Activated tunneling decay of metastable states

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It is shown that the Fokker-Planck equation that describes the motion of a Brownian particle in the presence of potential barriers can be reduced, if the interaction with the thermostat is weak enough, to an integral equation in terms of the energy variable, or else to a system of such equations. The basic small parameter is then the small ratio of the temperature to the barrier height. The proposed calculation scheme accounts in natural fashion for the quantum effects of above-barrier reflection and below-barrier passage. An exact solution is presented of the Kramers problem of the lifetime of a particle in a deep potential well. The problem of the lifetime of a particle in one of the minima of a two-well potential is formulated and solved. In a definite range of parameters, the results are useful for the description of the fluctuations of the phase shift of the order parameter in compact Josephson junctions.

Fluctuations in a number of physical systems are similar to Brownian motion of a particle in a certain potential relief. If the characteristic heights of the potential barriers exceed the temperature, the particle stays a long time near a certain local potential minimum before proceeding to the next minimum. States corresponding to energy minima are therefore metastable, and the lifetime in them is given by the Arrhenius law

$$\frac{1}{\tau}=\frac{\Omega}{2\pi}A\exp\left(-\frac{U_0}{T}\right),\qquad(1)$$

where Ω is the frequency of the particle oscillations at the bottom of the given potential well and U_0 is the height of the potential barrier. The nondimensional coefficient A is determined by the actual form of the potential, by the interaction with the thermostat, and also by the barrier transparency if quantum effects are taken into account.

The decay of metastable states via fluctuations and tunneling is attracting considerable interest at present in connection with experiments on Josephson junctions (see the review by Likharev[1]). In this article the lifetime of a particle in a potential well is calculated in the case when the particle leaves a single minimum of the potential, and when it goes over from one minimum of a two-well potential to the other.

The basic results in this field were obtained in 1940 by Kramers, who developed a theory of the absolute rates of chemical reactions under the assumption that molecule dissociation is the analog of the exit of a Brownian particle from a potential well.[2] In the Kramers model the motion of a Brownian particle is described by the Langevin equation

$$m\ddot{x}=-m\gamma\dot{x}-\partial U/\partial x+\eta(t),\qquad(2)$$

where m is the particle mass, γ the viscosity coefficient, $U(x)$ the potential, and η a random force whose correlator satisfies the fluctuation-dissipation theorem $\langle\eta(t)\eta(t')\rangle = 2m\gamma T\delta(t-t')$. The potential $U(x)$ is of the form shown in Fig. 1. The frequency Ω previously introduced and the frequency ω that will be used later are defined by

$$\Omega^2=U''(x_0)/m,\quad \omega^2=-U''(0)/m.$$

It can be seen that Ω and ω correspond to the minimum and maximum of the potential.

It is convenient to use, besides the Langevin equation, also its equivalent, the Fokker-Planck equation for the distribution function $f(p,x)$ in the momentum $p = m\dot{x}$ and in the coordinate x of the particle:

$$\frac{p}{m}\frac{\partial f}{\partial x}-\frac{\partial U}{\partial x}\frac{\partial f}{\partial p}=\gamma\frac{\partial}{\partial p}\left(pf+mT\frac{\partial f}{\partial p}\right).\qquad(3)$$

This equation is written under the assumption that the particle lifetime in the well exceeds all the characteristic time scales ($\omega\tau$, $\Omega\tau$, $\gamma\tau\gg1$), so that a quasistationary distribution is established in all the regions of the well. The time dependence of this distribution is given by

$$f(p,x,t)=f(p,x)e^{-t/\tau}.$$

The function $f(p,x)$ should be assumed normalized to unity, corresponding to the presence of one particle in the well at the initial instant of time. In the interior of the well, $f(p,x)$ should take the Boltzmann form

$$f(p,x)=\frac{\Omega}{2\pi T}\exp\left(-\frac{U_0+\varepsilon}{T}\right),$$
$$\varepsilon=\frac{p^2}{2m}+U(x)\qquad(\varepsilon<0,\ |\varepsilon|\gg T),\qquad(4)$$

