Quantum dynamics of an isotropic cosmological model

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The dynamical theory of a closed isotropic and homogeneous cosmological model filled with elastic gas is considered. The phenomenological equation of state of the gas at low densities corresponds to the case of dust, and at high densities to an ultrarelativistic gas. The quantization is done by choosing a parameter t that plays the part of time and can be expressed in terms of the dynamical variables of the system. The state vector satisfies a differential equation of first order in the time t, and this ensures the usual interpretation of the norm and expectation values of quantum-mechanical operators. After an appropriate canonical transformation, the Hamiltonian takes the form of an oscillator Hamiltonian. Operators of the scale factor, $\hat{a}$, and the proper time, $\hat{\tau}$, are introduced and their spectrum investigated. It is shown that when quantum-mechanical effects are taken into account a cosmological singularity can still be reached. The behavior of the observables in coherent packets is investigated.

1. INTRODUCTION

After Hawking and Penrose had proved the theorems establishing the unavoidability of a singularity in cosmology, interest in a quantum description of the region of the singularity increased appreciably. It has frequently been suggested that on extrapolation backward in time allowance for the quantum effects of the gravitational field will halt the collapse of the Universe at Planck radii $\sim 10^{-33}$ cm. However, there is as yet no consistent scheme for constructing a quantum theory of gravitation. In our view, the main difficulty here is related to the general covariance of the theory, which has the consequence that the total Hamiltonian of the system is equal to zero due to the arbitrariness in the choice of the parameter called "time," this creating the appearance that there is no dynamical development in the quantum case. The studies on the quantum theory of cosmological models,[1-3] beginning with DeWitt's fundamental paper, were methodological in nature rather than giving a possibility of analyzing the quantum behavior of the Universe near the singularity. Because the model considered here is strongly simplified and specific, the present paper also follows primarily methodological aims, but since the model is exactly solvable it permits some general conclusions to be drawn, the main one of which appears to be that when quantum phenomena are taken into account the Universe can still reach (and pass through) a singularity.

Before we attack the quantum theory, we consider from the clossical point of view an isotropic cosmological model with matter having the energy-momentum tensor of dust at low densities and of ultrarelativistic type at high densities (ensuring at high densities the ultrarelativistic relationship $\varepsilon = 3p$ between the energy density and the pressure).

2. CLASSICAL DYNAMICS

There have been numerous studies (see Ref. 4) of the classical dynamics of systems containing dust or radiation, and also compressible matter. We give only the simplest solutions relating to noninteracting dust and radiation and restrict ourselves to the case of a closed cosmological model. In this case, there are two integrals of the motion—the total mass M of the dust and, by virtue of the conformal invariance of the radiation, the "energy" E of the radiation; if the metric is chosen in the form

$$ds^2=a^2(\eta)\left[f^2(\eta)\,d\eta^2-\frac{dx^2+dy^2+dz^2}{(1+{}^1/_4(x^2+y^2+z^2))^2}\right] \quad (1)$$

this leads to the following dependence of the energy density on the scale factor a (Ref. 4):

$$\varkappa_0\varepsilon=3M/a^3+3E/2a^4. \quad (2)$$

The Friedmann equations in this case (when the condition $f(\eta) = 1$ and the Planck system of units are chosen) have the form

$$\dot{a}^2+(a-M)^2=E+M^2, \quad (3)$$

and its solutions are

$$a(\eta)=M[1-(1+E/M^2)^{1/2}\cos\eta], \quad (4)$$

$$\tau(\eta)=\int a\,d\eta=M[\eta+(1+E/M^2)^{1/2}\sin\eta], \quad (5)$$

where τ is the proper time.

The solutions (4) and (5) determine in parametric form the dependence of the scale factor a on the proper time τ. This dependence is shown graphically in Fig. 1 for different relationships between M and E. For $E/M^2 = 0$ we get a purely Friedmann solution, and for $M = 0$ the curve degenerates into the circle obtained in cosmological models with pure radiation.[5]

The most characteristic feature of the solutions is the transition to the region of negative values of the scale factor a. What is the geometrical and physical meaning of this region of solutions?

