Texture-spin waves in nonequilibrium states of superfluid ^{3}He-B

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(Submitted 12 December 1983)
Zh. Eksp. Teor. Fiz **86**, 2100–2110 (June 1984)

A novel type of spin waves in superfluid ^{3}He-B is studied, waves which may be regarded as a long-wave spatial modulation of the nonlinear magnetization ringing. The speed of propagation of these waves is anomalously small, by an order of magnitude smaller than the speed of usual spin waves in superfluid ^{3}He. Arguments are presented that these are the waves which were observed in the known experiments[1] on propagation of magnetic excitations in ^{3}He-B.

INTRODUCTION

Spin waves in nonequilibrium states of ^{3}He-B were first detected in Ref. 1, where the relaxation of magnetization in weak magnetic fields was studied. In order to detect these waves (which in the sequel will be called WSW waves) use has been made of general properties of the so-called WP mode of nonlinear magnetization ringing in ^{3}He-B. In Ref. 1 the WP-mode was generated in the following manner. A sample of ^{3}He-B was placed in a parallelepipedal cavity and an external field was applied parallel to one of the walls of the cavity. The magnitude of the field did not exceed that of the dipolar field, which in Ref. 1 was of the order of 10 gauss. After the system attained equilibrium the external field was switched of suddenly, i.e., over an interval much shorter than the relaxation time. This provoked a precession of the spin of the system, giving rise to a clear long lived signal. In a series of experiments the field was not switched off completely but only to some value H, called the residual field; an induction signal was observed also in this case. The high quality and the duration of the signal of the WP mode were used in Ref. 1 for the detection of the WSW waves, which could not be detected by usual methods. In Ref. 1 these waves were excited by a sudden change of the field component longitudinal to the direction of wave propagation at a point far removed from the pickup coil. The effect of propagation of a spin wave was estimated from its action on the WP mode. If the WSW wave arrives on time, the configuration required for the generation of the WP mode may be destroyed, and the signal of the WP mode is detected poorly or not at all. The results obtained in Ref. 1 show that there are two modes with a lifetime of several tens of milliseconds, and propagating in a small residual field of the order of 30% of the dipole field with a group velocity of several tens of cm/s. More precisely, at a pressure of 20.7 bar, a temperature $1-T/T_c = 0.02$, and a residual field of the order of 5 gauss, the speeds of propagation were 17 and 21 cm/s, i.e., differ by an order of magnitude from the corresponding speeds of the spin waves in spatially homogeneous ^{3}He-B at equilibrium.

The spin waves studied in the present paper seem to have a direct relation to the WSW waves. They may be considered as waves of spatial modulation of the WP mode, when over a sufficiently small scale the configuration of the spin and of the order parameter may be considered spatially homogeneous, whereas at the scale of the device there is no spatial homogeneity. The analysis of the magnitudes of the dynamical variables involved in the process shows (see below) that there exists a clear cut separation between the region of high frequencies, corresponding to the spatially homogeneous regime, and the region of low frequencies, corresponding to the propagation of large-scale spatial excitations. The group velocity of the waves studied in the present paper is given by the equation

$$v_g = 2\varkappa k c_\parallel^2/\gamma H, \tag{1}$$

where k is the wave number, γ is the gyromagnetic ratio for ^{3}He, $c_\parallel$ is the velocity of the longitudinal wave, H is the switched-off magnetic field required to generate the WP mode, $\varkappa$ is a dimensionless factor of order unity. It is interesting to note that for wavelengths of the order of 3.4 mm (the size of the generating coil in Ref. 1), a pressure of 20.7 bar, a temperature of $1-T/T_c = 0.02$, and a field $H = 5$ gauss (the data of Ref. 1) the velocity v_g turns out to be 13.8 cm/s (cf. infra, Sec. 3) in good qualitative agreement with the results of Ref. 1, namely 17 and 21 cm/s for the two observed modes.

1. LONG TEXTURE-SPIN WAVES IN ^{3}He-B

The order parameter for ^{3}He-B, $A_{ij} = (\Delta/\sqrt{3})e^{i\varphi}R_{ij}$ is parametrized in the present paper by the angle θ and the axis c_i of the rotation matrix

$$R_{ij} = \cos\theta\delta_{ij} + (1-\cos\theta)c_ic_j - \varepsilon_{ijk}\sin\theta c_k.$$

It is assumed that there exists a dependence only on one spatial variable z. The Leggett-Takagi (LT) equations of spin dynamics in superfluid ^{3}He in the presence of spatial inhomogeneities are given by[2–6]:

1) The Poisson brackets of the spin components and of the order parameters

$$[S_i(\mathbf{r}_1;t); S_j(\mathbf{r}_2;t)] = \varepsilon_{ijk}S_k(\mathbf{r}_1;t)\delta(\mathbf{r}_1-\mathbf{r}_2),$$
$$[S_i(\mathbf{r}_1;t); A_{jm}(\mathbf{r}_2;t)] = \varepsilon_{ijk}A_{km}(\mathbf{r}_1;t)\delta(\mathbf{r}_1-\mathbf{r}_2),$$
$$[A_{ij}(\mathbf{r}_1,t); A_{km}(\mathbf{r}_2,t)] = 0;$$

