Establishment of self-similar regimes of nonlinear random waves in dissipative media

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The conditions for the appearance of self-similar, strongly nonlinear regimes of one-dimensional acoustic turbulence are considered and the laws for the growth of the external turbulence scale and of energy attenuation are found. It is shown that, under certain conditions, a self-similar, strongly nonlinear dissipative structure of one-dimensional acoustic turbulence may be possible, irrespective of the magnitude of the initial Reynolds numbers and, in particular, even if the Reynolds number is less than unity. The results of numerical simulation are presented and the statistical characteristics of one-dimensional acoustic turbulence are determined.

1. The appearance of ordered structures is possible in nonlinear media with dissipation (see, for example, Refs. 1 and 2). These structures represent a succession of regions (domains) with regular behavior, alternating with randomly located zones of dissipation. In the present work, we consider the appearance and evolution of such structures using as the example the simplest equation of the theory of nonlinear waves—the Burgers equation[3]:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2},\quad u(x,t=0)=u_0(x), \tag{1}$$

where ν is the dissipation coefficient, $u_0(x)$ is a random function if the initial field has a noise character. This equation also describes two fundamental effects, characteristic of any turbulence—the nonlinear transfer of energy over the spectrum and damping of the energy in the region of small scales. It was for just this reason that it was proposed by Burgers as a model equation of hydrodynamic turbulence.[4,5] It was later shown that the description of one-dimensional acoustic waves in a compressible liquid reduces to Eq. (1). As a consequence of this, the Burgers equation has found widespread application in nonlinear acoustics,[6] in particular in the description of intense acoustic noise—the so-called one-dimensional acoustic turbulence (OAT).

Since external forces are lacking in (1), the evolution of the field $u(x,t)$ is completely determined by the initial conditions, while, because of the dissipation of energy, the amplitude of the field will decrease, i.e., degeneration of the turbulence will take place. Let u_0 and l_0 be the characteristic amplitude and spatial scale of the initial field. Then the dimensionless parameter $\mathrm{Re}_0\sim u_0 l_0/\nu$—the Reynolds number—will characterize the relative influence of the nonlinear and dissipative effects on the evolution of the field. If $\mathrm{Re}_0\gg 1$, then the random field is known to evolve into an ordered dissipative structure, which represents a succession of sawtooth pulses with equal slopes, separated by randomly distributed shock fronts, in the vicinity of which energy dissipation also occurs.[5,7,8] Under random initial conditions, the velocities of these discontinuities are also random, which leads to the effect of separated discontinuities and, consequently, to an increase in the characteristic scale of the field $l(t)$. The velocity of the individual discontinuity is proportional to the integral of the initial field $u_0(x)$ in some interval of order $l(t)$. Therefore, the tempo of the evolution of the field, because of the confluence of the discontinuities, is determined by the behavior of the energy spectrum of the initial field $g_0(k)$ near $k\sim 1/l(t)\to 0$. In particular, it has been shown (see, for example, Refs. 7 and 8) that the laws of increase in the scale and energy damping are different, depending on whether the spectrum at zero wave number $k=0$) is equal to or not equal to zero:

$$D=g_0(0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\langle u_0(x+\rho)\,u_0(x)\rangle d\rho. \tag{2}$$

Here

$$g_0(k)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\langle u_0(x+\rho)\,u_0(x)\rangle e^{ik\rho}\,d\rho$$

is the spectral density of the initial field.

Along with the establishment of local self-simularity—the universal structure of the field in each of the domains—it can be expected that the statistical characteristics of the field—its probability distributions, correlation functions, spectra—will also be self-similar. It has been shown (Refs 7–10) that at $D=0$ and $\mathrm{Re}_0\gg 1$, intermediate self-simularity[11] is realized for the field $u(x,t)$: all the statistical characteristics of the field are self-similar over some temporal interval $t_m<t<t_l$, where $t_m\sim l_0/u$ is the characteristic time of formation of a sawtooth structure, while t_l is the time of emergence into the linear regime. It has been shown for this case that, because of the effect of discontinuities, this time is extremely large: $t_l\sim t_m\exp(\mathrm{Re}_0^2)$. The hypothesis that the field is self-similar at $\mathrm{Re}_0\to\infty$ has been used in Refs. 12 and 13 for analysis of the laws of evolution at $D=0$ and $D\neq 0$. Here, however, an additional condition is required on the field spectrum, which is evident only at $D\neq 0$, since $g(o,t)=g_0(0)=\mathrm{const}$ is an invariant of the Burgers equation.