On the three-dimensional spatial sphere we introduce a parametrization by Euler angles θ, φ, ψ with three mutually orthogonal infinitesimally small displacements:

$$dl_1=a(\sin\theta\sin\psi d\varphi+\cos\psi d\theta),$$

$$dl_2=a(\sin\theta\cos\psi d\varphi-\sin\psi d\theta), \qquad dl_3=a(\cos\theta d\varphi+d\psi).$$

A change in the modulus of a leads to a change in the volume constructed using these elements and proportional to a^3. A

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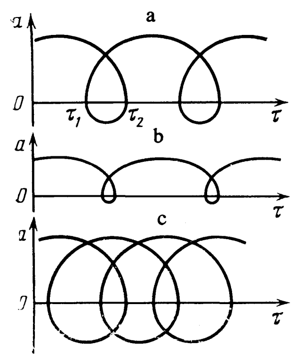

FIG. 1. Curves illustrating the dependence of the scale factor on the proper time in a closed isotropic cosmological model: a) E/M^2 of order unity, b) E/M^2 small, c) E/M^2 large.

change in the sign of a leads to a change in the direction of each of these displacements, i.e., it is a spatial reflection (P transformation); under it, the metric (1), which depends on a^2, does not change. In addition, for negative a the flow of the proper time ($d\tau = a d\eta$) is in the direction of its decrease, i.e., the change in the sign of a is simultaneously a local (with respect to the time) reversal of time (T transformation). The quantity $M \sim \rho a^3$ is an integral of the motion, and for positive a we have interpreted ρ as the density of the matter (particle number). When a changes sign the integral M is conserved, while ρ changes sign; this must necessarily be interpreted as a particle-antiparticle transformation (C transformation). Thus, at the point of the singularity there is a CPT transformation of the space, time, and matter.

Even in the classical case one can attempt to interpret the section of the curves with negative a as the dynamics of the development of an anti-Universe, without claiming that this expression has the usual meaning. Fig. 1a along the abscissa, we see that the section of the curve up to the proper time τ_1 corresponds to decrease in the scale factor a, i.e., to contraction of the Universe. At the time τ_1, a Universe–anti-Universe pair is created. At the proper time τ_2 the anti-Universe annihilates the originally contracting Universe, and the additionally created Universe continues to expand, repeating then the complete cycle. Although this interpretation is conditional, it should be pointed out that the solution we have obtained is an analytic solution of Einstein's equations without the introduction of any additional hypotheses into the theory of gravitation. In contrast, for example, DeWitt's subsidiary condition[1] postulating an infinitely high potential barrier at $a = 0$ that prevents penetration into the region $a < 0$ is an additional modification of the general theory of relativity.

We show that the existence of the loops is not peculiar to the model considered but is a characteristic type of solution in the general case. The disappearance of th loops (pure dust), like the degeneration into a circle (pure radiation) represent singular (limiting) points in the general solution. As is shown in Ref. 5, in the most general case when the densities increase without limit the leading term in the energy is $\sim a^{-4}$. As $a \to 0$, one can also ignore the term corresponding to the three-dimensional curvature of space, which is $\sim a^{-2}$, and therefore near $a = 0$ the cosmological equation has the

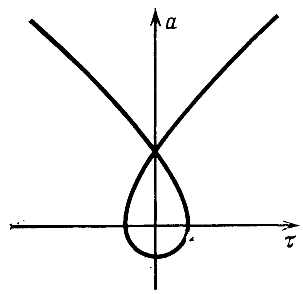

FIG. 2. Dependence of the scale factor on the proper time in an open isotropic cosmological model.

form

$$\frac{1}{a^2}\left(\frac{da}{d\tau}\right)^2 = \frac{E}{a^4}, \qquad a^2 = \pm 2E^{1/2}(\tau - \tau_0). \tag{6}$$

The solution (6) describes the appearance (for the plus sign) or disappearance (for the minus sign) of two branches of a parabola. On the transition to the region of values of a with large modulus, we go over to solutions of the type shown in Fig. 1. In the case of open isotropic models (for example, for three-dimensional curvature that vanishes) the solution given in Ref. 4,

$$\tau = 2(Ma - 2E)(Ma + E)^{1/2}/3M^2, \tag{7}$$

also has a loop (Fig. 2).

To construct the quantum theory of the model, we must formulate it in Lagrangian (and then in Hamiltonian) language.