2) The Leggett Hamiltonian with gradient terms

$$\hat{H} = \hat{H}_L + \hat{H}_\nabla, \quad \hat{H}_L = \frac{1}{2}\gamma^2\chi^{-1}\mathbf{S}^2 - \gamma\mathbf{HS} + U_D,$$

$$\hat{H}_\nabla = \int dz\left[\frac{1}{2}\gamma_1\left(\frac{\partial}{\partial z}R_{ij}\right)^2 + \frac{1}{2}\gamma_2\left(\frac{\partial}{\partial z}R_{i3}\right)^2\right];$$

3) The dissipative function

$$F = \frac{1}{2}\mu\left(\frac{\partial}{\partial\theta}U_D\right)^2.$$

 0038-5646/84/061221-06$04.00

Here γ, χ are the gyromagnetic ration and the susceptibility, U_D is the dipolar energy, $U_D = g_D(\cos\theta + \frac{1}{4})^2 + U_0$. It is convenient to use the dimensionless variables

$$\mathbf{S}_R = \gamma^2\chi^{-1}\Omega^{-1}\mathbf{S}, \quad t_R = \Omega t, \quad z_R = L^{-1}z, \quad \mathbf{H}_R = \gamma\Omega^{-1}\mathbf{H};$$

where Ω is the Leggett frequency, and L is the spatial scale. In the sequel the index R will be omitted everywhere. In the indicated variables the LT equations have the form

$$\frac{\partial}{\partial t} S_i = [\mathbf{S}\times\mathbf{H}]_i + \frac{16}{15}\sin\theta\left(\cos\theta + \frac{1}{4}\right)c_i$$

$$-\frac{1}{2}\left(\frac{c_\parallel}{L\Omega}\right)^2 \varepsilon_{ijk}R_{km}\frac{\partial^2}{\partial z^2}R_{jm} - \frac{c_\perp^2 - c_\parallel^2}{(L\Omega)^2}\varepsilon_{ijk}R_{k3}\frac{\partial^2}{\partial z^2}R_{j3},$$

$$\frac{\partial}{\partial t}c_i = \frac{1}{2}[(\mathbf{S}-\mathbf{H})\times\mathbf{c}]_i + \frac{1}{2}\operatorname{ctg}\frac{1}{2}\theta(\mathbf{S}-\mathbf{H})_i$$

$$-\frac{1}{2}\operatorname{ctg}\frac{1}{2}\theta[(\mathbf{S}-\mathbf{H})\times\mathbf{c}]c_i, \qquad (2)$$

$$\frac{\partial}{\partial t}\theta = (\mathbf{S}-\mathbf{H})\mathbf{c} + \frac{16}{15}\frac{\Gamma_\parallel}{\Omega}\sin\theta\left(\cos\theta + \frac{1}{4}\right).$$

Here $c_\parallel$ and $c_\perp$ are the speeds of the longitudinal and transverse spin waves (see Ref. 7), $\Gamma_\parallel$ is the width of the longitudinal NMR lines which determines the relaxation term.[2] In the sequel the coefficients in front of the gradient terms will be denoted by

$$K_1 = \frac{1}{2}\left(\frac{c_\parallel}{L\Omega}\right)^2, \quad K_2 = \frac{c_\perp^2 - c_\parallel^2}{(L\Omega)^2}.$$

In the weak-coupling approximation[7] the relation $2K_1 = K_2$ holds.

We estimate the magnitude of the terms entering into the LT equations in the range of temperatures and pressures investigated in Ref. 1: pressure 20.7 bar, temperature $1-T/T_c = 0.02$–0.05, external field of the order of 5 gauss. In this range the Leggett frequency is given by

$$\Omega = 2\pi\sqrt{12}(1-T/T_c)^{1/2}\cdot 10^5 \text{ [rad}\cdot\text{s]}^{13},$$

i.e., $\Omega = (3.078$–$4.867)\times 10^5$ rad/s. From the results of Ref. 2 [it suffices to compare equations (6.13) and (6.31) in Ref. 2] it follows that the relations $\Gamma_\parallel = (\alpha/3\pi^2)\Omega$, holds, where α is the constant in the relaxation law of the square of the period of the WP mode $f^{-2} = f_0^{-2} + \alpha t$ (see Ref. 2). In Ref. 1 the temperature-dependence $\alpha^{-2} = 5.8\times 10^{10}$ $(1-T/T_c)$ was found for α, hence $\Gamma_\parallel/\Omega = 0.305$.