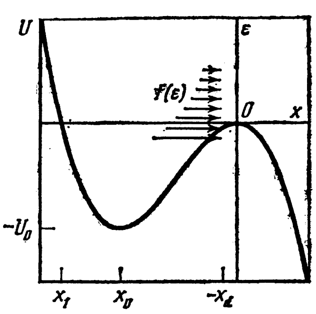

FIG. 1

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if it is assumed that a parabolic approximation can be used for the potential up to energies exceeding T. Outside the well there is only a flux of particles that go off to the infinity on the slope of the potential. This corresponds to the condition

$$f(p,x)\to 0,\quad x\to\infty. \tag{5}$$

Solution of Eq. (3) with boundary conditions (4) and (5) determines the particle flux from the well. At the chosen normalization this flux coincides with the reciprocal of the particle lifetime in the well:

$$j=\frac{1}{\tau}=\int_{-\infty}^{\infty} f(p,x)\frac{p}{m}dp,\quad x>0.$$

The flux-conservation condition guarantees independence of this expression of x, provided x is on the right of the region that determines the normalization of $f(p,x)$.

Kramers solved the Fokker-Planck equation and obtained the coefficient A for two limiting cases. For the region he obtained

$$A=(1+\gamma^2/4\omega^2)^{1/2}-\gamma/2\omega. \tag{6}$$

In the limit $\gamma\ll\omega$ it follows from this that $A\approx 1$. It is clear, however, that in the limit as $\gamma\to 0$ we should have $A=0$, since a classical Brownian particle leaves the well only under the influence of fluctuations. Accordingly, Kramers showed that the following relation holds in the limit of small γ:

$$A=\delta/T,\quad \delta\ll T,\quad \delta=\gamma S_0=2\gamma\int_{x_1}^{0}[-2mU(x)]^{1/2}dx, \tag{7}$$

where S_0 is the action, during the period, for a particle with energy on the barrier level in the absence of friction. The quantity S_0 appears in natural fashion if, allowing for Eq. (1), the energy loss per particle oscillation in the well is calculated in an approximation linear in γ:

$$\delta=2\gamma m\int_{x_1}^{0}\dot{x}\,dx=2\gamma\int_{x_1}^{0}p(x)\,dx=\gamma S_0.$$

Expression (7) is valid so long as $\gamma S_0\ll T$. In order of magnitude, $S_0\sim U_0/\omega$, so that on the side of small γ the value of A should approach unity at $\gamma\gg\omega T/U_0$. It follows from a comparison with (6) that A is close to unity in a wide range of γ,

$$A\approx 1,\quad \omega T/U_0\ll\gamma\ll\omega.$$

In this region of γ the particles that leave the well have a Boltzmann distribution that does not depend on γ. We recall that T/U_0 is the basic small parameter of our problem.

It can be seen that the solution obtained by Kramers is incomplete at least in two respects. First, there is no expression for A at $\delta\sim T$, i.e., the transition from the linear relation (7) to $A\approx 1$ at $\delta\gg T$ is not tracked. Second, Kramers's results pertain only to classical Brownian motion and take no account of, say, a quantum effect such as the finite transparency of the barrier. It is clear that with allowance for tunneling the particle flux may turn out to be larger than the maximum above-barrier flux, so that A will vary in the interval $(0,\infty)$, whereas in the classical case we always have $A\leqslant 1$.

It was shown in Refs. 3 and 4 that when account is taken of the quantum transparency of the barrier, expression (6) should be replaced by

$$A=\frac{\lambda^+}{\omega}\frac{\Gamma(1-\lambda^+/2\pi T)\Gamma(1-\lambda^-/2\pi T)}{\Gamma(1-\Lambda^+/2\pi T)\Gamma(1-\Lambda^-/2\pi T)} \tag{8}$$

where $\Gamma(x)$ is a gamma function, and Λ^t and λ^t are the eigenvalues of the linearized equation (2) near the minimum and maximum of $U(x)$:

$$\Lambda^\pm=-\gamma/2\pm(-\Omega^2+\gamma^2/4)^{1/2},\quad \lambda^\pm=-\gamma/2\pm(\omega^2+\gamma^2/4)^{1/2}. \tag{9}$$

To solve our problem in the entire range of γ we must therefore generalize expression (7), which Kramers obtained in the limit of low viscosity, to include the case of arbitrary δ/T with account taken of the finite character of the ratio ω/T that determines the quantum effects in the interaction of a particle with a barrier. We note that the result that we plan to obtain must be matched in the region $\omega T/U_0\ll\gamma\ll\omega$ with the result $A=(\omega/2T)/\sin(\omega/2T)$ that follows from (8).