3. EQUIVALENT DYNAMICAL MODEL WITH ELASTIC GAS

In the general theory of relativity, adiabatic irrotational motion of elastic gas can be described by means of a potential function $\sigma(x^i)$ that determines the specific enthalpy of the gas (cf. Ref. 6):

$$w = \left[g^{ij} \frac{\partial \sigma}{\partial x^i} \frac{\partial \sigma}{\partial x^j} \right]^{1/2}. \tag{8}$$

The action for the gas has the form

$$S_m = \int p(w) \sqrt{g}\, d^4x, \tag{9}$$

where the Lagrangian density is the pressure of the gas given (adiabatically) as a function of the enthalpy. Variation of S_m with respect to σ leads to the equations of motion

$$\frac{\partial}{\partial x^i}\left(\sqrt{g} \frac{dp}{dw} g^{ij} \frac{\partial \sigma}{\partial x^j} \frac{1}{w} \right) = 0. \tag{10}$$

Since $(dp/dw)_s = \rho$ is the gas density, and the vector field

$$u^i = g^{ij} \frac{\partial \sigma}{\partial x^j} \frac{1}{w} \tag{11}$$

satisfies the relation

$$g_{ij} u^i u^j = 1, \tag{12}$$

(u^i can be interpreted as the field of the 4-velocity of the gas), Eq. (10) takes the form of a mass conservation law:

$$\nabla_i (\rho u^i) = 0. \tag{13}$$

From (9), we obtain the energy-momentum tensor

$$T_{ik}=\frac{2}{\sqrt{g}}\frac{\partial(p\sqrt{g})}{\partial g^{ik}}=\frac{dp}{dw}\frac{1}{w}\frac{\partial\sigma}{\partial x^i}\frac{\partial\sigma}{\partial x^k}-pg_{ik}$$
$$=\rho w u_i u_k-pg_{ik}=(\varepsilon+p)u_iu_k-pg_{ik}, \quad (14)$$

where ε is the energy density. If we consider small oscillations of the gas in flat (or locally flat) space-time (representing σ in the form $\sigma = c^2t + \delta\sigma$), then Eq. (13), linearized with respect to $\delta\sigma$, is the d'Alembert equation with speed of sound

$$v^2=c^2\frac{\rho}{w}\frac{dw}{d\rho}=c^2\frac{dp}{d\varepsilon}. \quad (15)$$

For ultrarelativistic gas $\varepsilon = 3p$, i.e., the limiting speed of sound is $v^2 = c^2/3$. At the same time,

$$w\rho=\varepsilon+p=4p=dp/dw, \qquad p=w^4/4A^2, \quad (16)$$

$$\rho=\frac{w^3}{A^2}, \qquad w=(\rho A^2)^{1/3}, \qquad \varepsilon={}^3/_4A^{2/3}\rho^{4/3}. \quad (17)$$

For dust,

$$\varepsilon=\rho c^2. \quad (18)$$

We consider gas with compressibility law

$$\varepsilon=\rho c^2+{}^3/_4A^{2/3}\rho^{4/3}, \quad (19)$$

which ensures the asymptotic behaviors (17) and (18) at low and high densities, respectively. For the given case,

$$w=\frac{d\varepsilon}{d\rho}=c^2+(\rho A^2)^{1/3}, \qquad \rho=\frac{(w-c^2)^3}{A^2},$$
$$p=\int\rho\, dw=(w-c^2)^4/4A^2. \quad (20)$$

Assuming in the homogeneous and isotropic case $\sigma = \sigma(\eta)$ $(w = \dot\sigma/(af))$ and choosing the space-time metric in the form (1), we obtain the total action of the model in the Planck system of units:

$$S=\int\left(-\frac{\dot a^2}{2f}+f\frac{a^2}{2}+\frac{f}{4A^2}\left(\frac{\dot\sigma}{f}-a\right)^4\right)d\eta. \quad (21)$$

The canonical variables and Hamiltonian take the form

$$P_f=0,$$
$$P_\sigma=A^{-2}\left(\frac{\dot\sigma}{f}-a\right)^3, \qquad \dot\sigma=f(a+(A^2P_\sigma)^{1/3}),$$
$$P_a=-\frac{\dot a}{f}, \qquad \dot a=-fP_a, \quad (22)$$
$$H=f\left(-\frac{P_a^2}{2}-\frac{a^2}{2}+aP_\sigma+\frac{3}{4}A^{2/3}P_\sigma^{4/3}\right)\equiv fH_0(a,P_a,\sigma,P_\sigma),$$

and the equations of motion are

$$\partial H/\partial P_f=0, \quad f=\text{const}, \quad \partial H/\partial\sigma=0, \quad P_\sigma=\text{const},$$
$$\dot a=-P_a, \quad \dot P_a=a-P_\sigma, \quad \dot\sigma=a+A^{2/3}P_\sigma^{1/3}. \quad (23)$$