In the present work, the conditions for the appearance of the self-similar, strongly nonlinear regime of OAT are elucidated. This regime represents a series of sawtooth waves. We also find the laws for the growth of the external scale of turbulence and the damping of the energy. Here, for analysis of the OAT, we use both the exact solution of Eq. (1)

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and a smooth qualitative model describing the evolution of the field $u(x,t)$ in the form of a set of different, successively damped, weakly interacting, strongly nonlinear modes. It is shown that under certain conditions the establishment of a self-similar, strongly nonlinear dissipative structure of the OAT is possible regardless of the value of the initial Reynolds number, in particular, even if $\mathrm{Re}_0 \ll 1$.

2. Before proceeding to the asymptotic analysis of the establishment of the self-similar structure of OAT, we shall discuss the basic qualitative differences of the damping of the periodic (initially harmonic) signal and the noise field. For this purpose, we write down Eq. (1) in spectral form:

$$c(k,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,t)e^{ikx}\,dx, \tag{3}$$

$$\frac{\partial c(k,t)}{\partial t}+ik\int_{-\infty}^{+\infty} c(k-k')c(k')dk'=-\nu k^2 c(k,t). \tag{4}$$

If the signal initially has only a single spatial harmonic $k_0 = 2\pi/l_0$, then, as a result of the nonlinear self-action, generation of higher spatial harmonics $k = mk_0$ ($m = 2, 3, \ldots$) occurs. However, at $\mathrm{Re}_0 \ll 1$, the strong dissipation does not permit the nonlinear effect to develop and the wave is damped exponentially with the linear decrement $k_0^2\nu$. At $\mathrm{Re}_0 \gg 1$, the wave passes successively through three stages.[6] In the first stage, $t < t_n \sim l_0/u_0$, nonlinear generation of harmonics takes place and the profile of the wave is steepened. At $t > t_n$ discontinuities are formed in the wave and the wave has the form of a succession of periodic sawtooth pulses with equal slopes $u'_x = 1/t$ and a discontinuity amplitude $\sim l_0/t$ independent of the initial amplitude. Because of the growth of the width of the shock front $\delta \sim \nu t/l_0$ the effective Reynolds number

$$\mathrm{Re}\,(t)\approx l_0/\delta\approx (t_n/t)\,\mathrm{Re}_0$$

decreases, while at $t > t_l \approx t_n\,\mathrm{Re}_0$ the field emerges into a linear regime—the wave is transformed into a sinusoidal one and decays exponentially with linear decrement $k_0^2\nu$.

In the initial noise field with finite width of the spectrum, the picture of the evolution of the field is qualitatively different. This is connected with the fact that the nonlinear generation of the various harmonics and the strong dissipation of the fine-scale components lead to a relative increase in the large-scale components of the spectrum and, as a consequence, to a decrease in the relative role of linear dissipation. It follows from (4) that the spectral density at zero frequency, $g_0(0,t) = g_0(0)$, is an invariant of the Burgers equation. Therefore, if $g_0(0) \neq 0$, then in proportion to the damping of the fine-scale components, the basic energy field will become concentrated in the large-scale components, for which, as is shown below, the effective Reynolds number increases with decrease in frequency. Consequently, such noise sooner or later reaches a strongly nonlinear regime of propagation. If $g_0(0) = 0$, then, as is seen from (4), the behavior of the spectrum near the zero wave number will be determined by the relation between the law of behavior of $g_0(k)$ as $k \to 0$ and the effect of nonlinear generation of new large-scale components, which leads to the formation of a universal law $\propto k^2$ as $k \to 0$. However, even here the presence of large-scale components in the noise spectrum (which are present in the initial field or which appear as a result of nonlinear effects) leads to a relative increase in the role of nonlinear effects in the evolution of the field.

We assume that the initial noise spectrum has the form

$$g_0(k)=\alpha_n^2 k^n b_0(k),\quad 0<b_0(k)<\infty. \tag{5}$$

Here $b_0(k)$ is a sufficiently rapidly decreasing function as $k \to \infty$ with characteristic scale $k_* \sim 1/l_0$, approaching a constant value as $k \to 0$. We shall investigate below how the asymptotic behavior of the field depends on the exponent n in (5).