We begin with small γ. In this case, neglecting viscosity, the energy is conserved and in the next-order approximation the Brownian motion of the particle reduces to diffusion in energy, so that instead of $f(p,x)$ we must use the distribution function $f(\varepsilon)$ in energy, as was done earlier by Kramers. The energy ε, just as in (4), is measured from the top of the barrier. Allowing for quantum effects, the transparency of a parabolic barrier is $[1+\exp(-2\pi\varepsilon/\omega)]^{-1}$. If it is assumed that in the actual energy region the transparency is low and is given by the expression $\exp(2\pi\varepsilon/\omega)$, we can write for $f(\varepsilon)$ the expression

$$\delta\left(T\frac{\partial^2 f}{\partial\varepsilon^2}+\frac{\partial f}{\partial\varepsilon}\right)=f\exp(2\pi\varepsilon/\omega), \tag{10}$$

the solution of which with the boundary condition (4) is

$$f(\varepsilon)=\frac{\Omega}{\pi^2 T}\exp\left(-\frac{U_0}{T}-\frac{\varepsilon}{2T}\right)\sin\left(\frac{\omega}{2T}\right)\Gamma\left(1-\frac{\omega}{2\pi T}\right)\times\left(\frac{\omega}{2\pi(\delta T)^{1/2}}\right)^{\omega/2\pi T}K_\nu\left(\frac{\omega\exp(\pi\varepsilon/\omega)}{\pi(\delta T)^{1/2}}\right) \tag{11}$$

where K_ν is a modified Bessel function of imaginary argument. The significant values of ε correspond to the region where the argument of K_ν is of the order of unity. From the condition that the transparency coefficient be small at these energies we obtain the criterion $\delta\ll\omega^2/T$ for the validity of the solution obtained.

To calculate the particle flux with account taken of the ratio $pdp/m\equiv d\varepsilon$ we must integrate the right-hand side of (10) with respect to ε, using (11) for $f(\varepsilon)$. This yields

$$A=\left(\frac{\delta}{T}\right)^{1-\omega/2\pi T}\frac{\Gamma(1-\omega/2\pi T)}{\Gamma(1+\omega/2\pi T)}\left(\frac{\omega}{2\pi T}\right)^{\omega/\pi T}. \tag{12}$$

This equation generalizes the Kramers result (7) to include the quantum case. It is applicable so long as $\delta\ll\omega^2/T$. In our situation the significant energies are of the order of $(\omega/\pi)\ln[(T\delta)^{1/2}/\omega]$. Obviously, with decreasing δ, viz., at $\ln(\omega/\delta)\sim U_0/\omega$, the parabolic approximation we have used no longer holds, and the deviation of the potential from parabolic must be taken into account.

We proceed now to solve the problem with allowance for the finite δ/T. We show for this purpose that under the restriction $\delta \ll U$, i.e. at $\gamma \ll \omega$, we can derive from the Fokker-Planck equation (3), or from the Langevin equation (2), an integral equation that is valid for any relation between δ and T. Of course, the differential equation (10) is obtained from this integral equation in the limit of small δ.

In the considered limit $\gamma \sim \omega T/U_0 \ll \omega$ the motion of the particle breaks up into two qualitatively different and spatially separated stages. This corresponds to the fact that the characteristic well dimension $(U_0/m\Omega^2)^{1/2}$ exceeds in terms of the parameter $(U_0/T)^{1/2} \gg 1$ the width $(T/m\omega^2)^{1/2}$ that is essential for the formation of the above-barrier flux (we assume that $\Omega \sim \omega$). This means that the viscous friction and the fluctuations alter the energy of the particle as it moves in the main part of the well, but their relative influence on the motion near the barrier is small in terms of the parameter $\gamma/\omega \ll 1$. We can consequently assume that the particle is reflected from the barrier while possessing a definite energy, after which it oscillates in the well. The particle energy is altered by the friction and by the fluctuations, and with the particle with this altered energy makes the next attempt to go through the barrier.

