2)^{-1}\int_0^\infty j_l(qr)\,V(r)j_l(q'r)r^2dr, \qquad (2)$$

$j_l(x)$ is the spherical Bessel function, μ is the reduced mass, and l is the orbital angular momentum (the index l will be omitted for the sake of brevity). To obtain the Lippmann-Schwinger equation on the nonphysical sheet $I_-(\mathrm{Im}\sqrt{z}<0)$, we must perform the analytic continuation of the Cauchy-type integral on the right of (1). The analytic continuation of the Cauchy-type integral

$$\Phi(z)=\frac{1}{2\pi i}\int_0^\infty \frac{\varphi(\zeta)\,d\zeta}{\zeta-z} \qquad (3)$$

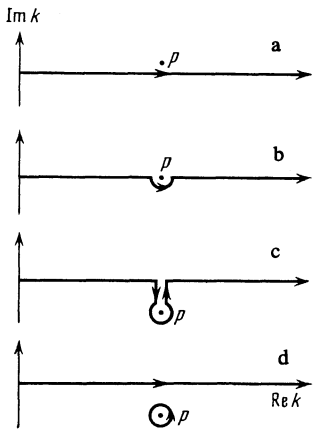

FIG. 1. Constrained deformation of the contour of integration for a Cauchy-type integral on the right of the Lippmann-Schwinger equation during the transition to the nonphysical sheet of energy z: a—z on physical sheet ($\mathrm{Im}\sqrt{z}>0$), b—$z=E$—real positive number, c—z on physical sheet ($\mathrm{Im}\sqrt{z}<0$).

from I_+ to I_- is the function (see, for example, Ref. 17)

$$\Phi_-(z)=\Phi(z)-\varphi(z), \qquad (4)$$

which is definable in the region in which $\Phi(z)$ and $\varphi(z)$ are analytic. From (4), we obtain

$$t_-(q,q';z)=V(q,q')+4\pi\int_0^\infty \frac{V(q,k)t_-(k,q';z)}{z-k^2/2\mu}k^2\,dk+i8\pi^2\mu pV(q,p)t_-(p,q';z), \qquad (5)$$

where $p=(2\mu z)^{1/2}$ is the arithmetic value of the root (i.e., $\sqrt{-1}=+i$). The derivation of (5) is a relatively clear illustration of the constrained deformation of the contour of integration in (1), in which the point $p=(2\mu z)^{1/2}$ is displaced from the upper half-plane of the complex variable k to the lower half-plane (Fig. 1). It is clear from Fig. 1 that the additional term in (5) is due to the residue of the integrand at $k=p$. Let us now transform (5) so that the equation does not include the unknown function $t_-(p,q';z)$ for complex momentum p. This will be done by obtaining $t_-(p,q';z)$ from (5) at $q=p$ and substituting the result in (5). The final result is the following equation, which has the same form as (1):

$$t_-\left(q,q';\frac{p^2}{2\mu}\right)=\hat{V}(q,q',p)+8\pi\mu\times\int_0^\infty \frac{\hat{V}(q,k,p)t_-(k,q';p^2/2\mu)}{p^2-k^2}k^2\,dk, \qquad (6)$$

but with the new "potential"

$$\hat{V}(q,q',p)=V(q,q')+V(q,p)V(p,q')\frac{i8\pi^2\mu p}{1-i8\pi^2\mu pV(p,p)}, \qquad (7)$$

i.e., with the replacement

$$V(q,q')\to\hat{V}(q,q',p). \qquad (8)$$

The connection between the t and t_- branches is made, as required, at the threshold branch point $z=0$ at which V and

$\hat{V}$ become identical. The addition to the potential is separable (i.e., nonlocal) and depends on the energy z. The latter is responsible for the more complicated analytic structure of $t_-(q,q';z)$ in z outside the energy surface. It is not difficult to see from (7), by taking into account the known properties of $V_l(q,q')$ [see (2)], that the actual potential satisfies the equation $\hat{V}^*(q,q'-p^*)=\hat{V}(q,q',p)$. The well-known symmetry in the disposition of the S-matrix poles relative to the imaginary axis of the complex variable p follows from this for the Hermitian Hamiltonian. We note that we did not use the unitarity condition in the derivation of (6), so that this result can be used for the complex potential as well. It is shown in Ref. 4 that the zeros of the denominator in (7) are not the poles of the t matrix. This is also clear from the relationship between t and t_- that can be obtained from the Low equation and unitarity condition (see, for example, Ref. 18). In our normalization, we have (real $p\geqslant 0$)

$$t_-(p,q';p^2/2\mu)=t(p,q';p^2/2\mu)\,[1-i8\pi^2\mu p t(p,p;p^2/2\mu)]^{-1}. \tag{9}$$

The analytic continuation of this relation to the entire complex energy plane will also enable us to find the poles of the S matrix as the zeros of the denominator of (9). However, this requires the initial evaluation of $t(p,p;p^2/2\mu)$ for the entire range of p, for which the pole is required, whereas once we have Eq. (6), it is sufficient to determine the zeros of its Fredholm determinant without solving the equation itself. Since $t(p,p;p^2/2\mu)\neq V(p,p)$, the zeros of the denominator in (7) do not coincide with the required poles of the S matrix. However, in the neighborhood of one of these zeros, it is better to consider the original equation (5) [augmented with (5) at the point $q=p$] instead of (6). The poles can also be sought by using the condition $\lambda(z)=1$, where $\lambda(z)$ is an eigenvalue of the kernel of the Lippman-Schwinger equation at energy z. The eigenvalues $\lambda(z)$ are obtained on I_- from the homogeneous equation [according to (6) and (7)]

$$\lambda^{-1}(z)X(q,q';p^2/2\mu)=R(q,q';p^2/2\mu)+S(q,q';p^2/2\mu), \tag{10}$$

$$R(q,q';p^2/2\mu)=8\pi\mu\int_0^\infty \frac{V(q,k)X(k,q';p^2/2\mu)}{n^2-k^2}k^2\,dk, \tag{11}$$

$$S\left(q,q'\cdot\frac{p^2}{2\mu}\right)=-\frac{i8\pi^2\mu pV(q,p)}{1-i8\pi^2\mu pV(p,p)}R\left(q,q'\cdot\frac{p^2}{2\mu}\right). \tag{12}$$

This form of the equations is convenient because it reveals the presence of potential singularities in the solution of the homogeneous equation (10). The dependence of $X(q,q';p^2/2\mu)$ on q appears explicitly in (12) through $V(q,p)$. Thus, the analytic continuation of the integral equation enables us not only to determine the energy poles of the S matrix, but also to establish the singularities of $X(q,q';z)$ in energy $z=p^2/2\mu$ and momentum q[q' is an external variable in (1), (6), and (10) and can assume any value]. The vertex function $g(q)$ for the decay of the corresponding unstable state has the same singularities in q, and the equation for this function is obtained from (9) by a limiting transition to the pole at $z=z_0$, using the well-known result[1)]

$$\lim_{z\to z_0}[t_-(q,q';z)\,(z-z_0)\,]=g(q)g(q'). \tag{13}$$

The equation for the vertex function has the form [see (13) and (10) for $\lambda(z_0)=1$]:

$$g(q)=4\pi\int_0^\infty \frac{\hat{V}(q,k;\sqrt{2\mu z_0})}{z_0-k^2/2\mu}g(k)\,k^2\,dk. \tag{14}$$

This is the analog of the Schrödinger equation in momentum representation for which the transition to the nonphysical sheet is also reduced to the replacement (8).