For the correct estimate of the magnitude of the dipolar term one must take into account the magnitude of the deviations of θ from the equilibrium value $\theta_0 = \arccos(-\frac{1}{4})$. In the absence of an external field one may assume that the system is near the regime of the WP mode; then the following formula derived by Brinkman[8] holds: $\theta - \theta_0 = \sqrt{3/5}(\gamma H/\Omega)^2$ (see also the following section), where H is the field being switched off. If there is a residual field of magnitude sufficiently small compared to the dipolar field, then the indicated formula remains valid in order of magnitude. For the data of Ref. 1 the field is of the order of 5 gauss, which in the range of temperatures under consideration is of the order of 30% of the dipole field. If one makes use of the Brinkman formula listed above, the dipole term will be of the order 0.04, the dissipative term is of the order 0.01, and the term involving the magnetic field is of the order of 0.09.

In order to estimate the gradient terms the spatial scale must be chosen correctly. The characteristic scales in Ref. 1 are of the order of several millimeters (the sizes of the coils are 3.4 mm, the distance between them 6.1 mm), and therefore one may choose 1 mm as the characteristic scale. In the weak coupling approximation this implies (see Sec. 3) that $2K_1 = K_2 = 3\times 10^{-5}$. Thus the gradient terms are by three orders of magnitude smaller than all other terms and can consequently be treated perturbatively. It is important for the sequel that the characteristic frequencies of spatially homogeneous spin dynamics in the range of temperatures and field strengths under consideration is of the order 10^4–10^5 rad/s, whereas the frequencies of spin waves at that scale of lengths (of the order of 1 mm) is of the order not exceeding 10^3 rad/s, since in the region of temperatures and pressures under consideration $c_\parallel \sim 200$ cm/s (Ref. 7, *vide infra*). One can thus carry out an averaging over the rapidly varying variables which are related to the spatially homogeneous dynamics, and derive equations for the mean values describing the spin waves on the configuration of the WP mode.

2. THE EQUATIONS FOR THE MEAN VALUES

We assume that the residual external fields are such that the term involving the magnetic field is much smaller than the dipole term and the relaxation term, and that in the first approximation the system may be considered spatially homogeneous.

The spin dynamics in ^{3}He-B in the absence of spatial inhomogeneities is characterized by the stationary solutions of the dissipation-free Leggett equations. To these solutions correspond long-lived modes of the nonlinear magnetization ringing. In the regime of switched-off external field there are only two such modes: the WP mode[8] and the attractor regime.[9,10] In the presence of a residual field there are several modes of stationary solutions, a complete set of which has been recently described by Fomin.[11] An essential trait of the long-lived modes is the fact that the system tends to reach a regime which is close to one of them over a time which is much shorter than the relaxation time. When an external field of magnitude smaller than a certain threshold value[1,9] is completely switched off, the system goes into the WP mode. If the residual fields is much smaller than the dipole field, one may assume as a first approximation that the system moves near a WP mode, and treat the influence of the residual field perturbatively.

The stationary solutions lose their meaning in the presence of sufficiently large gradient terms; this is a manifestation of the difference between the latter and the dissipative term, which does not destroy the long-lived modes related to stationary solutions, but only forces the system to be in the vicinity of one of them, whereas spatial inhomogeneities tend to force the system out of a long-lived mode. However, if the latter are sufficiently small compared to the dissipative term, their influence on the spin configuration and on the order parameter reduces only to a spatial modulation which is small in amplitude and can be taken into account perturbatively.

The method used in the present paper is based on the use of an integral of the spatially homogeneous dissipation-free Leggett equations, namely the Brinkman-Cross vector (cf. the review paper[12]) which has the form

$$\mathbf{J}=\sin^2\frac{1}{2}\theta\left[\operatorname{ctg}\frac{1}{2}\theta[\mathbf{S}\times\mathbf{c}]-\mathbf{S}+(\mathbf{S}\mathbf{c})\mathbf{c}\right].$$

In the presence of dissipation, external fields, or gradient terms the vector **J** is not conserved, and this allows one to use it for a quantitative estimate of the role played by the indicated factors.

It is very important that **J** allows one to describe completely the dynamics of the spatially homogeneous WP mode. In the regime of the WP mode the angle θ is a constant, the vector **c** is perpendicular to **J**, the spin **S** is perpendicular to **c** and forms the angle $\frac{1}{2}(\theta+\pi)$ with **J**. Both vectors **S** and **c** rotate around **J** with constant angular velocity

$$\boldsymbol{\omega}=-\frac{1}{2}\sin^{-2}\left(\frac{1}{2}\theta\right)\mathbf{J}. \tag{3}$$

The equations of the WP mode are obtained by equating to zero the right-hand sides of the nondissipative Leggett equations[8]; they yield an equation relating the magnitude of the vector **J** to the angle θ:

$$\mathbf{J}^2=-\frac{64}{15}\sin^4\frac{1}{2}\theta\left(\cos\theta+\frac{1}{4}\right). \tag{4}$$

The last equation may be considered as an equation of state for the WP mode. In turn the spin **S** and the axis **c** are determined by the vector **J** in the form

$$\mathbf{S}=-\mathbf{J}-\operatorname{ctg}{}^1/_2\theta[\mathbf{J}\times\mathbf{c}],\quad \mathbf{c}=\mathbf{u}\cos\psi+\mathbf{v}\sin\psi. \tag{5}$$

Here ψ is the phase and **u** and **v** are two vectors perpendicular to each other and to **J**. In the presence of dissipation the system will evolve in time remaining near the WP mode. We show that from the fact that the equations (4) and (5) are valid it follows that the linear Leggett-Takagi[2] relaxation law holds:

$$f^{-2}=f_0^{-2}+\alpha t,$$

where f is the frequency of the WP mode.