The condition $P_f = 0$ in (22) leads to $H = 0$; this last relation is a subsidiary condition (constraint) for the canonical variables a, P_a, P_σ. Taking into account this condition and adopting the "gauge" $f = 1$, we obtain the solutions

$$a=P_\sigma\left\{1+\left[1+{}^3/_2\left(\frac{A}{P_\sigma}\right)^{2/3}\right]^{1/2}\cos\eta\right\},$$
$$\tau=P_\sigma\left\{\eta+\left[1+{}^3/_2\left(\frac{A}{P_\sigma}\right)^{2/3}\right]^{1/2}\sin\eta\right\}. \quad (24)$$

When the notation is changed appropriately, the solutions (24) are identical to the solutions (4) and (5). The Friedmann case corresponds to $A = 0$.

4. REDUCED HAMILTONIAN DYNAMICS

We consider for the Hamiltonian system the extended phase space [7] that includes not only the coordinates and momenta $(x^i, P_i, 1\leqslant i\leqslant n)$ but also the time and energy as dynamical variables on an equal footing (we shall denote the complete set of canonical variables by z^α, $1\leqslant\alpha\leqslant 2n+2$). These variables have the Poisson brackets

$$\{x^i,x^j\}=0, \quad \{P_i,P_j\}=0, \quad \{x^i,P_j\}=\delta_j^i,$$
$$\{E,x^i\}=0, \quad \{E,P_i\}=0, \quad (25)$$
$$\{t,x^i\}=0, \quad \{t,P_i\}=0, \quad \{t,E\}=-1,$$

which in terms of the variables $z\alpha$ can be expressed in the form $\{z^\alpha, z^\beta\} = \omega^{[\alpha,\beta]}$ with corresponding nondegenerate skew-symmetric matrix $\omega^{[\alpha,\beta]}$. The existence of a constraint between the canonical variables that is specified by a function $F(z) = 0$ determines trajectories that lie entirely on the constraint surface:

$$\delta z^\alpha=\omega^{\alpha\beta}\frac{\partial F}{\partial z^\beta}\delta\eta,$$
$$\delta F=\frac{\partial F}{\partial z^\alpha}\delta z^\alpha=\omega^{\alpha\beta}\frac{\partial F}{\partial z^\alpha}\frac{\partial F}{\partial z^\beta}\delta\eta=0,$$
$$\delta x^i=\frac{\partial F}{\partial P_i}\delta\eta, \qquad \delta P_i=-\frac{\partial F}{\partial x^i}\delta\eta, \quad (26)$$
$$\delta E=\frac{\partial F}{\partial t}\delta\eta, \qquad \delta t=-\frac{\partial F}{\partial E}\delta\eta.$$

Equations (26) can be rewritten in parametric form with respect to the parameter η:

$$\frac{dz^\alpha}{d\eta}=\omega^{\alpha\beta}\frac{\partial F}{\partial z^\beta}\bigg|_{F=0}, \quad (27)$$

or, taking as independent variable the time t,

$$\frac{dx^i}{dt}=-\frac{\partial F/\partial t}{\partial F/\partial x^i}=\frac{\partial E}{\partial P_i}\bigg|_{F=0},$$
$$\frac{dP_i}{dt}=-\frac{\partial F/\partial t}{\partial F/\partial P_i}=-\frac{\partial E}{\partial x^i}\bigg|_{F=0}, \quad (28)$$

which go over into the usual Hamiltonian equations if the equation $F(x, P, E, t) = 0$ can be solved for E in the form $(E = \mathscr{H}(x, P, t)$. Equations (27) have the form of Hamilton equations with "Hamiltonian" $F(z)$ "canonically conjugate" to the nondynamical parameter η. In parametric form, the change in the function $f(z)$ with the parameter η is determined by the Poisson bracket (with E and t as dynamical variables together with x^i and P_i):

$$\frac{df}{d\eta}=\frac{\partial f}{\partial z^\alpha}\frac{dz^\alpha}{d\eta}=\omega^{\alpha\beta}\frac{\partial f}{\partial z^\alpha}\frac{\partial F}{\partial z^\beta}=\{f,F\}. \quad (29)$$