We introduce in accordance with the foregoing the distribution function of the particles incident on the barrier, given by the relation (see Fig. 1)

$$f(\varepsilon)=f(p,-x_d),\quad p>0,\quad \varepsilon=\frac{p^2}{2m}+U(-x_d),$$
$$\left(\frac{U_0}{m\Omega^2}\right)^{1/2} \gg x_d \gg \left(\frac{T}{m\omega^2}\right)^{1/2}. \tag{13}$$

The motion of the particle near the barrier can be regarded as nondissipative, so that we obtain for the rate of decay of the metastable state

$$\frac{1}{\tau}=\int_{-\infty}^{\infty}\frac{f(\varepsilon)\,d\varepsilon}{1+\exp(-2\pi\varepsilon/\omega)}. \tag{14}$$

We write down the equation for $f(\varepsilon)$ using the following considerations. A particle with energy ε' is reflected from the barrier with a probability $[1+\exp(2\pi\varepsilon'/\omega)]^{-1}$ and returns to the barrier after one oscillation in the well. The average energy lost thereby is $\delta=\gamma S_0$, and the mean squared energy spread is $2(\delta T)^{1/2}$, so that the probability of having an energy ε is given by the Gaussian expression

$$g(\varepsilon-\varepsilon')=(4\pi\delta T)^{-1/2}\exp\left[-(\varepsilon-\varepsilon'+\delta)^2/4\delta T\right]. \tag{15}$$

We express $f(\varepsilon)$ in the form of the condition for reproducing $f(\varepsilon)$ after reflection of the particle from the barrier and one oscillation in the well:

$$f(\varepsilon)=\int_{-\infty}^{\infty}\frac{g(\varepsilon-\varepsilon')f(\varepsilon')\,d\varepsilon'}{1+\exp(2\pi\varepsilon'/\omega)}. \tag{16}$$

Introducing the function

$$\varphi(\varepsilon)=f(\varepsilon)\left[1+\exp(2\pi\varepsilon/\omega)\right]^{-1}, \tag{17}$$

we rewrite (16) in the form

$$\varphi(\varepsilon)\left(1+\exp\frac{2\pi\varepsilon}{\omega}\right)=\int_{-\infty}^{\infty}g(\varepsilon-\varepsilon')\varphi(\varepsilon')\,d\varepsilon'. \tag{18}$$

The Fourier transformation

$$\varphi(\lambda)=\int_{-\infty}^{\infty}\exp\left[\left(i\lambda+\frac{1}{2}\right)\frac{\varepsilon}{T}\right]\varphi(\varepsilon)\,d\varepsilon \tag{19}$$

permits the integral equation (18) to be reduced to the multiplicative-difference equation

$$\varphi\left(\lambda-2\pi i\frac{T}{\omega}\right)=-\varphi(\lambda)\left\{1-\exp\left[-\frac{\delta}{T}\left(\lambda^2+\frac{1}{4}\right)\right]\right\}. \tag{20}$$

The boundary condition for this equation is obtained from the boundary condition (4) and is tantamount to the requirement that $\varphi(\lambda)$ have at $\lambda=-i/2$ a pole of the form

$$\varphi(\lambda)\approx-\frac{i\Omega\exp(-U_0/T)}{2\pi(\lambda+i/2)},\qquad |\lambda+i/2|\ll 1. \tag{21}$$

To solve (20) we factor the multiplicand of $\varphi(\lambda)$:

$$1-\exp\left[-\frac{\delta}{T}\left(\lambda^2+\frac{1}{4}\right)\right]=G_+(\lambda)G_-(\lambda), \tag{22}$$

where

$$G_\pm(\lambda)=\exp\left\{\pm\int_{-\infty}^{\infty}\frac{d\lambda'}{2\pi i}\ln\frac{\left[1-\exp\left[-\frac{\delta}{T}\left(\lambda'^2+\frac{1}{4}\right)\right]\right]}{(\lambda'-\lambda\mp i0)}\right\} \tag{23}$$

Those branch points of the integrand in (23) which are closest to the real axis are located at $\lambda'=\pm i/2$. This means that the functions $G_+(\lambda)$ and $G_-(\lambda)$ are analytic at $\mathrm{Im}\lambda>-1/2$ and $\mathrm{Im}\,\lambda<1/2$, respectively, and the intersection of the analyticity regions is a strip $|\mathrm{Im}\lambda|<1/2$.