It follows from the equations (2) that, in the presence of dissipation but in the absence of an external field and of spatial gradients, the vector **J** satisfies the equation

$$\frac{\partial}{\partial t}\mathbf{J}=\frac{8}{15}\frac{\Gamma_{\parallel}}{\Omega}\sin\theta\left(\cos\theta+\frac{1}{4}\right)[\cos\theta[\mathbf{S}\mathbf{c}]-\mathbf{S}\sin\theta].$$

The right-hand sides of the last equation are small, and we average them over the period of the basic solution of the unperturbed problem for which **J** is a constant of the motion. This basic solution is the WP mode defined by the equations (4) and (5). Making use of the averages

$$\langle[\mathbf{S}\mathbf{c}]\rangle=\operatorname{ctg}({}^1/_2\theta)\mathbf{J},\quad \langle\mathbf{S}\rangle=-\mathbf{J}$$

and of the equation (4) for the angle θ we obtain the equations for the mean value $\langle\mathbf{J}\rangle$ (the symbol $\langle\rangle$ will be omitted in the sequel)

$$\frac{\partial}{\partial t}\mathbf{J}=-\frac{1}{4}\frac{\Gamma_{\parallel}}{\Omega}\cos^2\frac{1}{2}\theta\sin^{-4}\left(\frac{1}{2}\theta\right)J^2\mathbf{J}. \tag{6}$$

For fields of the order of 30% of the dipole field θ is close to the equilibrium value $\theta_0=\arccos(-\frac{1}{4})$. Therefore, considering θ as constant, we have

$$J^{-2}=J_0^{-2}+\left[\frac{1}{2}\frac{\Gamma_{\parallel}}{\Omega}\cos^2\frac{1}{2}\theta\sin^{-4}\frac{1}{2}\theta\right]t.$$

This yields the linear law

$$f^{-2}=f_0^{-2}+\alpha t,\quad \alpha=8\pi^2\cos\left(\frac{1}{2}\theta\right)\Gamma_{\parallel}/\Omega.$$

For $\theta=\arccos(-\frac{1}{4})$ we have $\Gamma_{\parallel}=(\alpha/3\pi^2)\Omega$, in complete agreement with the relation obtained above from the microscopic equations of Ref. 2. By means of similar reasoning one can see that the deviations $\delta\theta$ from the equilibrium values exhibit a linear behavior of the type $\delta\theta^{-1}=\delta\theta^{-1}+\beta t$. It is important to note that the equation for the time dependence of θ is obtained from the equation (6) for the mean **J** and the equation of state (4) for the WP mode. A direct averaging of the LT equation for θ is incorrect, and leads to a wrong result: the relaxation time of θ to equilibrium comes out to be several microseconds, whereas it is well known, and also from experiment,[1] that this time coincides with the lifetime of the WP mode, and is of the order of several milliseconds.

It follows from the above considerations that it is convenient to characterize the spatial modulation of the WP mode through the dependence of the vector **J** on the point in space. In order to derive the appropriate equation we first list the exact (not the averaged) equations for the vector **J**, equations which follow from the complete LT equations (2). They can be written in terms of Poisson brackets in the following compact form:

$$\frac{\partial}{\partial t}J_i=-[J_i;\mathbf{HS}]+[J_i;\hat{\mathbf{H}}_\nabla]+\frac{1}{2}\{\cos\theta[\mathbf{S}\mathbf{c}]_i-S_i\sin\theta\}\frac{\partial}{\partial t}\theta. \tag{7}$$

The first two terms of the equations (7) are generated by the gradient part of the Hamiltonian and by the external field; each of them is considered a small perturbation of the original Hamiltonian $\hat{H}_L$. It should be noted that $[\mathbf{J};\hat{H}_L]=0$. The third term is generated by the dissipative term in Eqs. (2). The right-hand sides of Eq. (7) are averaged over the basic solution (4), (5) of the unperturbed Hamiltonian system with Hamiltonian $\hat{H}_L$. Thus a method analogous to the adiabatic approximation of quantum mechanics is used. It is worth noting that this method was used already by Laplace for a calculation of planetary orbits.