Moreover, Eqs. (27) determine extremals of the functional

$$S=\int\left[P_i\frac{dx^i}{d\eta}-E\frac{dt}{d\eta}-F(x,P,E,t)\right]d\eta, \quad (30)$$

where P_i and E must be expressed in terms of $dx^i/d\eta$ and $dt/d\eta$ by means of the equations

$$dx^i/d\eta = \partial F/\partial P_i, \quad dt/d\eta = -\partial F/\partial E. \tag{31}$$

Thus, we have a correspondence between the dynamic and parametric forms of the equations determining the dynamics of the system with the same action (30). If the action is such that the Hamiltonian, conjugate to the parameter η, vanishes (constraint condition), we are dealing with a pseudodynamical system, in which η is not a dynamical parameter. At the same time if originally there were $2n$ variables that depend on η, then the extended phase space is reduced—it has dimension $2n$, and not $2n+2$, as in an ordinary Hamiltonian system. For reduction of the system to the ordinary dynamic form it is necessary to choose among the $2n$ dynamical variables one (t) as the time (pseudotime) and $n-1$ dynamical variables, whose Poisson brackets with t vanish. Simultaneously, we must find the Hamiltonian, conjugate to t, and the $n-1$ momenta conjugate to the dynamical variables, with corresponding Poisson brackets between all the variables. Thus, the system actually has $n-1$ dynamical degrees of freedom.

In the quantum theory, the state vector is determined as a function of these $n = 1$ dynamical variables, is normalized with respect to them, and evolves in the pseudotime. The expectation value of any operator for given value of the pseudotime is expressed in terms of an integral with respect to the $n-1$ variables.

Thus, to determine the quantum dynamics it is necessary to choose one of the dynamical variables as a pseudotime. The dynamical development of the state vector is then determined by the dependence of the momentum conjugate to it (the energy) on the other dynamical variables. This dependence can be obtained from the constraint $F(z) = 0$.

If as such a variable we take a variable $t(z)$ such that

$$\{t, F\} = dt/d\eta = 1 \tag{32}$$

(we shall call it the kinematic time), then the constraint equation is linear in the momentum canonically conjugate to it. Indeed, if we introduce $n-1$ variables $y^i(z)$ and $\pi_i(z)$ with the usual mutual Poisson brackets and having vanishing Poisson brackets with t and P_t, and $\{t, P_t\} = -1$, then for $\Phi(z) = F + P_t$ we have $\{t, \Phi\} = 0$, i.e., $\Phi(z)$ is a function of only y^i, π_i, and t, and the constraint equation takes the form

$$P_t = \Phi(y, \pi, t), \tag{33}$$

i.e., it is linear in P_t. In the quantum theory, this leads to a differential equation of first order in the kinematic time for the state vector and to the usual interpretation of the norm and expectation values of the operators. All this will be demonstrated for the cosmological model defined above.

It should be noted that in a series of papers[8] devoted to the dynamical description of dust, Lund also obtained a quantum-mechanical equation of first order in the time. However, in his papers the order of the equation depends essentially on the law of compressibility, first order corresponding to dust.

5. QUANTUM THEORY OF THE COSMOLOGICAL MODEL

The vanishing of the Hamiltonian (22) leads to the conclusion that the variables a, P_a, σ, and P_σ form an extended phase space, and the proper time τ and the kinematic time t, regarded as dynamical variables, must be functions of these variables. Thus, the relation $d\tau = af d\eta$, regarded as a dynamical relation,

$$d\tau/d\eta = af = \{\tau, H\}, \tag{34}$$

leads to the representation of the proper time and the dynamical variables additional to it in, for example, the form

$$\tau = \frac{\sigma + (A/P_\sigma)^{2/3} P_a}{1 + (A/P_\sigma)^{2/3}}, \quad P_\tau = P_\sigma + 3A^{2/3} P_\sigma^{1/3},$$
$$y = a + 3A^{2/3} P_\sigma^{1/3}, \quad \pi_y = P_a. \tag{35}$$

In the general case, the Poisson bracket

$$\{a, \tau\} = [1 + (P_\sigma/A)^{2/3}]^{-1} \tag{36}$$

is nonzero (it is zero only in the purely Friedmann case $A = 0$); this indicates that these quantities cannot be measured simultaneously in the quantum theory, and figures of the type shown in Fig. 1 have meaning only for the expectation values.