We introduce a new function

$$\psi(\lambda)=\frac{1}{G_-(\lambda)}\prod_{n=1}^{\infty}\frac{G_+(\lambda+2\pi inT/\omega)}{G_-(\lambda-2\pi inT/\omega)}. \tag{24}$$

It can be directly verified that

$$\psi\left(\lambda-2\pi i\frac{T}{\omega}\right)=\psi(\lambda)G_+(\lambda)G_-(\lambda)$$
$$=\psi(\lambda)\left\{1-\exp\left[-\frac{\delta}{T}\left(\lambda^2+\frac{1}{4}\right)\right]\right\}. \tag{25}$$

It follows from a comparison of (25) with (20) and (21) that $\varphi(\lambda)$ differs from $\psi(\lambda)$ by a function that reverses sign following a shift by $2\pi i/T\omega$ and has a pole at $\lambda=-i/2$. Obviously, this function is $1/\sinh[\omega(\lambda+i/2)/2T]$. Thus, the solution of Eq. (20) with boundary condition (21) is

$$\varphi(\lambda)=-\frac{i\Omega\omega\psi(\lambda)\exp(-U_0/T)}{4\pi T\,\mathrm{sh}[\omega(\lambda+i/2)/2T]\psi(-i/2)}. \tag{26}$$

When (23) is allowed for, the infinite product (24) is transformed into

$$\psi(\lambda)=\exp\left\{\frac{\omega}{T}\int_{-\infty}^{\infty}\frac{d\lambda'}{4\pi i}\ln\left[1-\exp\left[-\frac{\delta}{T}\left(\lambda'^2+\frac{1}{4}\right)\right]\right]\mathrm{cth}\frac{\omega(\lambda'-\lambda)}{2T}\right\},\quad \mathrm{Im}\,\lambda<0. \tag{27}$$

The time τ is connected with the function $\varphi(\lambda)$ by the relation

$$1/\tau=\varphi(i/2-2\pi iT/\omega), \tag{28}$$

which is obtained when account is taken of the definition (14) and of the substitution (17). For the sought quantity A we obtain after substituting (26) and (28) and allowing for (1)

$$A=\frac{\omega\psi(i/2-2\pi iT/\omega)}{2T\sin(\omega/2T)\psi(-i/2)}.$$

Substituting here expression (27), we get ultimately after simple transformations

$$A=\frac{\omega/2T}{\sin(\omega/2T)}\exp\left\{\frac{\omega\sin(\omega/2T)}{2\pi T}\int_{-\infty}^{\infty}d\lambda\ln\left[1-\exp\left[-\frac{\delta}{T}\left(\lambda^2+\frac{1}{4}\right)\right]\right]\Big/\left(\operatorname{ch}\frac{\lambda\omega}{T}-\cos\frac{\omega}{2T}\right)\right\}. \tag{29}$$

Neglecting quantum effects ($\omega/T=0$) we have

$$A_{\rm cl}=\exp\left\{\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\ln\frac{\left[1-\exp\left[-\frac{\delta}{T}\left(\lambda^2+\frac{1}{4}\right)\right]\right]}{\left(\lambda^2+\frac{1}{4}\right)}\right\}. \tag{30}$$

In the case of small dissipation $\delta\ll T$ this leads to the Kramers result (7). Allowing for the next term we obtain in this limit

$$A_{\rm cl}\approx\frac{\delta}{T}\left[1+\zeta\left(\frac{1}{2}\right)\left(\frac{\delta}{\pi T}\right)^{1/2}\right],\quad \delta\ll T, \tag{31}$$

where $\zeta(x)$ is the Riemann zeta function, $\zeta(1/2)=-1.46$. We note that (31) is not analytic in δ in accord with the fact that reversal of the sign of the viscosity coefficient changes qualitatively the properties of a Brownian particle. In the limit $\delta\gg T$ it follows from (30) that

$$A_{\rm cl}\approx1-2(T/\pi\delta)^{1/2}\exp(-\delta/4T),\quad \delta\gg T.$$

In the case $\delta\ll\omega^2/T$ Eq. (16) goes over into the differential equation (10), and the result for A is given by (12). The general expression (29) enables us, in the limit δ, $\omega\ll T$, to obtain the following expansions:

$$A\approx\frac{\delta}{T}\left\{1+\zeta\left(\frac{1}{2}\right)\left(\frac{\delta}{\pi T}\right)^{1/2}+\frac{\omega}{2\pi T}\left[\ln\frac{\omega^2}{4\pi\delta}+2C\right]+\frac{\pi\delta}{12\omega}\right\},$$

$$\frac{\delta}{T}\ll\frac{\omega^2}{T^2}\ll1,$$

$$A\approx\frac{\delta}{T}\left\{1+\zeta\left(\frac{1}{2}\right)\left(\frac{\delta}{\pi T}\right)^{1/2}+\frac{\omega}{2\pi T}\left[\ln\frac{\omega^2}{4\pi\delta}+2C\right]+\frac{1}{24}\left(\frac{\omega}{T}\right)^2\left(\frac{T}{\pi\delta}\right)^{1/2}\right\},\quad\frac{\omega^2}{T^2}\ll\frac{\delta}{T}\ll1,$$

where $C\approx0.577$ is the Euler constant. It can be seen that in both cases the quantum corrections do not exceed in order of magnitude the classical correction $\sim(\delta/T)^{3/2}$.