In the present paper the dynamics of the spin and of the order parameter are considered in the linear approximation, i.e., the deviations of **J** from some value $\mathbf{J}_0$ which does not depend on the point of space, are considered small and the products of gradients are neglected. It is assumed that everything depends only on one spatial coordinate z and the derivatives with respect to z are denoted by primes: $f'=\partial f/\partial z$. With these assumptions the averaged equation (7) has the form

$$\frac{\partial}{\partial t}J_i=\left(1+\frac{1}{2}\cos\theta\right)H\varepsilon_{3ik}J_k-2K_1(1-\cos\theta)\langle[\mathbf{c}\mathbf{c}'']_i\rangle$$
$$-\frac{1}{4}K_2\varepsilon_{3ij}w_jw_3\theta''\sin\theta$$

$$+\frac{1}{2}K_2\{\varepsilon_{3ij}\langle c_3c_j''\rangle[\sin^2\theta+\cos\theta(1-\cos\theta)^2]$$
$$-\varepsilon_{3ij}\langle c_3''c_j\rangle(1-\cos\theta)^2$$
$$-\delta_{i3}\langle[\mathbf{cc}'']_3\rangle\sin\theta-\langle c_3^2[\mathbf{cc}'']_i\rangle(1-\cos\theta)^3 \qquad (8)$$
$$+\langle c_3c_i[\mathbf{cc}'']_3\rangle[(1-\cos\theta)^2-\sin^2\theta(1-\cos\theta)]\}$$
$$-\frac{1}{4}\frac{\Gamma_{\parallel}}{\Omega}\cos^2\frac{1}{2}\theta\sin^{-4}\frac{1}{2}\theta J^2J_i.$$

Here $w_i = J_i/|\mathbf{J}|$, ε_{ijk}-is the totally antisymmetric third rank tensor (the Levi-Civita symbol), $\langle\rangle$ denotes averaging over the period of the WP mode. In order to calculate the averages $\langle c_3c_j''\rangle$, $\langle c_3''c_j\rangle$, etc. a convenient parametrization of the vectors **u**, **v**, and **w** which define the configuration of the WP mode [cf. Eq. (5)] at a point of space, is very important. The vectors **u**, **v**, and **w** form an orthonormal frame which, when moving from point to point, undergoes a three-dimensional rotation R. In the linear approximation the rotation matrix is close to the unit matrix $R = 1 + \delta R$. To the infinitesimal rotation δR corresponds a vector $\delta\alpha$ such that the change of a vector $\mathbf{X}$ under such a rotation R is $\delta\mathbf{X} = \delta\boldsymbol{\alpha}\times\mathbf{X}$. Thus, the derivative of the vector $\mathbf{X}$ with respect to z will have the form $\mathbf{X}' = \delta\boldsymbol{\beta}\times\mathbf{X}$ with $\boldsymbol{\beta} = \delta\alpha/\delta z$. In the linear approximation, when the squares of gradients (i.e., terms quadratic in β) are neglected, we obtain for the second derivative with respect to z the result $\mathbf{X}'' = \mathbf{A}\times\mathbf{X}$. With respect to the frame **u**, **v**, **w** it is convenient to choose the gauge for the vector $\mathbf{A}$ such that $\mathbf{A}\cdot\mathbf{w} = 0$ and then $\mathbf{u}'' = -(\mathbf{uw}'')w$, $\mathbf{v}'' = -(\mathbf{vw}'')\mathbf{w}$. In calculating the averages in Eq. (8) the values of the coordinates of the vectors **u**, **v**, **w** may be set equal to the coordinates of constant vectors $\mathbf{u}_0$, $\mathbf{v}_0$ $\mathbf{w}_0$, around which they oscillate. From the invariance of the system with respect to rotations around the z axis it follows that **w** can be selected in the form $\mathbf{w} = (0,w_2,w_3)$. Indeed, the right-hand sides of the averaged equations (cf. infra) depende only on the coordinates u_3, v_3, and w_3 of the frame **u**, **v**, **w**.

The results of the averaging of the right-hand side of the equations (7) is conveniently represented in the form of an equation for the magnitude $J = |\mathbf{J}|$ and two equations for the coordinates w_u, w_v of the vector **w** relative to the axes $\mathbf{u}_0$, $\mathbf{v}_0$. In doing this it is convenient to use the general formulas

$$\frac{\partial}{\partial t}X=\mathbf{n}\frac{\partial}{\partial t}\mathbf{X},\quad \frac{\partial}{\partial t}n_i=X^{-1}(\delta_{ij}-n_in_j)\frac{\partial}{\partial t}X_j,$$

where $\mathbf{X}$ is a time-dependent vector, X is its magnitude, and **n** is the unit vector $\mathbf{n} = X^{-1}\mathbf{X}$.

In this section we consider the case when the residual field **H** is absent; the influence of the residual field is discussed in Section 3.