We first consider the case of small initial value of the Reynolds number ($\mathrm{Re}_0 \sim u_0 l_0/\nu \ll 1$). Then we neglect the nonlinear effects from the initial state and obtain the following expression from (1) for the spectral density:

$$g(k,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\langle u(x+\rho,t)u(x,t)\rangle e^{ik\rho}\,d\rho=g_0(k)e^{-2\nu k^2 t}. \tag{6}$$

Because of the damping of the fine-scale components of the spectrum at $t > l_0^2/\nu$ the behavior of the field will be determined only by the form of the large-scale power portion of the initial spectrum (5). Here the increase in the characteristic spatial scale of the field $l(t)$ and the energy damping $\sigma^2(t) = \langle u^2(x,t)\rangle$ as a result of the coordination of the fine scales are determined by the laws

$$l(t)\approx(\nu t)^{1/2},\quad \sigma^2(t)\approx\alpha_n^2(\nu t)^{-(n+1)/2} \tag{7}$$

while the actual value of the Reynolds number changes as

$$\mathrm{Re}\,(t)\approx\sigma(t)l(t)/\nu=(\alpha_n/\nu)\,(\nu t)^{(1-n)/4}. \tag{8}$$

The qualitative difference of the spectra with exponents $n < 1$ and $n > 1$ follows immediately from (8). If the initial spectrum increases sufficiently slowly in the region of large scales ($n > 1$), then the effective Reynolds number decreases with passage of time and, consequently, the nonlinear state of development of the OAT can be realized only if the initial Reynolds number is large. If now $n < 1$, then, because of the greater damping of the fine-scale components, the relative role of the large-scale components increases, and the effective Reynolds number also increases. Consequently, even at $\mathrm{Re}_0 \ll 1$, the wave emerges here into the nonlinear regime of propagation.

3. We now consider the other limiting case $\mathrm{Re}_0 \gg 1$, and we shall show that the resultant nonlinear structure is stable at $n < 1$, i.e., it never departs from the linear regime. As is well known, the exact solution of the Burgers equation has the form[14,15]:

$$u(x,t)=\frac{\left[\int_{-\infty}^{+\infty}\frac{x-y}{t}\exp\left(\frac{-S(x,y,t)}{2\nu}\right)dy\right]}{\left[\int_{-\infty}^{+\infty}\exp\left(\frac{-S(x,y,t)}{2\nu}\right)dy\right]}, \tag{9}$$

$$S(x,y,t)=\frac{(x-y)^2}{2t}+S_0(y),\quad S_0(y)=\int^{y} u_0(x)\,dx. \tag{10}$$

Here $S_0(y)$ is known as the action of the initial field.[7,12] As

$\nu \to 0$, a contribution to the integral (9) is made only in the vicinity of the point where $S(x,y,t)$ takes on a minimal value, which allows us to write the solution of the Burgers equation in the form[5,12]:

$$u(x,\ t)=[x-y(x,\ t)]/t, \tag{11}$$

where $u(x,t)$ is the coordinate of the absolute minimum of $S(x,y,t)$. At $t \gg t_n$ the function $y(x,t)$ is a discontinuous, piecewise constant function of x, while the field $u(x,t)$ transforms into a series of sawtooth pulses with the same slope $u'_x = 1/t$.[7,8] As an estimate for the external scale of turbulence $l(t) \sim |x-y|$, we can take the condition that the parabola in (10) and the value of the initial action are of the same order, i.e.,

$$[d_S(l(t))]^{1/2} \sim l^2/t, \quad d_S(\rho)=\langle (S_0(x+\rho)-S_0(x))^2\rangle, \tag{12}$$

where $d_s(\rho)$ is the structure function of the initial action. Its behavior is different at $n<1$ and at $n>1$. If $n>1$, then the dispersion of the initial action is limited and $d_s(\rho \to \infty) = 2\sigma_s^2$, if $n<1$, then the structure function increases according to the power law $d_s(\rho) \approx \alpha_n^2 \rho^{1-n}$. Correspondingly, we have from (12) for the external scale of the OAT in these cases,

$$l(t)=\begin{cases}(\sigma_S t)^{1/2}, & n>1\\ (\alpha_n t)^{2/(3+n)}, & n<1\end{cases}. \tag{13}$$