If $1-\omega/2\pi T\ll1$ the bulk of the flux is due to tunneling, so that

$$A\approx\frac{1}{1-\omega/2\pi T}+\int_{-\infty}^{\infty}\frac{d\lambda}{4\pi}\ln\left\{1-\exp\left[-\frac{\delta}{T}\left(\lambda^2+\frac{1}{4}\right)\right]\right\}\Big/\operatorname{ch}2\pi\lambda.$$

The negative correction from the second term can be large only if $\delta\ll T$. In this case

$$A\approx\frac{1}{1-\omega/2\pi T}-\frac{1}{8\pi}\ln\frac{T}{\delta}.$$

Expressions (29) and (30) yield the complete solution of the Kramers problem in the quantum and classical cases, and the expressions that follow them give the expansions for A in limiting cases. These results were obtained assuming validity of the classical treatment in the main part of the well. This assumption is justified because for particles with energy ε close to the top of the barrier the period of the oscillations diverges logarithmicaly in the absence of friction, and the distance between the quasiclassical-quantization levels decreases and is of the order of $\omega/\ln(U_0/\varepsilon)$.

The exact expression obtained above for A is valid at sufficiently high temperatures $T>\omega/2\pi$ and at not too low values of the viscosity coefficient $\ln(\omega/\gamma)\ll U_0/\omega$. If one of these criteria is violated, it is necessary to allow explicitly for the non-parabolicity of the barrier. A problem of this type was investigated in an exponential approximation by Larkin and Ovchinnikov[5] using a differential equation of the type (10) employed in the limit $\delta\ll T$. Sütö and the author (Ref. 6) investigated the case of extremely low viscosity, when γ is comparable with the rate of tunneling decay of levels that are close to the bottom of the well. In this case it turns out that at $T^*>T>\Omega/\ln(U_0/\Omega)$ we get the activation relation

$$\tau\propto\exp\frac{T^*}{T},\quad T^*\approx\frac{2\Omega\ln(\gamma\tau_0)}{\ln(U_0/\Omega)},$$

and it is assumed that $U_0\gg T^*\gg\Omega/\ln(U_0/\Omega)$. The transition to a pure tunneling decay from the ground state takes place at $T\sim\Omega/\ln(U_0/\Omega)$.

We have assumed so far that after overcoming the barrier the particle leaves the potential well forever. Assume now that the decrease of the potential on the right of the barrier, shown in Fig. 2, gives way at some instant to an increase, so that the potential has two minima and one maximum, as shown in Fig. 2. If the dimensions of the right-hand well are large enough, the particle that lands in the well loses

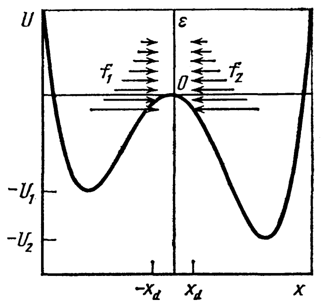

FIG. 2

in one oscillation an energy larger than T, and cannot return to the left-hand well. In this case our previously assumed one-well approximation is valid. If, however, a particle with energy on the barrier level loses in one oscillation in the left and right wells energies δ_1 and δ_2 comparable with T, it is necessary, when calculating the lifetime of the particle in the left-hand well, to take into account the possibility of its return from the right-hand well.

We introduce in analogy with (13) the functions $f_1(\varepsilon)$ and $f_2(\varepsilon)$, which yield the densities of the particles incident on the barrier from left and right wells, respectively (see Fig. 2). The function $f_1(\varepsilon)$ is formed by left-well particles with a period earlier than those reflected from the barrier, and right-well particles with period earlier than those passing through the barrier into the left well. The function $f_2(\varepsilon)$ is formed similarly, so that the system of equations for the two functions is

$$f_1(\varepsilon)=\int_{-\infty}^{\infty}\frac{g_1(\varepsilon-\varepsilon')}{1+\exp(2\pi\varepsilon'/\omega)}\left[f_1(\varepsilon')+f_2(\varepsilon')\exp\frac{2\pi\varepsilon'}{\omega}\right]d\varepsilon',$$

$$f_2(\varepsilon)=\int_{-\infty}^{\infty}\frac{g_2(\varepsilon-\varepsilon')}{1+\exp(2\pi\varepsilon'/\omega)}\left[f_1(\varepsilon')\exp\frac{2\pi\varepsilon'}{\omega}+f_2(\varepsilon')\right]d\varepsilon', \quad (32)$$

where g_1 (g_2) differs from (15) in that δ is replaced by δ_1 (δ_2).