In the determination of the virtual pole, we must take into account the cut of $V(p,p)$ that begins at the point $p=-i\beta$, where $V(p,p)$ has a logarithmic branch point for the potential in the Yukawa form or in the form of a superposition of Yukawa potentials. In the analytic continuation in p^2 (or p), we must bypass the corresponding branch point. The trajectory of the pole of the t matrix for real energies $z_0=p^2/2\mu$ is then found to lie on the sheet of the logarithm that corresponds to the real branch of the logarithm for $p=-i|p|$. We can thus follow the trajectory of the pole as the factor λ in front of the potential varies along the entire imaginary axis of p (the real axis of z on the nonphysical sheet).

Equation (7) enables us to formulate a further theory of symmetry, but now relative to the real axis of p. When the sheet I_+ contains a bound state for $p=i\beta$, where $i\beta$ is a singular point of the potential (i.e., $V_0(p,p)\to\infty$ as $p\to i\beta$), a virtual pole (zero of the Jost function) should occur at $p=-i\beta$. This result is a consequence of the fact that the additional term in (7) is zero at the singularity of $V(p,p)$ and $\hat{V}(q,q',p)=V(q,q')$. We recall [see (2)] that $V(-q,-q')=V(q,q')$. Because of the symmetry of the zeros and poles of the S matrix, $g(-i\beta)=0$. The theorem is valid for analytic potentials, since the truncation of $V(r)$ at any finite distance takes all the singularities of $V(q,q')$ to infinity. We note that rigorous results on the nature of the trajectory of the pole for $V(r)$ that vanishes rigorously for $r\geqslant R$, were obtained in Ref. 21.

3. GENERALIZATION OF THE FOURIER TRANSFORMATION, THE NORMALIZATION RULE FOR THE WAVE FUNCTION, AND THE EVALUATION OF MATRIX ELEMENTS TO THE CASE OF QUASISTATIONARY STATES

Having found the vertex function $g(q)$ from (14) to within a constant factor, we can normalize it through the t matrix [see (13)] as suggested in Ref. 4. However, it is also possible to perform a direct normalization[22] of the wave function $\varphi_k(q)$ which is related to $g(q)$ by the well-known expression

$$\varphi_k(q)=-g(q)/(q^2-k^2). \tag{15}$$

The poles of $\varphi_k(q)$ at the points $q=\pm k$ determine the asymptotic behavior of the radial wave function if the singularities of $g(q)$ lie further away from the point $q=0$. For the sake of simplicity, we confine our attention to the S wave (generalization to $l>0$ is elementary).

We shall look upon the unstable virtual state as an "absent bound state", and will start with the known integral transformation for the bound-state wave function for which we must take $k=i\varkappa$, in (15), where $\varkappa=(2\mu E_b)^{1/2}$ and E_b is the binding energy ($\varkappa>0$). Using the equation $g(q)=g(-q)$

[see (14) and (2)], we can write the Fourier transformation in the form

$$\psi_\varkappa(r)=\frac{C}{r}\int_{-\infty}^{\infty}e^{iqr}\frac{g(q)}{q^2+\varkappa^2}\,q\,dq, \tag{16}$$

where C is the normalization constant.

If we close the contour of integration in the upper half-plane, on which we can discard the integral over the large half-circle, we obtain the well-known asymptotic expression determined by the residue at the pole at $q=i\varkappa$,

$$\psi_\varkappa(r)\sim\frac{\text{const}}{r}e^{-\varkappa r}\quad\text{for}\quad r\to\infty. \tag{17}$$

The contour also runs around the singularity of $g(q)$. To obtain the analog of (16) for the virtual state, we perform the analytic continuation in the coupling constant λ in the Hamiltonian $H=H_0+\lambda V$. As λ decreases from the region in which $\lambda>\lambda_0$ to λ_0, for which the bound state has zero energy, the poles of $\varphi_\varkappa(q)$ at the point $i\varkappa$ and $-i\varkappa$ move toward one another, approaching the contour of integration from different directions and clamp it down. As λ is reduced still further, the poles change places and take the contour of integration with them (see the contour shown in Fig. 2a), for otherwise the integral will not exist. The generalized Fourier transformation is thus found to have the form

$$\psi_\varkappa(r)=\frac{C}{r}\int_{\Gamma}e^{iqr}\frac{g(q)}{q^2+\varkappa^2}\,q\,dq. \tag{18}$$

This asymptotic expression is now determined by the residue at $q=-i\varkappa(\varkappa>0)$:

$$\psi_\varkappa(r)\sim\frac{\text{const}}{r}e^{\varkappa r}\quad\text{for}\quad r\to\infty. \tag{19}$$

In other words, we have obtained the well-known Gamow asymptotic formula with an exponential increase at infinity [the use of (16) would have resulted in bound-state asymptotic behavior]. Any matrix element containing the wave function of a virtual or resonance state can be transformed in a similar way. Modification of the matrix element

$$F=\int_0^\infty f(q)\,dq$$

(see Fig. 2) reduces to

$$\tilde F=F+i\pi[\operatorname{Res} f(q)|_{q=-i\varkappa}-\operatorname{Res} f(q)|_{q=i\varkappa}]. \tag{20}$$

This expression must also be used for normalization. In that case,

$$f(q)=q^2g^2(q)/(q^2+\varkappa^2)^2. \tag{21}$$

The additional term in (20) reflects none other but the fact of analytic continuation that takes the pole of the S matrix from the point $p=i\varkappa$ to $p=-i\varkappa$. Since $f(q)$ in (21) is even, the modified normalizing integral can be written in the form

$$\tilde N=N+2i\pi\operatorname{Res} f(q)|_{q=q_{res}}. \tag{22}$$

This expression is available for any even function $f(q)$ (it is also given in Ref. 23 without derivation) and N is the usual integral. It is clear that all that we have said so far is also valid for a resonance when $\varkappa$ is complex ($l\neq0$) and the trajectory of the pole has a different shape (see Fig. 2b). Equation (14) can be used in practical calculations of the residues in (20) and (22). Integrals evaluated with the aid of (20) or (22) are finite, independently of the position of the pole. It is known (see, for example, Ref. 19) that the method of regularization in r-space depends on the position of the pole. For example, the Zel'dovich regularization procedure,[24] which uses the factor $\exp(-\varepsilon r^2)$ (with passage to the limit as $\varepsilon\to0$ after evaluation), is valid only for resonances for which $|\operatorname{Re} q_{res}|>|\operatorname{Im} q_{res}|$.