In the variables (35), the constraint equation has a rather complicated form. We therefore introduce canonical variables associated with the kinematic time t, choosing as variables

$$t = \frac{\sigma - P_a}{P_\sigma + A^{2/3} P_\sigma^{1/3}}, \quad P_t = \frac{P_\sigma^2}{2} + \frac{3}{4} A^{2/3} P_\sigma^{4/3},$$
$$x = a - P_\sigma, \quad P_x = P_a,$$
$$\{t, H\} = dt/d\eta = f, \quad \{t, x\} = 0, \quad \{t, P_x\} = 0. \tag{37}$$

In these variables, the constraint equation ($H_0 = 0$, see (22)) takes the form

$$P_t = P_x^2/2 + x^2/2, \tag{38}$$

i.e., in the variables x, t the cosmological model is described by an oscillator Hamiltonian. The general solution of the quantum problem,

$$\Psi(x, t) = \sum_{n=0}^{\infty} C_n U_n(x) e^{i(n+1/2)t}, \tag{39}$$

is normalized with respect to the variable x (U_n are normalized wave functions of the stationary states of the oscillator). It is this circumstance—the normalization of the wave function with respect to one and not two variables—that leads to ordinary quantum dynamics of the system despite the existence of the constraint, which appears formally as vanishing of the Hamiltonian. The quantity (38) is an integral of the motion in the cosmological model, restoring the energy conservation law for the given case.

Taking into account the relation (37) between P_t and P_σ, we obtain a discrete spectrum of the operator P_σ:

$$(P_\sigma)_{n,m} = \omega_n \delta_{n,m}, \tag{40}$$

where ω_n is a positive root of the equation

$$\omega_n^2 + {}^3/_2 A^{2/3} \omega_n^{4/3} = 2n + 1. \tag{41}$$

The solutions of this equation for the lowest values of n are given in Table I. We note that P_σ determines the total mass of the gas in the Universe in accordance with

$$M = (3\pi c\hbar/2k)^{1/2} P_\sigma, \tag{42}$$

where the mass is measured in dimensional units.

TABLE I. Eigenvalues ω_n for different values of A.

A	n					
	0	1	2	3	4	5
0	1	1.73	2.24	2.65	3	3.32
0.25	0.76	1.43	1.90	2.28	2.62	2.92
1	0.56	1.12	1.53	1.88	2.18	2.46
10	0.22	0.50	0.72	0.91	1.09	1.25

In the variables x and t, the scale factor a and the proper time τ are represented by

$$a=x+P_\sigma,\quad \tau=P_\sigma t+P_x,$$
$$\{\tau,t\}=-t\frac{dP_\sigma}{dP_t}=\frac{t}{P_\sigma+A^{2/3}P_\sigma^{1/3}}. \tag{43}$$

The Poisson bracket $\{\tau, t\}$ vanishes only for $t=0$ (this is the condition of the choice of the initial value of t), in agreement with Misner's conclusion[8] that the proper time has an operator nature in the quantum theory of gravity.

An important question about the quantum behavior of the system is that of the spectrum of the operator $\hat{a}$. In the energy representation, the matrix elements of $\hat{a}$ have the form

$$a_{kn}=\omega_n\delta_{k,n}+(n/2)^{1/2}e^{it}\delta_{k,n-1}+((n+1)/2)^{1/2}e^{-it}\delta_{k,n+1}. \tag{44}$$

To establish the nature of the spectrum of $\hat{a}$, numerical computer methods can be employed. This analysis shows that if the matrix is truncated at large n ($40\leqslant n\leqslant 500$) the following occurs: For $A=0$, the eigenvalues of the matrix are positive and crowd together with increasing dimension of the matrix, the lowest level tending monotonically to zero (its values for matrices of ranks 40, 100, 500, respectively, are $1.37\cdot 10^{-2}$, $3.6\cdot 10^{-3}$, $5.2\cdot 10^{-4}$ in units of the Planck length). For $a>0$, the levels pass over to the region of negative values and crowd together with increasing rank of the matrix. This corresponds to the fact that for $A=0$ the spectrum of the operator $\hat{a}$ is continuous and positive with lower limit $a=0$. For $A>0$ the spectrum is continuous, but a may be either positive or negative. We arrive at the same conclusions if we make a semiclassical analysis of the spectrum (the idea of this analysis was proposed by A. M. Satanin). Indeed, regarding $\hat{a}$ as the Hamiltonian of some dynamical system, let us investigate the phase trajectories for constant value of a. We have