At finite viscosity ($\nu \neq 0$) the width of the shock front (the internal scale of turbulence) $\delta \sim \nu t/l(t)$ will increase. However, because of the effect of the discontinuities, the external scale $l(t)$—the characteristic distance between discontinuities— will also increase. For the effective Reynolds number $\mathrm{Re}(t) = l(t)/\delta(t)$, which is equal to the ratio of the external scale to the internal, we have here

$$\mathrm{Re}(t) \sim \begin{cases}\mathrm{Re}_0=\text{const}, & n>1\\ t^{(1-n)/(n+3)}, & n<1\end{cases}. \tag{14}$$

Thus, if $n<1$, then the effective Reynolds number increases even in the nonlinear state, i.e., the regime of strongly nonlinear sawtooth waves turns out to be structurally stable and never departs from the linear damping regime. For the case $n \geqslant 2$, the most rigorous estimate show that the Reynolds number decreases but, because of the effect of the discontinuities, this increase is logarithmically slow.

4. We now show rigorously that in the case $n<1$, the establishment of a strongly nonlinear regime actually takes place, and also that this regime is self-similar. If the initial spectrum of the field is determined by the expression (5), then at $n<1$, the structure function is represented in the form

$$d_S(\rho)=\langle [S_0(x+\rho)-S_0(x)]^2\rangle=\alpha_n^2 a(\rho)\rho^{1-n}. \tag{15}$$

Here $a(\rho \gg l_0) = \text{const}$, and for definiteness we set $a(\infty)=1$. For analysis of the field, we use the exact solution of the Burgers equation (9), (10). We introduce new variables in (9) and (10):

$$x=\xi l(t), \quad y=\eta l(t). \tag{16}$$

Introducing the effective Reynolds number as $\mathrm{Re}(t) = l^2(t)/\nu t$, we can rewrite the solution of (9) and (10) in the form

$$u(x,t)=\frac{l(t)}{t}\left\{\int_{-\infty}^{+\infty}(\xi-\eta)\exp[-\mathrm{Re}(t)\tilde{S}(\xi,\eta,t)]d\eta\right\} \times\left\{\int_{-\infty}^{+\infty}\exp[-\mathrm{Re}(t)\tilde{S}(\xi,\eta,t)]d\eta\right\}^{-1}, \tag{17}$$

$$\tilde{S}(\xi,\eta,t)=\frac{(\eta-\xi)^2}{2}+\tilde{S}_0(\eta,t), \quad \tilde{S}_0(\eta,t)=\frac{t}{l^2(t)}S_0(\eta l(t)). \tag{18}$$

The function $\tilde{S}(\eta,t)$ in (18) possesses the following structure function:

$$\tilde{d}_s(\tilde{\rho},t)=\langle[\tilde{S}_0(\eta+\tilde{\rho},t)-\tilde{S}_0(\eta,t)]^2\rangle=(t^2/l^4)d_s(\tilde{\rho}l) =(t^2\alpha_n^2/l^{3+n})a(\tilde{\rho}l)\tilde{\rho}^{(1-n)}. \tag{19}$$

Substituting $l(t)$ from (13) here at $n<1$, we have $(t^2\alpha_n^2/l^{3+n})=1$, in (19) while the effective Reynolds number increases with increase in t according to the law (14). At $\mathrm{Re}(t) \gg 1$ the contribution to the integral (17) is made only by the point where the function $\tilde{S}(\xi,\eta,t)$ reaches a minimum and the solution is rewritten in the form

$$u(x,t)=\frac{l(t)}{t}[\xi-\eta(\xi,t)]=\frac{l(t)}{t}\tilde{u}(\xi,t), \tag{20}$$

where $\eta(\xi,t)$ is the coordinate of the absolute minimum of $\tilde{S}(\xi,\eta,t)$. At $t>t_n$, the function $\eta(\xi,t)$ is piecewise constant and consequently the field $\tilde{u}(\xi,t)$ represents a series of sawtooth pulses in the variables ξ with a shock-front width that decreases as $1/\mathrm{Re}(t)$. On going to $u(x,t)$, the width of the shock fronts will increase; however, their relative width [in comparison with the external scale $l(t)$] decreases, which allows us to speak of a strongly nonlinear regime of evolution of the wave. It follows from (19) that at $\rho > l_0/l(t) \ll 1$ the structure function $\tilde{S}_0(\eta,t)$ has the form

$$\tilde{d}_S(\tilde{\rho},t)=\tilde{\rho}^{(1-n)}, \tag{21}$$

i.e., it does not depend on the time and does not possess spatial scales. This means that at $l(t) \gg l_0$ and $\mathrm{Re}(t) \gg 1$, the statistical characteristics of $u(x,t)$ will be determined by the function $\eta(\xi,t)$, the statistical properties of which, in turn, do not change with passage of time. Consequently, all the statistical characteristics of the field $u(x,t)$, in accord with (20), become self-similar and are determined by a single scale—the external scale of the OAT, $l(t)$, which increases according to the law (13) because of the effect of the discontinuities.