4. MULTICHANNEL PROBLEM. STRONG CHANNEL COUPLING

The rule for the analytic continuation of the Lippmann-Schwinger equation can readily be generalized to the set of integral equations in the so-called strong channel coupling method for the matrix elements of the T operator

$$T_{\alpha'\alpha}(k_{\alpha'},k_\alpha;z)=V(k_{\alpha'},k_\alpha)$$
$$-4\pi\sum_{\alpha''=1}^{n}\int_0^\infty\frac{V(k_{\alpha'},k_{\alpha''})\,T_{\alpha''\alpha}(k_{\alpha''},k_\alpha;z)}{k_{\alpha''}^2/2\mu_{\alpha''}-(z-Q_{\alpha''})}k_{\alpha''}^2\,dk_{\alpha''}. \tag{23}$$

where $Q_\alpha=m_\alpha-m_1$ is the threshold of the $(1,\alpha)$ reaction and m_α is the total mass in channel α. The lightest-particle channel is labeled with the index 1. Only binary channels are considered in the method of strong channel coupling.[25,26] On the energy surface,

$$k_\alpha^2/2\mu_\alpha=z-Q_\alpha, \tag{24}$$

where μ_α is the reduced mass and z is the energy.

According to (23), the functions $T_{\alpha'\alpha}$ have n cuts along the real z axis, beginning with the thresholds $z=Q_\alpha$, and, correspondingly, 2^n sheets of the Riemann surface. The channel indices on the operators V are omitted for the sake of brevity (the angular momentum indices are also omitted). The additivity of the different cuts in (23) enables us to pass to the nonphysical sheet relative to any of them. Let $\{\beta\}$

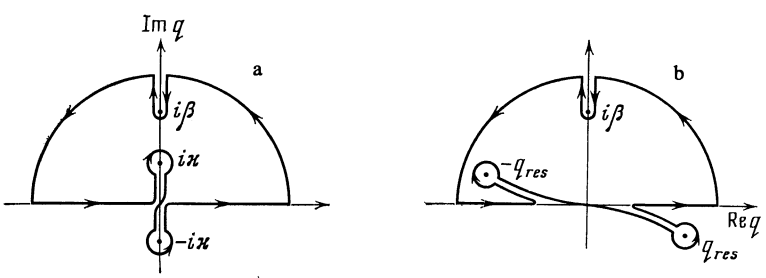

FIG. 2. Contour of integration for the evaluation of a matrix element containing a Gamow function: a—virtual state, b—resonance state.

represent the set of channels for which (23) is to be analytically continued to the nonphysical sheets. We can then use matrix algebra to show that the result of this continuation is equivalent to the replacement of the potential matrix $V(k_{\alpha'},k_\alpha)$ with the following expression:

$$\mathcal{V}(k_{\alpha'},k_\alpha)=V(k_{\alpha'},k_\alpha)+i8\pi\sum_{\beta\beta'}\mu_\beta p_\beta V(k_{\alpha'},p_\beta)W^{-1}_{\beta\beta'}V(p_{\beta'},k_\alpha), \tag{25}$$

where W^{-1} is the inverse of W:

$$W_{\beta\beta'}=\delta_{\beta\beta'}-i8\pi^2\mu_{\beta'}p_{\beta'}V(p_\beta,p_{\beta'}), \tag{26}$$

$$p_\beta=[2\mu_\beta(z-Q_\beta)]^{1/2}. \tag{27}$$

Just as in the single-channel case, the additional term in (25) is nonlocal (takes the form of a sum of separable terms) and depends on the energy z. Restriction to binary channels is not fundamental, as can be seen by considering the Faddeev equations.

5. FADDEEV EQUATIONS. TWO-PARTICLE CUT

We shall now consider a simple example of the Faddeev equation for three identical spinless particles whose interaction is described by the separable potential[2)]

$$\langle\mathbf{p}'|V|\mathbf{p}''\rangle=\lambda\chi(\mathbf{p}')\chi^*(\mathbf{p}''). \tag{28}$$

The problem of scattering of a particle by the bound subsystem reduces to the solution of the one-dimensional integral equation for the particle amplitude $X_L(q,q';E+i\varepsilon)$ (see, for example, Ref. 27):

$$X_L(q,q';E+i\varepsilon)=2Z_L(q,q';E+i\varepsilon)$$

$$+8\pi\int_0^\infty q''^2\,dq''Z_L(q,q'';E+i\varepsilon)\tau\left(E+i\varepsilon-\frac{3q''^2}{4m}\right)X_L(q'',q';E+i\varepsilon). \tag{29}$$

Here, we use the notation defined in Ref. 27: L is the total angular momentum of the system,

$$Z_L(q,q';E+i\varepsilon)=\frac{1}{2}\int_{-1}^{1}dyP_L(y)Z(\mathbf{q},\mathbf{q}';E+i\varepsilon), \tag{30}$$

$$y=\mathbf{q}\mathbf{q}'/qq', \tag{31}$$

$$Z(\mathbf{q},\mathbf{q}';E+i\varepsilon)=\frac{\chi^*(\mathbf{q}'+\mathbf{q}/2)\chi(-\mathbf{q}'/2-\mathbf{q})}{E+i\varepsilon-[\mathbf{q}^2+(\mathbf{q}+\mathbf{q}')^2+\mathbf{q}'^2)/2m}. \tag{32}$$

The function $\tau(z)$ associated with the two-body t matrix has a pole at $z=-\varepsilon_b$ (ε_b is the binding energy of the subsystem):

$$\tau(z)=\{(z+\varepsilon_b)C(z)\}^{-1},\quad z(q)=E-3q^2/4m, \tag{33}$$

$$C(z)=-\int d\mathbf{p}\frac{\chi(\mathbf{p})\chi^*(\mathbf{p})}{(z-p^2/m)(\varepsilon_b+p^2/m)}. \tag{34}$$