$$a-P_x=P_\sigma,\quad \frac{P_\sigma^2}{2}+\frac{3}{4}A^{2/3}P_\sigma^{4/3}=\frac{P_x^2}{2}+\frac{x^2}{2},$$
$$x-a=y,\quad P_x=P_y, \tag{45}$$
$$^3/_2A^{2/3}y^{4/3}-ay-P_y^2=a^2,$$

from which it can be seen that in the plane (y, P_y) the phase trajectories go away to infinity for any value of a and the parameter A. For A = 0, we have $a=x+(x^2+P_x^2)^{1/2}>0$ for all x and P_x.

Thus, even in the Friedmann model (dust) none of the quantum processes can hinder the existence of states for which the value of the scale factor a is arbitrarily near zero. At the same time, in this model penetration into the region of negative a does not occur even without the DeWitt "wall" (the spectrum of $\hat{a}$ is positive).

We now consider the proper-time operator $\hat{\tau}$. We also estimate its spectrum by the quasiclassical method:

$$\hat{\tau}=\hat{P}_\sigma t+\hat{P}_x(t)=\hat{P}_\sigma t+\hat{P}_x^0\cos t-\hat{x}^0\sin t. \tag{46}$$

Here, P_x^0 and $\hat{x}^0$ are constant, time-independent matrices with the commutation relations of a coordinate and momentum. We have separated the time dependence explicitly, since the spectrum of the operator $\hat{\tau}$ depends on the time, and we shall regard t simply as a parameter. Substituting P_σ, expressed by means of (38), in (46), and taking into account (37), we obtain

$$tP_\sigma=\tau+x^0\sin t-P_x^0\cos t,$$
$$t^2(P_x^{02}+x^{02})=(\tau+x^0\sin t-P_x^0\cos t)^2$$
$$+^3/_2t^{2/3}A^{2/3}(\tau+x^0\sin t-P_x^0\cos t)^{4/3}.$$

Collecting on one side of the equation the terms quadratic in P_x^0 and x^0, we obtain

$$(t^2-\cos^2 t)P_x^{02}+(t^2-\sin^2 t)x^{02}+2\sin t\cos tP_x^0x^0$$
$$=\tau^2+2\tau(x^0\sin t+P_x^0\cos t)+^3/_2t^{2/3}A^{2/3}(\tau+x^0\sin t+P_x^0\cos t). \tag{47}$$

We consider first of all the case $A=0$ (dust). Then (47) determines a curve of second order with discriminant

$$(t^2-\cos^2 t)(t^2-\sin^2 t)-\sin^2 t\cos^2 t=t^2(t^2-1), \tag{48}$$

i.e., for $-1<t<1$ (the initial phase of the time is distinguished by the time of commutation of τ with t) we have hyperbolas—the spectrum of the operator $\hat{\tau}$ is continuous and lies in the interval from $-\infty$ to $+\infty$. For $t>1$, the spectrum is discrete and in accordance with semiclassical quantization, when the area of the ellipse is $\pi(2n+1)$, we obtain

$$\tau_n=(2n+1)^{1/2}(t^2-1)^{1/4}t^{-1/2}. \tag{49}$$

For $t<-1$, the eigenvalues τ_n have the same modulus but negative sign, as can be seen from (46). This means that any wave packet contains states with different τ_n varying in accordance with the law (49) with the passage of time; in addition, any wave packet spreads in the τ space.

In the case $A\neq 0$, we cannot calculate the spectrum, but qualitatively it stays the same as for $A=0$, since the terms with x and P_x for $A\neq 0$ have degree lower than the second.

It should be noted that the actual values of τ do not have a particular physical meaning. In the problem we consider, only the specific enthalpy is related to the proper time, $w=d\sigma/d\tau$, and the derivatives with respect to the proper time are expressed in terms of a.

TABLE II. Values of $\langle\omega\rangle$, D_ω, Q in coherent states for different values of A and $R=\mu\sqrt{2}$.