It is seen from (20) that the correlation function and the energy spectrum of the turbulence in the self-similar stage can be represented in the form

$$B(\rho,t)=\langle u(x+\rho,t)u(x,t)\rangle=\frac{l^2(t)}{t^2}\langle y^2\rangle R\left(\frac{\rho}{l(t)}\right), \tag{22}$$

$$g(k,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}B(\rho,t)e^{ik\rho}d\rho=\frac{l^3}{t^2}\langle y^2\rangle\tilde{g}(kl(t)), \tag{23}$$

$$\tilde{g}(\tilde{k})=\frac{1}{2\pi}\int_{-\infty}^{+\infty}R(z)e^{i\tilde{k}z}dz.$$

The dimensionless functions $R(z)$ ($R(0)=1$), $\tilde{g}(\tilde{k})$, and the constant $\langle y^2\rangle$ entering into (22) and (23) all depend only on the exponent n in the growth law of the spectrum near the zero wave numbers (5). The specific form of these functions can be found, for example, by statistical treatment of the numerical model of the dimensionless field $u(\xi) = \xi - \eta(\xi)$ (20). A number of properties of these functions can be established by using certain general properties of the OAT. First, we note that in the realization of the field $u(x,t)$ there are discontinuities in the self-similar state. Their appearance leads to a universal, shortwave asymptote of the energy spectrum of the field[7-9,12] $\tilde{g}(\tilde{k}) = \lambda_n \tilde{k}^{-2}$. Consequently, with accuracy to within a numerical factor λ_n that depends on the exponent n of the initial spectrum (5), the energy spectrum of the OAT, regardless of the fine structure of the initial spectrum, will have the universal asymptote $\tilde{g}(\tilde{k}) \sim \tilde{k}^{-2} (\tilde{k} \gg 1)$.

In the region of large scales, as follows from (4), the nonlinearity and the dissipation cannot change the form of the spectrum at $n < 1$. Therefore, at $n < 1$, the dimensionless spectrum $g(k)$ has the same asymptotic behavior of the spectrum as $k \to 0$ as the initial spectrum: $\tilde{g}(\tilde{k}) = \beta_n \tilde{k}^n$. We note that by substituting this relation in (23) we obtain an equation for the external scale of turbulence $l(t)$, whose solution naturally leads to the previously obtained law of growth of the external scale of OAT (13).

5. The presence of a single characteristic scale in the field $u(x,t)$ at the discontinuity stage allows us to construct a simple qualitative model of the evolution of the turbulence with $n < 1$. For this purpose, we replace the continuous spectrum $g_0(k)$ by a discrete set of modes—the spatial harmonics with wave numbers k_m and amplitudes A_m:

$$k_m = k_0 \varepsilon^{-m}, \quad A_m = \alpha_n k_0^{(n+1)/2} \varepsilon^{-m(n+1)/2}. \tag{24}$$

We choose the amplitudes of the harmonics from the condition that the mean spectral density of the harmonics in the interval $\Delta_m = k_{m+1} - k_m$ be identical with the noise spectral density: $A_m^2 = g_0(k_m) \Delta_m$. We set $\varepsilon \gg 1$, so that the harmonics are sufficiently spread out over the spatial spectrum. The energy of an individual mode is conserved at $t < t_m$, where t_m is the characteristic time of development of the nonlinearity of the mth mode:

$$t_m = 1/k_m A_m \approx (\alpha_n k_0^{(n+3)/2} \varepsilon^{-m(n+3)/2})^{-1}. \tag{25}$$

At $t > t_m$, the mode transforms into a series of sawtooth pulses with characteristic period $l_m \sim 1/k_m$ and its energy decays in jumps. For the energy of the mode, we write out the following approximate expression:

$$E(t) \approx \begin{cases} A_m^2, & t < t_m, \\ 1/k_m^2 t^2, & t > t_m. \end{cases} \tag{26}$$