Thus, the following two-particle propagator appears in (29):

$$[z(q)+\varepsilon_b]^{-1}=[E+i\varepsilon-3q^2/4m+\varepsilon_b]^{-1}=[E_{12,3}+i\varepsilon-3q^2/4m]^{-1}, \tag{35}$$

where $E_{12,3}=E+\varepsilon_b$ is the energy of relative motion of particle 3 about the center of mass of the bound pair $\{12\}$. Correspondingly, the function $X_L(q,q';E)$ acquires the right root cut that begins at $E=-\varepsilon_b$. By introducing the "potential"

$$V_L(q,q';E)=2Z_L(q,q';E)C^{-1}(z(q')), \tag{36}$$

we obtain the Lippmann-Schwinger equation, whose analytic continuation in the variable $E_{12,3}$ gives us an equation of the same form as (29), but with a new inhomogeneous term ($p=[2\mu(E+\varepsilon_g)]^{1/2}$):

$$\hat{Z}_L(q,q';E)=Z_L(q,q';E)+F(q,p;E)Z_L(p,q';E) \tag{37}$$

and potential

$$\hat{V}_L(q,q';E)=V_L(q,q';E)+F(q,p;E)V_L(p,q';E), \tag{38}$$

$$F(q,p;E)=\frac{i8\pi^2\mu pV_L(q,p;E)}{1-i8\pi^2\mu pV_L(p,p;E)},\qquad \mu=2m/3. \tag{39}$$

For the same reason as for the Lippmann-Schwinger equation, the denominator in (39) is not a pole of the S matrix. Similarly, near the zero of the denominator in (39), we must consider the equation augmented with the equation at the point $q=p$. The resulting extended equation can be used to find virtual and resonance states by standard methods (see also Ref. 15). At the point of the corresponding pole, we obtain a homogeneous equation for the vertex function of the (123)→{12} + 3 decay, similar to equation (4) in the two-body problem and, consequently, for the Gamow wave function of the resonance (or virtual) state than can also be normalized when the modification considered above is taken into account or the t matrix is employed.

As in the two-body problem, the analytic properties of the solution of the three-particle homogeneous equation are determined by the properties of the kernel of the integral equation, i.e., the potential (36). In contrast to the two-particle potential, whose singularities do depend on the specific model, the nearest singularities of the exchange potential $Z(\mathbf{q},\mathbf{q}'';E)$ are determined by the real physical properties of the three-particle system that are independent of the specific form of the short-range pair interaction. The virtual pole (zero of the Jost function), that appears as a result of the symmetry theorem (Sec. 2) at the point $p=-i\beta$, at which $V(p,p;E)$ is singular if a bound-state pole appears at $p=i\beta$, reflects the dynamic properties of the system, since $\beta=\frac{2}{3}(m\varepsilon_b)^{1/2}$ [see (43)].

6. FADDEEV EQUATIONS. THREE-PARTICLE CUT

The three-particle cut is connected with the three-particle propagator that appears in both $\tau(z)$ and $Z_L(q,q'';E)$. The right cut of the pair t matrix that appears in $\tau(z)$, or specifically in $C(z)$ given by (34), induces a three-particle cut corresponding to the so-called direct term in the unitarity relation (Fig. 3b). We shall now show that the cross term in the unitarity relation on the three-particle cut is due to the function $Z_L(q,q'';E)$, i.e., the projection onto the L th partial wave of the exchange potential describing the interaction between the bound pair $\{12\}$ and the spectator particle 3, shown in Fig. 4. In contrast to the two-body problem, the potential Z_L is an explicit function of energy. Its singularities are determined by the Legendre function of the second kind $Q_L(y)$ (if we ignore the singularities of the vertex functions χ that do not lead to a right cut in E), where

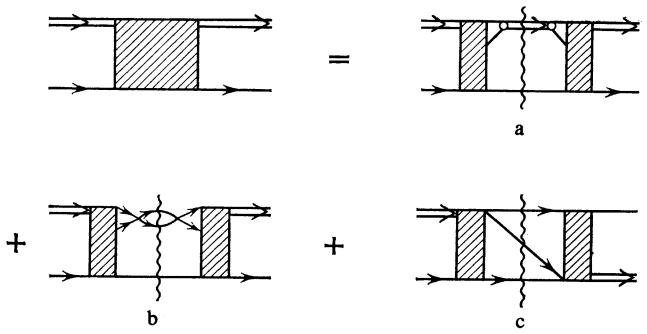

FIG. 3. Feynman diagram for the scattering of a particle by a bound pair, illustrating the appearance of the two-particle cut (a) and three-particle cuts corresponding to direct (b) and cross (c) terms in the unitarity relation.

$$y=(mE-q^2-q''^2)/qq''. \tag{40}$$

This function has a logarithmic cut along the real axis of y between -1 and $+1$, which induces cuts in $Z_L(q,q'';E)$ on the complex plane of q''. The positions of these cuts are determined from the condition that the denominator in (32) is zero:

$$q''^2+qyq''-(mE+i\varepsilon-q^2)=0, \tag{41}$$

from which we have

$$q''=-qy/2\pm(q^2y^2/4+B^2+i\varepsilon)^{1/2}, \quad B^2=mE-q^2. \tag{42}$$

The cuts join the branch points in pairs:

$$q_{min}^{(1)}=-q/2+(q^2/4+B^2+i\varepsilon)^{1/2}, \quad q_{max}^{(1)}=q/2+(q^2/4+B^2+i\varepsilon)^{1/2}; \tag{43a}$$

$$q_{min}^{(2)}=-q/2-(q^2/4+B^2+i\varepsilon)^{1/2}, \quad q_{max}^{(2)}=q/2-(q^2/4+B^2+i\varepsilon)^{1/2}, \tag{43b}$$

where we take into account the sign of the imaginary additional term for real $E(>0)$. The cuts are illustrated schematically[27] in Figs. 5a, b, and c for

$$\text{a)}\ 0\leqslant q\leqslant(mE)^{1/2}, \quad \text{b)}\ \sqrt{mE}\leqslant q\leqslant(4mE/3)^{1/2}, \quad \text{c)}\ q\geqslant(4mE/3)^{1/2}. \tag{44}$$

In region (b), the cuts clamp down the contour of integration with respect to q'' (at the point $q''=R=(q^2-mE)^{1/2}$), which leads to the appearance of the logarithmic cut in E (for real $E\geqslant 0$). Since there is no mirror symmetry in the position of the cuts in region (b), a change in the sign of $i\varepsilon$ that is equivalent to the reflection of the cuts in the real axis of q'' leads to a discontinuity in X_L on the three-particle cut in $E(>0)$. The initial momentum q on $X_L(q,q';E)$ can assume any positive value in the equation [because of integration with respect to q'' in (29)], so that (44b) can be satisfied from

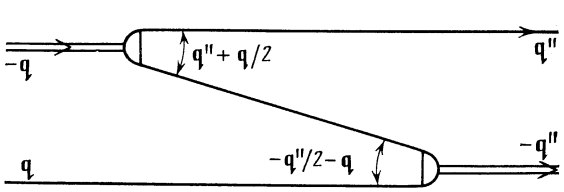

FIG. 4. Feynman graph for the exchange potential $Z(\mathbf{q},\mathbf{q}';E)$ in the Faddeev equations.