A	R	⟨ω⟩	D_ω	Q
	0	1	0	0
	0.28	1.03	0.021	0.20
0	0.57	1.11	0.078	0.39
	1.41	1.64	0.32	0.79
	7.07	7.11	0.50	1.02
	0	0.76	0	0
	0.28	0.79	0.017	0.19
0.25	0.57	0.87	0.065	0.36
	1.41	1.35	0.27	0.73
	7.07	6.57	0.47	1.00
	0	0.56	0	0
	0.28	0.58	0.012	0.16
1.0	0.57	0.67	0.047	0.27
	1.41	1.06	0.20	0.63
	7.07	5.88	0.42	0.97

It is interesting to consider stationary and coherent states of a wave packet. In an arbitrary state (39), we obtain for the expectation values of a and τ and their dispersions

$$\langle a\rangle=\langle\omega\rangle-R\cos(t-t_1),\quad \langle\tau\rangle=\langle\omega\rangle t-R\sin(t-t_1),$$
$$D_a=D_\omega+D_x(t)-Q\cos(t-t_2)-2R\langle\omega\rangle\cos(t-t_1),\quad (50)$$
$$D_\tau=D_\omega t^2+D_{P_x}(t)-Q\sin(t-t_2)-2R\langle\omega\rangle t\sin(t-t_1),$$

where

$$\langle\omega\rangle=\sum_{n=0}^{\infty}\omega_n|C_n|^2,\quad Re^{it_1}=-\sum_{n=0}^{\infty}\left(\frac{n+1}{2}\right)^{1/2}\dot{C}_{n+1}C_n,\quad (51)$$
$$Qe^{it_2}=-\sum_{n=0}^{\infty}(\omega_n+\omega_{n+1})\left(\frac{n+1}{2}\right)^{1/2}\dot{C}_{n+1}C_n.$$

In the special case of stationary states,

$$\langle n|\hat{a}|n\rangle=\omega_n,\quad \langle n|\hat{\tau}|n\rangle=\omega_n t,\quad D_a=D_\tau=n+{}^1/_2.$$

This means that in these states the expectation value of $\hat{a}$ is always greater than zero and the proper time on the average always "flows forward."

For coherent states (for $t_1=t_2=0$)

$$C_n=\exp\left(-\frac{\mu^2}{2}\right)\frac{(-\mu)^n}{(n!)^{1/2}}.\quad (52)$$

At large μ, the centers of the packets move along the classical trajectories ($\langle\omega\rangle<R$), but it can be seen from the values of $\langle\omega\rangle$, D_ω, R, and Q given in Table II for $A=0$, 0.25, and 1 that for small $\mu(\langle\omega\rangle>R)$ the centers of the packets do not enter the region of negative a but move along truncated cycloids. For $A=0$, this occurs for any coherent packet (Fig. 3).

6. CONCLUSIONS

Although the model is strongly simplified, it contains a number of characteristic features of cosmology near a singularity.[10] Thus, if in the case $A>0$ a simple topology is specified initially, the topology is nevertheless nontrivial from the point of view of the proper time and varies with the time, namely, at different instants of the proper time there are different numbers of disconnected "Universes" and "anti-Universes," though it is true that the difference between their numbers is conserved.

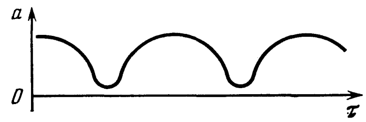

FIG. 3. Dependence of the mean value of the scale factor on the mean value of the proper time in a coherent state with small value of the classical amplitude.

The most characteristic feature of the solution is the existence of a region with negative values of the scale factor a; as shown above, this can be interpreted as an actually realized CPT transformation of the space, time, and matter. Such behavior is manifested not only in the present model but is characteristic of cosmological models with elastic matter.

The positivity of the spectrum of the operator $\hat{a}$ in the case of dust in the absence of a wall at $a=0$ indicates that the theory does not require any addtions of such type. At the same time, wave packets contain states with values of a arbitrarily near zero and the quantum-mechanical effects do not eliminate the singular state; however, in a real wave packet an infinitesimally small fraction of the system is in such a state, as, for example, in the ground state of the hydrogen atom there is a probability for finding the particle at $r=0$, a region with infinite value of the potential.

In the case of elastic matter, the wave packets can be at least partly in the region of negative a, which can be interpreted as the unavoidable presence of both matter and antimatter near the singularity.

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