The energy decay of an individual mode is connected with the transfer of its energy upward in the spectrum. However, in the case of a finite width of the spectrum of the mode, part of the energy is carried into the long wavelength region of the spectrum, while the intensity of the new components is proportional to k^2. Consequently, if the spectrum $g_0(k)$ in the long wavelength region is sufficiently intense ($g_0(k) \sim k^n$, $n < 1$), then the newly appearing components have a smaller intensity than $g_0(k)$. The transfer of energy into the long wavelength region can be neglected here and we can assume that the amplitudes of the large-scale modes do not change, without beginning their nonlinear decay. The interaction of the modes naturally affects the laws of mode transformation. However, at $\varepsilon > 1$, when the individual modes are widely distributed over the spectrum, this interaction can be neglected if we limit ourselves to the consideration of the energy characteristics only. Actually, in correspondence with (2), the fine-scale modes have practically no effect on the large scale. The large scale modes lead only to local transfer of the higher modes as a whole, i.e., to their spatial modulation.[6,16,17] This allows us to assume that at the instant of time t, the energy of the wave $E(t)$ is the sum of the energies of the modes that have not yet begun to be damped, and the energy of the modes that are represented in the sawtooth wave is approximately equal to

$$E(t) = \sum_{m=-\infty}^{m_*(t)} A_m^2 + \sum_{m=m_*(t)}^{t} \frac{1}{k_m^2 t^2}. \tag{27}$$

Here $m_*(t)$ is the boundary number of the mode, which begins to be damped at the time t. This number is equal to the solution of the equation $t_m = t$ (25). It follows from this equation that at time t the characteristic period of the m_*th mode is equal to

$$l_{m_*}(t) \approx 1/k_{m_*} = \varepsilon^{m_*}/k_0 = (\alpha_n t)^{2/(n+3)}, \tag{28}$$

which coincides with the result obtained from an exact solution of the Burgers equation. For the energy of the wave (27), this scale turns out to be the largest energy-containing scale, since modes with $m < m_*$ are already damped, while the modes with $m > m_*$ possess less energy. Consequently, we have [from (28)] the result that the energy of the field is damped approximately as

$$E(t) \approx l_{m_*}^2/t^2 \sim t^{-2(n+1)/(n+3)}. \tag{29}$$

The field $u(x,t)$ here has the form of a series of sawtooth pulses with characteristic period $l(t)$ (18), against the background of which there exists a ripple of discontinuities with smaller periods and with energies much less than the energy of the fundamental mode. As the wave propagates, the energy-containing mode develops a longer and longer period.

6. The self-similarity of the field at the stage of developed discontinuities allows us effectively to use numerical methods of investigation of the OAT. Actually, it is seen from (29) that the field $u(x,t)$ in this stage is expressed in terms of the dimensionless field $\tilde{u}(\xi,t)$, the statistical properties of which do not depend on the time in the self-similar stage. Therefore, carrying out a statistical treatment of the results of the numerical experiment on the modeling of the field $\tilde{u}(\xi,t) = \xi - \eta(\xi,t)$, where $\eta(\xi,t)$ is the coordinate of the absolute minimum of the function $\tilde{S}(\xi,\eta,t)$ (18):

$$\tilde{S}(\eta, \xi) = {}^1/_2 (\eta - \xi)^2 + \tilde{S}_0(\eta), \tag{30}$$

we have succeeded in finding the form of the correlation function, the spectrum and other statistical characteristics of the OAT in the self-similar stage. This is possible because the dimensionless function $\tilde{S}_0(\eta)$ entering into (18) and (30) possesses the structure function (21) in the self-similar stage, and which is time-independent. Consequently, we can assume in (30) that $\tilde{S}_0(\eta)$ is also time-independent.

In the present work, the result is given of a numerical

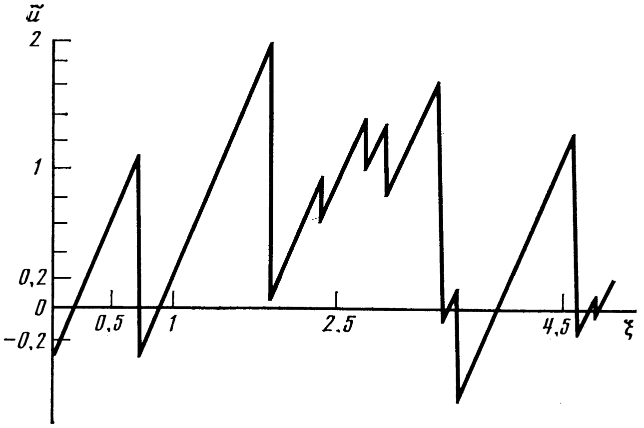

FIG. 1. Typical form of realization of OAT.