$E=0$ onward. The resulting cut corresponds to the cross term in the unitarity relation (see Fig. 3c).

So far, we have examined the region of real positive E. We shall now perform the analytic continuation in E into the neighboring nonphysical sheet relative to the threshold $E=0$, using the method of constrained contour deformation. Figure 1 illustrates this for the Lippmann-Schwinger equation. In contrast to the case of the Lippmann-Schwinger equation, we now have the displacement not of one point (propagator pole in the Lippmann-Schwinger equation), but a whole line (logarithmic cut of the integral containing the three-particle propagator). We shall move in the plane of the complex variable E on the nonphysical sheet through the point $E=0$ and along a certain ray at the angle -2φ. This will correspond to a ray (at the angle $-\varphi$) on the q'' plane, passing through the point $p=(mE)^{1/2}$. Suppose that the point q lies on this ray. The cuts of $Z_L(q,q'';E)$ on the plane of q'' will differ from those shown in Fig. 5 only by the rotation of the coordinate axes through the angle φ in the anticlockwise direction. This is why the above motion of the point E toward the ray containing the segment $[0,\sqrt{mE}\,]$ will be accompanied by the emergence on the q'' plane of the logarithmic cut of the function $Z_L(q,q'';E)$ connecting the branch points (43a). The result of this will be that, in addition to the integral along the real axis of q'' we shall have a further integral over the contour $L(E)$ around this cut (see Fig. 6, $|q|^2<|mE|$). In fact, the region $C(q,E)$, where disc $Z_L\neq 0$, will provide a contribution. As the point q moves along the ray between $q=0$ and $q=(mE)^{1/2}$, the segment $C(q,E)$ is displaced along this ray, and its length varies. The branch points of Z_L corresponding to (a) $y=+1$ and (b) $y=-1$ occupy the lower and upper parts of the ellipse, which are separated by the points ($q^2=0,\tilde{q}^2=mE$) on the $q^2\tilde{q}^2$ plane (see Fig. 7). The regions $C(q,E)$ in this figure correspond to the vertical arrows. It is clear that $C(q,E)$ contracts to the point $\tilde{q}=(mE)^{1/2}$ as $q\to 0$. Moreover, $C(q,E)$ does not cross the cut of $\tau(z)$, which is located on positive $z\geqslant 0$. The position of the latter [$K(E)$ in Fig. 6] does not depend on q and is given by

$$\text{Im}\,(E-3q''^2/4m)=0, \quad \text{Re}\,(E-3q''^2/4m)\geqslant 0. \tag{45}$$

It is only for $q^2=mE/3$ that the C and K curves have one common point: $\tilde{q}^2=4mE/3$. The analytic continuation of the Faddeev equation in E requires, generally speaking, the elucidation of the analytic properties of $X_L(q,q';E)\equiv X_L(q,E)$ as a function of the complex variables q and E.

To achieve maximum simplification of our problem, we confine our attention to the homogeneous equation for the eigenvalues $\Lambda(E)$ of the kernel of the Faddeev equation, which has the following form on the physical sheet (we omit the index L and use a system of units in which $\hbar=m=c=1$)

$$\Lambda^{-1}(E)X(q,E)=R(q,E), \tag{46}$$

$$R(q,E)=8\pi\int_0^\infty q''^2dq''\tau(E-3q''^2/4)Z(q,q'';E)X(q'';E). \tag{47}$$

In accordance with the foregoing, Eq. (46) changes in two respects as E moves onto the nonphysical sheet:

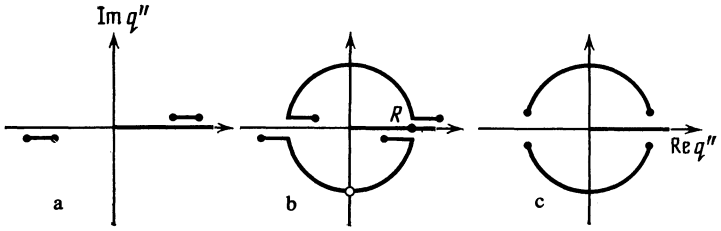

FIG. 5. Scheme of logarithmic cuts of the function $Z_L(q,q'';E+i\varepsilon)$ where q,E are real positive numbers: a) $q\leqslant(mE)^{1/2}$, b) $(mE)^{1/2}\leqslant q\leqslant(4mE/3)^{1/2}$, c) $q>(4mE/3)^{1/2}$.

1. Instead of $\tau(E-3q''^2/4)$, we have $\tau_-(E-3q''^2/4)$, i.e., the t matrix in the kernel of the equation must be taken on the nonphysical sheet of the subsystem energy.

2. The right-hand side of (46) acquires an additional term in the form of an integral of disc $Z(q,\tilde q;E)$ with respect to $C(q,E)$. Hence, on the nonphysical sheet, we obtain

$$\Lambda^{-1}(E)X_-(q,E)=R_-(q,E)+S(q,E), \tag{48}$$

where R_- differs from R since the substitution $\tau\to\tau_-$ has been introduced, and

$$S(q,E)=8\pi\int\limits_{C(q,E)}\tilde q^2\,d\tilde q\,\tau_-\left(E-\frac{3}{4}\tilde q^2\right)\operatorname{disc}Z(q,\tilde q;E)X_-(\tilde q,E). \tag{49}$$