Under the formulated assumptions, the equations for J, w_u, w_v have the form

$$\frac{\partial}{\partial t}J=-\left[\frac{5}{4}+\frac{65}{64}(u_3^2+v_3^2)+\frac{15}{32}w_3^2\right]K_2\psi''$$
$$-\frac{5}{32}K_2w_3(v_3w_u''-u_3w_v'')-\frac{6}{25}\frac{\Gamma_{\parallel}}{\Omega}J^3, \qquad (9)$$

$$J\frac{\partial}{\partial t}w_u=A_{uu}w_u''+A_{uv}w_v''+\frac{\sqrt{15}}{16}w_3v_3K_2\theta''+\frac{5}{32}w_3u_3\psi'', \qquad (10)$$

$$J\frac{\partial}{\partial t}w_v=A_{vu}w_u''+A_{vv}w_v''-\frac{\sqrt{15}}{16}w_3u_3K_2\theta''+\frac{5}{32}w_3v_3\psi'', \qquad (11)$$

where the coefficients A_{uu}, A_{uv}, A_{vv}, and A_{vu} in the weak-coupling approximation ($2K_1 = K_2$) is given by the formulas

$$A_{uv}=K_2\left[\frac{5}{8}+\frac{515}{1024}v_3^2+\frac{365}{1024}u_3^2+\frac{25}{64}w_3^2+\frac{25}{1024}(v_3^2-u_3^2)\right],$$
$$A_{vu}=-K_2\left[\frac{5}{8}+\frac{515}{1024}u_3^2+\frac{365}{1024}v_3^2+\frac{25}{64}w_3^2+\frac{25}{1024}(u_3^2-v_3^2)\right], \qquad (12)$$
$$A_{uu}=-A_{vv}=\frac{75}{512}K_2u_3v_3.$$

The coefficients in Eq. (12) have been calculated for the equilibrium value $\theta = \arccos(-\frac{1}{4})$, since in the linear approximation one can ignore the small deviations of θ from the equilibrium value.

It is convenient to reduce Eq. (9) to an equation for the phase ψ. For this we note that **J** may be decomposed into a sum $\mathbf{J} = \mathbf{J}_{SH} + \delta\mathbf{J}$, where $\mathbf{J}_{SH}$ obeys the spatially homogeneous relaxation equation (6) and $\delta\mathbf{J}$ corresponds to the spatial inhomogeneities. In the linear approximation we obtain for $|\delta\mathbf{J}|$ the equation

$$\frac{\partial}{\partial t}\delta J=-K_2\left[\frac{5}{4}+\frac{15}{32}w_3^2+\frac{65}{64}(u_3^2+v_3^2)\right]\psi''$$
$$-\frac{5}{32}K_2w_3(v_3w_u''-u_3w_v'')-\frac{18}{25}\frac{\Gamma_{\parallel}}{\Omega}J_{SH}^2\delta J. \qquad (13)$$

According to Eq. (3) the resolution of **J** into a $\delta\mathbf{J}$ and $\mathbf{J}_{SH}$ corresponds to a resolution of the angular velocity $\omega = \omega_{SH} + \delta\omega$ and of the phase $\psi = \psi_{SH} + \delta\psi$ into a spatially homogeneous and dephasing parts. From Eq. (3) and Eq. (4) we get

$$-\frac{1}{2}\sin^{-2}\left(\frac{1}{2}\theta\right)\frac{\partial}{\partial t}\delta J=\frac{\partial^2}{\partial t^2}\delta\psi$$
$$-\frac{1}{2}\frac{\Gamma_{\parallel}}{\Omega}J^4\cos^3\frac{1}{2}\theta\sin^{-5}\frac{1}{2}\theta$$
$$\times\left[2J^2\operatorname{ctg}\frac{1}{2}\theta+\frac{64}{15}\sin\theta\sin^4\frac{1}{2}\theta\right]^{-1}\frac{\partial}{\partial t}\delta\psi.$$

Note that $\psi'' = \delta\psi''$. Making use of the relation between δJ and $\delta\psi$ given above, we obtain the phase shift equation

$$\frac{\partial^2}{\partial t^2}\delta\psi-K_2\left[1+\frac{3}{8}w_3^2+\frac{13}{16}(u_3^2+v_3^2)\right]\delta\psi''$$
$$-\frac{1}{8}K_2w_3(v_3w_u''-u_3w_v'')+\frac{\Gamma_{\parallel}}{\Omega}J^2Z\frac{\partial}{\partial t}\delta\psi=0,$$

$$Z=\frac{18}{15}-\frac{1}{2}J^2\cos^3\frac{1}{2}\theta\sin^{-5}\frac{1}{2}\theta$$
$$\times\left[\frac{64}{15}\sin^4\frac{1}{2}\theta\sin\theta+2J^2\operatorname{ctg}\frac{1}{2}\theta\right]^{-1}. \qquad (14)$$

For $\theta = \arccos(-\frac{1}{4})$ and switched-off dipole fields of the order of 30% the dipole field, we have $Z\simeq 0.3$. One can eliminate θ'' from the equations (10) and (11). For this purpose one must first express θ'' in terms of J'' with the aid of Eq. (4), and then in terms of the phase shift $\delta\psi$ by means of Eq. (3). We finally obtain the equations for w_u and w_v in the form

$$J\frac{\partial}{\partial t}w_u=A_{uu}w_u''+A_{uv}w_v''-\frac{3}{8}K_2w_2w_3\frac{\partial}{\partial t}\delta\psi''+\frac{5}{32}K_2w_2w_3\delta\psi'',$$
$$J\frac{\partial}{\partial t}w_v=A_{vu}w_u''+A_{vv}w_v''+\frac{3}{8}K_2w_2w_3\frac{\partial}{\partial t}\delta\psi''+\frac{5}{32}K_2w_2w_3\delta\psi'', \quad (15)$$

where the coefficients A_{uu}, A_{uv}, A_{vu}, A_{vv} are given by Eqs. (12). It should be noted that their form depends on the choice of **u**, **v**. The most symmetric expressions for the coefficients are obtained if one chooses **u**, **v** such that $u_3=v_3$, $u_1=-v_1$. Then one obtains the equality $A_{uv}=-A_{vu}$.