experiment in the case $n = 0$, when the random function $\tilde{S}_0(n)$ is a Wiener process with structure function $d_s(\rho) = |\rho|$. The modeling of (30) has been carried out with a step in η equal to $\Delta = 0.05$. The Wiener process has been approximated in this case by the discrete series

$$\tilde{S}_{0,m} = \tilde{S}_0(m\Delta) = \sum_{i=1}^{m} \psi_i = \tilde{S}_{0,m-1} + \psi_m, \quad (31)$$

where ψ_i are independent random numbers possessing a normal distribution with a variance determined from the condition of identity of the structure function of the discrete series and the Wiener process:

$$\langle \tilde{S}^2_{0,m} \rangle = \langle \psi_m{}^2 \rangle m = d_S(\rho = m\Delta) = m\Delta.$$

Thus, the procedure of estimating the dimensionless field $\tilde{u}(\xi) = \xi - \eta(\xi)$ has ended in the construction of a random realization of the action $S(\eta,\xi)$ (30) and the estimate of the absolute minimum of this series $\eta = \eta(\xi)$. A typical shape of the realization is shown in Fig. 1.

The statistical characteristics of the field have been determined as a result of the treatment of an ensemble consisting of N independent realizations. In the numerical experiment, the following characteristics of the OAT were found: the one-dimensional probability function and its first cumulants—$\varkappa_n$, the correlation function, and the laws of decay of

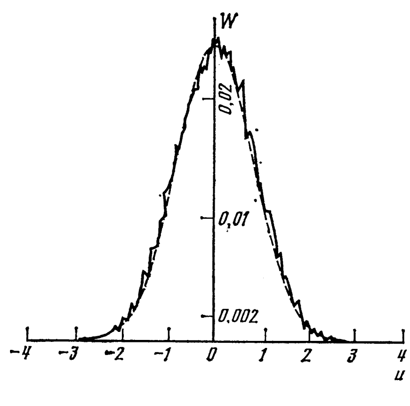

FIG. 2. One-dimensional probability distribution of OAT (the Gaussian distribution possessing the same variance is given by the dashed curve).

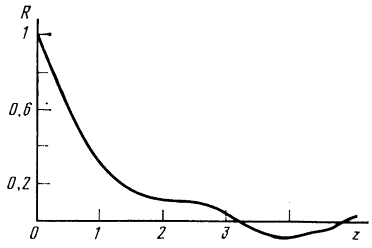

FIG. 3. Dimensionless correlation coefficient of OAT.

the energy spectrum. The probability distribution (see Fig. 2) was determined by the averaging of 100,000 realizations consisting of a single point $\tilde{u}(\xi = 0) = -\eta$. In the numerical experiment, the variance $\varkappa_2 = \langle \tilde{u}^2 \rangle = 0.699$ entering as a dimensionless parameter $\langle y^2 \rangle = \langle \tilde{u}^2 \rangle$ in (22) and (23), and the cumulant coefficients $\gamma_n = \varkappa_n / \varkappa_2^{n/2}$, that describe the decrease of the departure of the distribution from Gaussian[18] were determined. In contrast to the case $D = 0$, where the one-dimensional distribution is asymptotically Gaussian,[9] the probability distribution $W(u)$ differs from Gaussian, albeit not strongly. In particular, the excess coefficient, equal to $\gamma_4 = -0.0149$, shows that the distribution has a more typical vertex than the Gaussian distribution with the same dispersions.

The dimensionless correlation coefficient $R(z) = \langle \tilde{u}(z + \xi)\tilde{u}(\xi) \rangle$ was determined numerically as a result of double averaging: over the individual realizations $\tilde{u}(\xi)$ and over the ensemble of $N = 10\,000$ realizations. The form of the correlation coefficient is shown in Fig. 3. As is seen, the correlation coefficient is nonanalytic near zero, which is directly connected with the presence of discontinuities in the realizations of the random field of the OAT. From the results of the numerical experiment, it follows that near zero $R(z) \approx 1 - 0.87|z|$. This leads to the following power asymptote of the dimensionless energy spectrum of the turbulence: $g(k) \approx 0.419k^{-2}$.

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