In deriving (48), we have naturally confined our attention to values of q and E for which $X_-(\tilde q,E)$ does not have singularities on $C(q,E)$. The singularities of $X_-(q,E)$ manifest themselves in $S(q,E)$, just as in the case of the Lippmann-Schwinger equation. It is striking that disc $Z(q,\tilde q,E)$ is separable in q and $\tilde q$, since the argument of the vertex functions $\chi(\mathbf{p}_1)$ and $\chi^*(\mathbf{p}_2)$, i.e.,

$$p_1(y)=[{}^1\!/_3(q^2/4+\tilde q^2+q\tilde q y)]^{1/2},$$
$$p_2(y)=[{}^1\!/_3(q^2+\tilde q^2/4+q\tilde q y]^{1/2}, \tag{50}$$

appear in disc $Z(q,\tilde q;E)$ for $y=\bar y=(E-q^2-\tilde q^2)/q\tilde q$, for which the denominator of the three-particle propagator is found to vanish, and the variables $p_1(\bar y)$ and $p_2(\bar y)$ are given by

$$p_1=[{}^1\!/_3(E-3q^2/4)]^{1/2},\quad p_2=[{}^1\!/_3(E-3\tilde q^2/4)]^{1/2}, \tag{51}$$

i.e., they are expressed in terms of the pair energies $z(q)$ and $z(\tilde q)$ for the spectator momenta q and $\tilde q$. For $L\neq0$, the function disc $Z_L(q,\tilde q;E)$ will also contain the Legendre polynomial $P_L(\bar y)$, i.e., the sum of the product of different powers of q and $\tilde q$. The function $S(q,E)$ will then explicitly contain $\chi[{}^4\!/_3(E-3q^2/4)]^{1/2}$ as a factor, i.e., it will have distant (in the case of short-range forces) potential singularities, and can be written as the sum of integrals whose singularities in q are determined by the position of the cut $C(q,E)$ itself, since the functions under the integral signs do not depend on q. Nonpotential singularities of $S(q,E)$ appear when the subsystem has a virtual or resonance pole, i.e., $\tau_-(z)$ has a pole at $z=\nu_p$, that generates the two-particle propagator of the resonance-plus-particle system. $S(q,E)$ is then a Cauchy-type integral whose logarithmic singularities[28] (see below) for $q^{(a)}_{p1,2}$ and $q^{(b)}_{p1,2}$ correspond to the pole of $\tau_-(z)$ at one of the ends of integration.[3)] Finally, $S(q,E)$ has a root cut in the pair energy $z(q)=E-3q^2/4$, since $\sqrt z$ appears in the limits $q^{(a)},q^{(b)}$ of integration along the curve $C(q,E)$ (43):

$$q^{(a,b)}=\mp q/2+(z(q))^{1/2}, \tag{52}$$

although $C(q,E)$ does not, in fact, lie on the root cut of $\tau(z)$ ($z>0$, and real), which has already been noted above. It is convenient to introduce the replacement $\tilde q\to y$ in the integral (49) for $S(q,E)$, using the relation

$$\tilde q(y)=-qy/2+(q^2y^2/4+E-q^2)^{1/2}.$$

The limits of integration are then constants (for $|q|\leqslant\sqrt E$ on the segment $[0,\sqrt E\,]$ and real q): $-1\leqslant y\leqslant1$, and the dependence on q becomes explicit. The result is

$$S(q,E)=i\pi\chi(\mathbf{p}_1(q))\int\limits_{-1}^{1}dy\,w^{-1/2}\tilde q(q/2-w^{1/2})\tau_-(z(\tilde q))\chi^*\times(\mathbf{p}_2(\tilde q))P_L(y)X_-(\tilde q,E), \tag{53}$$

where

$$w=q^2y^2/4+E-q^2,\quad \tilde q=-qy/2+\sqrt w,\quad z(\tilde q)=E-3\tilde q^2/4.$$

The singularities of $S(q,E)$ in q that are nearest to the physical region are determined as indicated above, by the pole of $\tau_-(z(\tilde q))$ at $z=\nu_p$, i.e., by the function $Q_L(\hat y_p)$, where

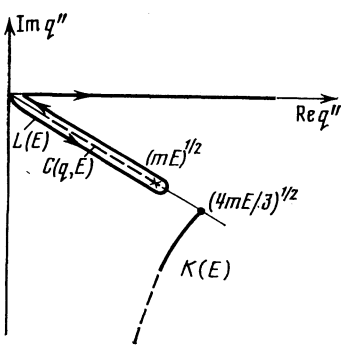

FIG. 6. Scheme of singularities in q'' of the kernel of the Faddeev equation on the neighboring nonphysical sheet of E relative to the threshold at $E=0$.

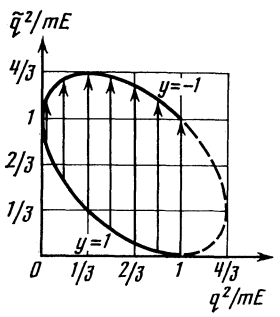

FIG. 7. Regions of integration $C(q,E)$ with respect to the variable $\tilde q$ in $S(q,E)$ for fixed complex E for points q on the segment $[0,(mE)^{1/2}]$. In the region bounded by the broken curve, the logarithmic cut of Z_L departs from the ray containing this segment (see Fig. 5b).

$$\hat{y}_p=[(4\nu_p-E)/3-q^2]/2q[(E-\nu_p)/3]^{1/2}. \quad (54)$$

The resulting logarithmic branch points

$$q^{(a)}_{p1,2}=[(E-\nu_p)/3]^{1/2}\pm\nu_p^{1/2}, \quad q^{(b)}_{p1,2}=-[(E-\nu_p)/3]^{1/2}\pm\nu_p^{1/2} \quad (55)$$

connect the cuts corresponding to the curve $C(q,E)$, i.e., the variation of y from -1 to $+1$, for which

$$q_{p1,2}(y)=-y[(E-\nu_p)/3]^{1/2}\pm[(E-\nu_p)(y^2-1)/3+\nu_p]^{1/2}. \quad (56)$$

The analogous singularities of $X(q,E)$ for $\mathrm{Im}\sqrt{E}>0$ are found in Ref. 30 by a method that seems to us less clear.

It has been assumed above that $X_-(q,E)$ does not have singularities on $C(q,E)$. Once the nearest singularities of $S(q,E)$ connected directly with the kernel of the equation have been obtained, we can find the remaining singularities in q by successively substituting into $S(q,E)$ the terms of $X_-(q,E)$ with the known singularities of $S(q,E)$. In each case, the problem reduces to analyzing an integral of a known function between finite limits.