The equations (14) and (15) split if the vector $\mathbf{w}=\mathbf{w}_0$ is parallel to the z axis: $u_3=v_3=0$. One should keep in mind that the coefficients in front of the gradient terms contain the coordinates of the vectors $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{w}_0$ (*vide supra*).

The equations (14) and (15) represent a complete set of linearized averaged equations for the phase shift $\delta\psi$ and for the oscillations of the precession axis **w**.

The equations (14) and (15) yield a dispersion law for the waves proportional to $\exp i(kz-wt)$. For this purpose it is convenient to choose the gauge of the frame **u**, **v**, **w** so that $u_3=0$. Then $v_3=w_2$, $A_{uu}=-A_{vv}=0$. The coefficients A_{uv} and A_{vu} are of the order of K_2 and one can neglect the small difference between their magnitudes and set $A_{uv}=-A_{vu}=\Omega=K_2$.

After the indicated simplifications the dispersion law takes the form

$$(J^2\omega^2+A_{uv}A_{vu}k^4)\left(\omega^2-Bk^2-iZ\frac{\Gamma_\parallel}{\Omega}J^2\omega\right)+{}^1/_8K_2(w_2w_3)^2k^2({}^5/_{32}K_2A_{uv}k^4-{}^3/_8K_2k^2J\omega^2)=0, \quad (16)$$
$$B={}^1/_8K_2(11+7w_2^2). \quad (17)$$

We consider the case when **w** is collinear with z, i.e., $w_2=0$. The dispersion law splits into an equation for the phase shift and an equation for the oscillations of the vector **w**:

$$J^2\omega^2-\Omega^2k^4=0, \quad \omega^2-Bk^2-i(\Gamma_\parallel/\Omega)J^2Z\omega=0. \quad (18)$$

In the situation considered in the present paper (*vide supra*), we have $B\sim K_2\sim 3\cdot 10^{-5}$; $(\Gamma_\parallel/\Omega)J^2Z\sim 0{,}01–0{,}008$, whence $B\ll(\Gamma_\parallel/\Omega)J^2Z$. Thus, the second equation (18) admits only decaying modes for the phase shift. From the first equation (18) follows the existence of an undamped mode of oscillations of the precession axis $\mathbf{w}=\mathbf{J}/J$. The group velocity of the corresponding waves is $v_g=2k\Omega J^{-1}$; in dimensional units it is given by Eq. (1).

The configuration with $w_2\neq 0$ can be taken into account perturbatively. It is easy to see that the phase-shift mode will be damped in this case too. The dispersion law for the propagating mode of oscillations of the vector **w** has (up to second order terms in K_2 and w_2) the form

$$J^2\omega^2=\Omega^2k^4(1-i\xi w_2^2),$$

where $\xi\sim 0.01$ for the external switched-off fields considered in the present paper (see Section 1). Thus, for the data considered here,[1], the damping is still sufficiently small so that waves can propagate.

3. THE INFLUENCE OF A RESIDUAL EXTERNAL FIELD

We assume that there is a residual external field H_e which is much smaller than the dipole field, and is parallel to the z axis. We assume that the vector **w** oscillates around the z axis, i.e., $\mathbf{w}_0=\hat{\mathbf{z}}$. In this special case the equations for the phase shift $\delta\psi$ and for the vector are uncoupled. Repeating the reasoning of the preceding section we obtain from the equation (8) an equation for the phase shift of the form

$$\frac{\partial^2}{\partial t^2}\delta\psi-\frac{11}{8}K_2\delta\psi''+\frac{\Gamma_\parallel}{\Omega}J^2Z\frac{\partial}{\partial t}\delta\psi=0,$$

where Z is given by the same formula as in Eq. (14). It is easy to see that the phase-shift mode is damped. The two equations for w_u and w_v are conveniently written in terms of a single equation, by introducing the complex function $\Phi=w_u+iw_v$. The equation for Φ which is equivalent to the two equations for w_u and w_v, has the form

$$i\frac{\partial}{\partial t}\Phi=\frac{65}{64}K_2J^{-1}\Phi''+\frac{7}{8}H_e\Phi. \quad (19)$$

Here all coefficients are calculated in the weak-coupling approximation, for $\theta=\arccos(-\frac{1}{4})$.