The particle is initially in the left well, therefore Eqs. (32) must be supplemented with the boundary conditions

$$f_1(\varepsilon)\approx\frac{\Omega}{2\pi T}\exp\left(-\frac{U_0+\varepsilon}{T}\right),\quad f_2(\varepsilon)\approx 0\quad (-\varepsilon\gg T).$$

The rate of departure of the particles from the left well is then given by

$$\frac{1}{\tau_1}=\int_{-\infty}^{\infty}\frac{f_1(\varepsilon)-f_2(\varepsilon)}{1+\exp(-2\pi\varepsilon/\omega)}d\varepsilon. \quad (33)$$

Introducing the functions

$$\varphi_{1,2}(\varepsilon)=f_{1,2}(\varepsilon)[1+\exp(2\pi\varepsilon/\omega)]^{-1} \quad (34)$$

and using the Fourier transformation (19), we obtain the system of equations

$$\varphi_1(\lambda-2\pi iT/\omega)=-[1-g_1(\lambda)]\varphi_1(\lambda)+g_1(\lambda)\varphi_2(\lambda-2\pi iT/\omega),$$

$$\varphi_2(\lambda-2\pi iT/\omega)=g_2(\lambda)\varphi_1(\lambda-2\pi iT/\omega)-[1-g_2(\lambda)]\varphi_2(\lambda), \quad (35)$$

where

$$g_{1,2}(\lambda)\equiv\exp\left[-\frac{\delta_{1,2}}{T}\left(\lambda^2+\frac{1}{4}\right)\right]. \quad (36)$$

To calculate τ_1 it is suffices, according to (33), to know only the difference $\varphi_1-\varphi_2$. The equation for it can be easily found by solving the system (35) for $\varphi_{1,2}(\lambda-2\pi iT/\omega)$ and forming the required difference. We then obtain

$$\varphi_1(\lambda-2\pi iT/\omega)-\varphi_2(\lambda-2\pi iT/\omega)$$
$$=-\frac{[1-g_1(\lambda)][1-g_2(\lambda)]}{1-g_1(\lambda)g_2(\lambda)}[\varphi_1(\lambda)-\varphi_2(\lambda)]. \quad (37)$$

This equation is solved in trivial fashion, since the coefficient of $\varphi_1-\varphi_2$ in the right-hand side is made up of functions such as (22), and can therefore be directly factorized. In full analogy with the transition from (20) to (29), we obtain

$$\frac{1}{\tau_1}=\frac{\Omega_1}{2\pi}A(\delta_1,\delta_2)\exp\left(-\frac{U_0}{T}\right),$$

where

$$A(\delta_1,\delta_2)=A(\delta_1)A(\delta_2)/A(\delta_1+\delta_2), \quad (38)$$

with $A(\delta)$ given by (29). We indicate by way of example that in the case $\delta_{1,2}\ll\omega^2/T$ we have according to (12)

$$A(\delta_1,\delta_2)=\left[\frac{\delta_1\delta_2}{(\delta_1+\delta_2)T}\right]^{1-\omega/2\pi T}\frac{\Gamma(1-\omega/2\pi T)}{\Gamma(1+\omega/2\pi T)}\left(\frac{\omega}{2\pi T}\right)^{\omega/\pi T}.$$

It follows from (38) that the transition to the one-well approximation takes place at $\delta_2\gg\delta_1$ when $A(\delta_2)\approx A(\delta_1+\delta_2)$.

The quantities τ and τ_1 obtained above must be regarded as the parameters of the elementary processes and used subsequently in the phenomenological equations for the populations. Assume, for example, that there is no barrier, and that $U(x)$ tends with increasing x to zero in such a way that the integral (7) for the action converges. In this case, of course, we must set $\omega=0$ and use expression (30) for A.