We are now in a position to write down the analytic representation for $X_-(q,E)$ that explicitly takes into account the nearest singularities found above:

$$X_-(q,E)=\sum_{n=1}\Big[a_n+\sum_p b_n^{(p)}Q_L(\hat{y}_p)\Big]\chi(p_1,\beta n), \quad (57)$$

where Σ_p indicates summation over the poles. The basis of the expansion in (57) is constructed on the basis of the vertex function $\chi(p_1,\beta)$, where $\chi(p_1,\beta)\equiv\chi(p_1)$, and β^{-1} characterizes the interaction range. This ensures that $X_-(q,E)$ decreases with increasing q for large q. Similarly to the number of quadrature nodes in the integral equation, the number of basis functions in the sum over n is determined by the required precision. An analogous basis arises in a natural manner, for example, when the two-body problem is solved by the spectral method described by Martin.[31] When the finite sums over n are used, the substitution of $X_-(q,E)$ in the form given by (57) into (48) gives a set of linear algebraic equations for the parameters a_n and $b_n^{(p)}$, which can be solved in the usual way. Since the position of the three-particle pole in the energy is independent of q, the simplest way is to expand the left- and right-hand sides of (48) into power series around the point $q=0$. The number of terms in the expansion in q^k is then determined by the number of unknown linear parameters a_n and $b_n^{(p)}$ for a fixed maximum number of terms in the sum over n. In the limit as $q\to 0$, the cut $C(q,E)$ contracts to the point $\tilde{q}=\sqrt{E}$, as already noted. The function $X_-(\sqrt{E},E)$ then appears on the right-hand side of (48) and has singularities in E that are connected with the singularities of $S(q,E)$ in the variable q. Substituting $q_{p1}^{(a,b)}=\sqrt{E}$ in (55), we obtain the logarithmic cut in E for the first term of the expansion of $X_-(q,E)$ in powers of q along the segment between the points $E=\nu_p$ and $E=4\nu_p$. In addition, the function $\tau_-(E/4)$ that appears in $S(\sqrt{E},E)$ has a pole at $E=4\nu_p$. The next terms of the expansion in powers of q contain the derivatives at the point $q=\sqrt{E}$. This means that the position of the singularities does not change but they become stronger (instead of logarithm, we have a pole and, instead of a pole of order K, we have a pole of order $K+1$, and so on). Thus, the segment on the plane between the points ν_p and $4\nu_p$ is a forbidden

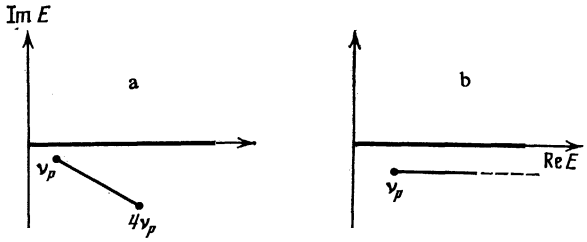

FIG. 8. Structure of nearest singularities of $S(q,E)$ (a) and $R_-(q,E)$ (b) in the variable E.

region for the analytic continuation in E. The cut $[\nu_p, 4\nu_p]$ (Fig. 8a) is analogous to the cut in E that appears in $R_-(q,E)$ and begins at the point $E=\nu_p$, running to the right along the line parallel to the real axis of E (Fig. 8b). The cut of $S(q,E)$ is limited because the region $C(q,E)$ is finite for fixed q and E. It is important to emphasize the additivity of the above singularities of $X_-(q,E)$ in E, which is due to the form of (48) which contains the sum $R_-(q,E)+S(q,E)$ on the right-hand side. We shall illustrate the foregoing by the simple example of the Yamaguchi potential acting only in the S state, which has the form given by (28) with $\chi(\mathbf{p})=(\mathbf{p}^2+\beta^2)^{-1}$. In this case,

$$\mathrm{disc}\, Z_0(q,\tilde{q};E)=\frac{2i\pi}{q\tilde{q}}\left(E+\beta^2-\frac{3q^2}{4}\right)^{-1}\left(E+\beta^2-\frac{3\tilde{q}^2}{4}\right)^{-1}. \quad (58)$$

Hence, the first term in the expansion for $S(q,E)$ into a series in powers of q is

$$S(0,E)=\lim_{q\to 0}S(q,E)=2i\pi\sqrt{E}\,\frac{\tau_-(E/4)}{(E/4+\beta^2)(E+\beta^2)}X_-(\sqrt{E},E). \quad (59)$$

The fact that this expression is finite for $E\neq 0$ is due to the fact that, although $C(q,E)$ contracts to a point as $q\to 0$, and the length decreases in proportion to q because $q_b-q_a=q$ [see (52)], we have disc $Z_0\sim 1/q$ [see (58)]. Equation (59) shows that the function $S(0,E)$ has the following singularities: (1) a root branch point at $E=0$ due to the factor $\sqrt{E}$; (2) a logarithmic cut along the segment $\nu_p\leqslant E\leqslant 4\nu_p$ due to the singularities of $X_-(\sqrt{E},E)$ in the first variable ($q=\sqrt{E}$); (3) a pole at $E=4\nu_p$ due to $\tau_-(E/4)$; (4) distant potential singularities at $E=-\beta^2$ and $E=-4\beta^2$ due to the vertex functions, and (5) singularities of $X_-(\sqrt{E},E)$ in the second variable (E).

As expected, the branches X and X_- corresponding to different sheets of the Riemann energy surface become identical at the branch point at $E=0$, where $S(q,E)$ vanishes more rapidly than $X(q,E)$. As noted above, the right-hand side will also contain the derivatives of $X_-(q,E)$ with respect to q at the point $q=\sqrt{E}$, which enhances the strength of the singularity but has no effect on the region of singularities in E.

Thus, when we consider the analytic continuation of the Faddeev equation to the neighboring nonphysical sheet of E relative to the three-particle threshold at $E=0$, we must bypass the ray segments $\nu_p\leqslant E\leqslant 4\nu_p$. We note, finally, that, to cross the neighboring sheet relative to the two-particle cut of the function $R_-(q,E)$ which is of the resonance-plus-particle type, we must implement the same procedure that was

used on the physical sheet of E relative to the two-particle cut, where the bound state of the subsystem $\{12\}$ appears instead of the resonance or the virtual state.

To conclude this section, we note that, for fixed E and varying q, the $C(q,E)$ curve "floats" on the complex plane of q'' in a somewhat erratic manner, so that Eq. (48) cannot be solved by the usual determinant method, using some kind of quadrature, since it turns out to be nonclosed with respect to the set of quadrature nodes $\{q_i\}$ on the entire contour. This means that the method of solution proposed above, which explicitly uses the more important analytical properties of $X_-(q,E)$, is probably the most natural approach. It can also be used in calculations on the physical sheet of E. As shown in Ref. 30, the implementation of the determinant method is related to the necessity of solving the equations for the two auxiliary functions.