In form Eq. (19) coincides with the one-dimensional Schrödinger equation. It yields the dispersion law

$$\omega=-\frac{65}{64}K_2J^{-1}k^2+\frac{7}{8}H_e.$$

The corresponding group velocity is $v_g=(65/32)K_2J^{-1}H$. In dimensional units it is given by the same equation (1) as in the absence of a field.

CONCLUSION

In the present paper we have considered a type of spin waves which is specific for a superfluid liquid with a complicated order parameter, such as ^{3}He-B. The situation studied here is not similar to spin waves in usual magnetic materials, in the sense that the waves discussed in the present paper exist only in nonequilibrium states of ^{3}He-B generated by the WP mode of the spatially homogeneous spin dynamics. For the case of the Brinkman-Smith mode a similar phenomenon has already been studied by Fomin.[6] It is possible that in addition to the gap-free spin- and order-parameter waves studied in the present paper there may exist other types of waves; Fomin conjectured that there may exist high-frequency gap modes with a low velocity of propagation. This interesting suggestion deserves to be investigated.

As Volovik pointed out to the author, the equations derived in this paper for the averaged characteristics are reminiscent of the equations for the orbital dynamics of liquid crystals.[14] This circumstance seems natural, since the vector **J**, whose average lies at the basis of the derivation of the equations, is reminiscent of an angular momentum [cf. Eq. (3)].

The peculiarity of the processes considered in the present paper manifests itself with particular clarity in their large-scale, averaged characteristics, described by waves both in the spin and in the order parameter. It is as if we were dealing with a new system, which arises on the basis of the superfluid ^{3}He. This circumstance might serve as the source of some not completely useless physical analogies, like the above-mentioned analogy with liquid crystals.

The waves under discussion can apparently be generated in experimental situations by a sudden switching-off of an external field for ^{3}He-B placed in a parallelepipedal cavity, i.e., in exactly the same manner in which the WP mode, on which they propagate, is generated. As explained in Section 3, the presence of a sufficiently small and appropriately oriented magnetic field does not impede their existence. If these conditions can be weakened, then they are an adequate model of WSW waves.[1] In this respect it is encouraging that their group velocity agrees well with the propagation velocity of WSW waves (see the Introduction). At the present time this is the only existing theoretical explanation of the small velocity of WSW waves (see Ref. 1). It should, however, be noted that there are two disagreements with the results of Ref. 1: that paper recorded two modes with velocities of propagation differing by 20–30%; and the temperature dependence of the observed velocities was proportional to $(1-T/T_c)^{1/2}$, whereas from Eq. (1) of the present paper it follows that there is a linear dependence on $1-T/T_c$. It is also important to note that in Ref. 1 the residual fields are comparable in order of magnitude with the dipole field.

The indicated difficulties do not seem to be an insurmountable obstacle, but are rather a stimulus for further investigations. They can be interpreted qualitatively within the framework of the present paper. First of all, for residual fields of the order of 30% of the dipole field which was used in Ref. 1, the basic stationary solution is an analog of the WP mode in a magnetic field, mode which exhibits two branches (Ref. 11), which split from the single branch of the WP mode which exists in the completely switched-off field. The residual fields in Ref. 1 were sufficiently large that this splitting became noticeable. Unfortunately in Ref. 1 it was not indicated which configuration of residual fields was used: whether it was parallel or antiparallel to the switched-off field, and this determines the branch of the stationary solution and corresponds to the wave mode which is the spatial modulation of the stationary solution. Secondly, as follows from its derivation, the equation (1) for the group velocity as applied to the situation in Ref. 1, has, essentially, a qualitative character. In this respect one may consider it satisfactory, since it yields correct orders of magnitude for the temperature range studied in Ref. 1.

It is interesting to note the following circumstance also. It is known that the observed frequency of the WP mode differs from the theoretical value $(2/5)^{1/2}H$, Ref. 8, by approximately 7.6% (Ref. 12), a discrepancy which is entirely within the experimental limits. One may assume, that the reason for the discrepancy are the texture-spin waves considered in the present paper. Indeed, as was pointed out to the author by A. F. Andreev, such waves must lead to a shifts of the frequency of the spatially homogeneous ringing of the magnetization (the formation of a zone structure in the spectrum), and this is, possibly, the reason for the indicated disagreement with experiment.

Apparently analogs of the waves considered in the present paper may appear also in zones of instability of spatially homogeneous regimes, considered in Refs. 5 and 15; owing to the nonlinear character of their dispersion law the appearance of an instability may lead to phenomena reminiscent of weak turbulence.[16] The possibility of occurrence of turbulence-like regimes in ^{3}He-A was pointed out earlier in Ref. 17.

The considerations expounded here allow one to assume that the texture-spin waves considered in the present paper are interesting from a theoretical point of view and useful for the analysis of experiments.

In conclusion the author would like to thank A. F. Andreev, Yu. M. Bruk, G. E. Volovik, and I. A. Fomin for discussions and critical remarks.

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Translated by Meinhard E. Mayer