The particles leaving the well will become thermalized in accord with Eq. (3), and after times $t\gg\gamma^{-1}$ will have a Boltzmann distribution in velocity, while their density $n(x,t)$ will obey the diffusion equation with a diffusion coefficient $D=T/m\gamma$. The particle flux into the well will be proportional to $n(0,t)$, and the proportionality coefficient can be related with the well characteristic from balance considerations under equilibrium conditions. Accordingly, to find the well population $n(t)$ we must solve the system

$$\frac{dn(t)}{dt}=-\frac{n(t)}{\tau}+A\left(\frac{T}{2\pi m}\right)^{1/2}n(0,t),$$
$$\frac{\partial n(x,t)}{\partial t}=D\frac{\partial^2 n(x,t)}{\partial x^2}$$

with the boundary condition

$$-D\frac{\partial n(x,t)}{\partial x}\bigg|_{x=0}=\frac{n(t)}{\tau}-A\left(\frac{T}{2\pi m}\right)^{1/2}n(0,t).$$

The solution is easily obtained by using Laplace transforms. Allowing for the exponential smallness of $1/\tau$ compared with the remaining quantities, we obtain ultimately

$$n(t)=\frac{2}{\pi}\int_0^{\infty}\frac{\exp(-z^2t/t_0)}{z^2+1}dz,\quad t_0=\frac{2\pi\gamma}{\Omega^2}\exp\frac{2U_0}{T}.$$

Equally worthy of attention here are both the nonexponential relation $n(t)$ and the fact that the activation dependence of the characteristic time t_0 is determined by double the barrier height.

The foregoing results can apparently be applied to such experimental systems as a compact Josephson junction and a superconducting ring closed by a Josephson junction. If the total current through the junction is given, the fluctuations of the phase φ of the order parameter are described by the equation[7]

$$C\frac{\Phi_0}{2\pi}\ddot{\varphi}+\frac{\Phi_0}{2\pi R}\dot{\varphi}+I_c\sin\varphi=I+I_f(t),$$

where C is the capacitance of the junction, Φ_0 the magnetic-flux quantum, R the resistance of the junction in the normal state, I_c the junction critical current, and $I_f(t)$ the fluctuation current. The potential $U(\varphi)$ and the viscosity coefficient are given in this case by

$$U(\varphi)=-\frac{\Phi_0}{2\pi}[I_c\cos\varphi+I\varphi],\qquad \gamma=\frac{1}{RC}.$$

If the current I is close to critical the shape of the potential well is close to a cubic parabola. We then obtain for the quantities that enter in Eqs. (1), (14), and (34)

$$U_0\approx\frac{2^{5/2}}{3\pi}\Phi_0 I_c\left(1-\frac{I}{I_c}\right)^{3/2},\quad \Omega^2=\omega^2=\frac{2^{1/2}\pi I_c}{C\Phi_0}\left(1-\frac{I}{I_c}\right)^{1/2},$$

$$\delta\approx 1{,}1\frac{\Phi_0}{R}\left(\frac{\Phi_0 I_c}{C}\right)^{1/2}.$$

We note that for a strong enough dissipation the superconducting-current damping time was calculated by Larkin and Ovchinnikov.[8]

The fluctuations of the magnetic flux through a superconducting ring closed by a Josephson junction are equivalent to the motion of a Brownian particle in a potential constituting a superposition of a parabola and a sinusoid. By variation of the external magnetic field the potential can be so transformed that it suffices to take only two minima into account. In this case the probability of the transition between the minima is given by (38). In a definite parameter range the decay of the metastable state of the superconducting ring can be described also in the one-well approximation, when expression (29) is valid. We note that fluctuations in a superconducting ring were analyzed in the high-viscosity limit by Kurkijarvi.[9]

It appears that Josephson junctions are the most suitable experimental objects for the observation of the decay of metastable states.[10] We note nonetheless that the problem of departure of a Brownian particle from a potential well is being actively discussed in connection with chemical reactions in a condensed phase,[11] thermal desorption of atoms from a surface,[12] catalysis on a surface,[13] and dynamics of a charge-density wave.[14]

It was shown thus in this article that the Fokker-Planck equation for the motion of a Brownian particle having a low enough viscosity, in the presence of potential barriers, is equivalent to an integral equation in the energy variable or else to a system of such equations. A Fourier transformation changes these equations into multiplicative-difference equations that can be solved by a factorization method, as was done above, or by another method.[15] The described scheme is suitable for the calculation of the decay rate of metastable states of a Brownian particle in one-well and two-well potentials with account taken of the tunneling transparency of the barrier.

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