7. VIRTUAL STATES OF THE TRITON

We have calculated the positions of the virtual pole of the nd system for the quantum numbers of the triton (^{3}H) with allowance for the modification (38) and (39) of the kernel of the Faddeev equation during the transition to the nonphysical sheet relative to the threshold of the $n+d$ channel at $E=-\varepsilon_d$, where ε_d is the deuteron binding energy. The position of this pole was first established in Ref. 32 as a result of a fit to the experimental doublet S phase at low energies. The N/D method was used for the same purpose in Ref. 10. The virtual pole of the triton was found in Refs. 10 and 33 by analytic continuation of the t matrix evaluated on the physical sheet with the aid of the three-particle equations (the authors of Ref. 33 used the method put forward in Ref. 34, which is analogous to the Kowalski method[35]) but, because of technical difficulties, the analysis was confined to separable potentials. Since the determination of the virtual level with the aid of our equations is no more complicated than the determination of the binding energy of the triton, we were able to perform the calculations for the "realistic" Malfliet-Tjon (MT) local potential (variants I and III in Ref. 36). The starting point was the set of equations for the bound triton [Eqs. (8) in Ref. 3], obtained by Bateman's method in which the potential was approximated by a sum of separable terms.[4)] In accordance with (38) and (39), the following substitution was made in these equations[3] [i,j are the numbers of the Bateman cuts and $\lambda=s(t)$ is a singlet (triplet) in the spin of $\{12\}$]:

$$K_{ji}^{\lambda\lambda'}(Q,p)\to\hat{K}_{ji}^{\lambda\lambda'}(Q,p), \tag{60}$$

$$\hat{K}_{ji}^{\lambda\lambda'}(Q,p)=K_{ji}^{\lambda\lambda'}(Q,p)-V_j^{\lambda t}(Q,q)\frac{a^{\lambda\lambda'}iqC}{1+iqC\mathcal{F}^{tt}(q,q)}V_i^{\lambda' t}(p,q),$$

where

$$C=4/3\pi^2m^2,\ a^{tt}=a^{ts}=a^{st}=1,\ a^{ss}=9,$$

$$V_i^{\lambda\lambda'}(q_1,q_2)=\sum_l R_l K_{il}^{\lambda\lambda'}(q_1,q_2),$$

$$\mathcal{F}^{tt}(q,q)=\sum_{m,k}R_m K_{mk}^{tt}(q,q)R_k,$$

$$R_i=A_i^{(t)}(-1)^i/[\Delta_t'(-\varepsilon_d)]^{1/2},\quad q=[(4m/3)(E+\varepsilon_d)]^{1/2}.$$

The quantities R_i define the residue of the pair t matrix for np scattering at the pole corresponding to the bound deuteron [see Ref. 3, Eqs. (10)–(13)].

The calculated position B_v of the virtual triton level below the elastic threshold, i.e., for the three-particle energy $E_v=-\varepsilon_d-B_v$ ($\varepsilon_d=2.31$ MeV for the MT potential) is given in Table I together with the results of the other calculations,[10,32,33] cited above. The corresponding triton binding energies E_T are given for convenience.

It is clear from Table I that B_v is sensitive to the potential, and our result for the local central potential containing repulsion at short distances is close to the experimental result.[32]

8. CONCLUSION

We have thus been able to formulate new integral equations for the virtual and resonance states, which are based on the analytic continuation of the integral equations of the theory of scattering. This constitutes a direct generalization of the bound-state problem to the case of unstable states. It is important to note that the method that we have developed can also be used to find amplitude singularities that are hidden in the original equation, and this is essential for the practical implementation of analytic continuation itself. The equations for the three nucleon system have been used for the first time to find the energy of the antibound state of the triton for a "realistic" potential with repulsion, and the result of the calculations is in agreement with experiment. There are numerous other possible physical applications of the method. They include the discretization of the basis in calculations of nuclear–reaction cross sections with allowance for the continuous spectrum in the final state. An interesting problem for the three-body system is the analysis of the trajectories of poles whose nature was established by Efimov.[38] Sufficiently complete and reliable theoretical information about the poles of the three-neutron system is not as yet available (see Refs. 39 and 40). Further theoretical studies of dibaryon resonances will be necessary (see Ref. 13

TABLE I.

	[32], experiment	[10], N/D- method	[10]	[33]	present work
B_v, MeV	0,515	0,482	0.489	0,381	0,502
E_T, MeV	8,48	8,48*	8,48*	11	8,23

*The experimental value of E_T was used in Ref. 10 to find the parameters of a separable potential yielding the best fit.

and the references therein). Finally, there is considerable interest in resonances exhibited by nuclear systems containing an antinucleon,[41–43] especially in connection with the experiments on the antiproton storage ring (LEAR) that began at CERN in 1983. The above method of analytic continuation of integral equations will probably be useful for the various resonance phenomena that are frequently encountered in physics generally. Its universality is due to the fact that it can be used with any integral equation with a Cauchy kernel. It is important that our method provides a recipe for finding not only the poles of the S matrix, but also the normalized Gamow wave functions of quasistationary states,[5)] and we have pointed out the rules for the evaluation of the matrix elements involving these functions.

The authors are greatly indebted to V. B. Belyaev, L. D. Blokhintsev, V. N. Efimov, and L. D. Faddeev for fruitful discussions. They are particularly indebted to I. S. Shapiro for reading this paper in manuscript and for contributing a number of valuable suggestions.

[1)]The fact that the pole term of the t matrix can be written in the form of (13) (without the complex conjugate) is closely connected with the use in the extended completeness condition of the so-called biorthogonal system of resonance (Gamow) wave functions (see, for example, Ref. 19). It appears in a natural manner in S-matrix theory where this term corresponds to the Feynman pole diagram in the s channel (see, for example, Ref. 20).

[2)]The complex conjugation symbol refers to the spherical harmonic in $\chi(\mathbf{p})$.

[3)]It is precisely these singularities (which play an important role when the two-particle subsystem has a level close to the threshold) that are responsible for the effective long-range interaction in three-particle systems, which determines their dynamic properties at low energies. The physical nature of the effective long-range interaction is examined for the three-nucleus system in Ref. 29.

[4)]The author of Ref. 3 appears to be the first to report calculations on the "singlet deuteron", based on the analytic continuation of the t matrix, found by solving the Lippmann-Schwinger equation with different Bateman-separable local potentials. An analogous method, using some other methods of approximation to the t matrix by a finite sum of separable terms, was recently employed in Ref. 37.

[5)]The formalism that can be used with Gamow wave functions is now well developed. An extended bibliography on this equation is given, for example, in Ref. 44.

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Translated by S. Chomet