Stochastic instability of rays and the speckle structure of the field in inhomogeneous media

S. S. Abdullaev and G. M. Zaslavskiĭ

L. V. Kirenskiĭ Institute of Physics, Siberian Branch of the Academy of Sciences of the USSR, Krasnoyarsk

(Submitted 2 April 1984)
Zh. Eksp. Teor. Fiz. **87**, 763–775 (September 1984)

It is shown that the dynamic stochastization of rays in regularly inhomogeneous waveguide media produces a speckle structure over distances of the order of the correlation uncoupling length for ray oscillation phases in the plane perpendicular to the axis of the waveguide channel. The statistical properties of the speckle structure, the characteristic dimensions of spots, the uncoupling length for ray oscillation phases, and the ray diffusion length are determined in the shortwave approximation. The characteristic values of these parameters are reported for the case of the acoustic waveguide channel with a periodically inhomogeneous surface.

1. INTRODUCTION

Studies of the propagation of waves to great distances in inhomogeneous media (for example, in the ionosphere[1] or the ocean[2]) show the problem to have definite peculiarities. The fact that the waves propagate to great distances means that small inhomogeneities in the medium become significant because their effect is cumulative. Even in the geometric-optics approximation, such cumulative effects may turn out to be important and may appear well before this approximation ceases to be valid. The effect of small inhomogeneities on wave propagation can be estimated by applying nonlinear Hamiltonian dynamics to rays. This approach reveals new physical properties of rays such as "nonlinear resonances" between a ray and a periodic inhomogeneity, stochastic instability and diffusive drift of a ray out of the waveguide channel, and so on.[3,4]

The stochastic instability of rays arises in media with regular (nonrandom) inhomogeneities and is not only the strongest instability but also a relatively typical situation, for example, in waveguide channels with two-dimensional cross section of arbitrary form (Ref. 4).[1)] One of the important manifestations of stochastic instability of rays is the considerable wavefront distortion at sufficiently long distances from the point of entry of the radiation. This wavefront distortion is the subject of the present paper.

It is well known that, when a rough surface on which the dimensions of inhomogeneities exceed the radiation wavelength is illuminated by coherent laser radiation, a deterioration is observed in the image of the surface at long enough distances. The image becomes spotty (a speckle structure appears), and this is due to interference efffects (see, for example, Ref. 5). When the source of radiation is completely coherent, the speckle structure can be "deciphered" despite its complicated nature by using adaptive systems to produce wavefront reversal. The reconstruction of the original image is, however, incomplete because of the onset of wave-surface catastrophes.[6,7]

When the radiation entering the waveguide channel is initially noncoherent, the speckle structure is smeared out and the wavefront cannot be reversed. The loss of reversibility occurs over distances of the same order as the length over which spatial coherence is lost. Stochastic instability then plays an exclusive role since ray instability develops with exponential rapidity.[8,3]

It is thus clear that stochastic instability of rays that results from even slight inhomogeneity in the waveguide channel leads to a twofold effect, namely: (1) the speckle distribution becomes a random and (2) any slight broadening of the signal at entry leads to an irreversible smearing of the image.

These two effects are investigated below.

In addition, it will be useful to consider the analogy with similar problems in quantum mechanics. The possibility of quasirandom wave-function profiles in quantum-mechanical K-systems, i.e., systems with stochasticity in the classical limit, was discussed in Refs. 9 and 10. A picture of nodal lines of wave functions in quantum-mechanical K-systems that was close to the random speckle structure was obtained as a result of a numerical analysis in Refs. 11 and 12.

2. QUALITATIVE CONSIDERATIONS

Consider a waveguide medium whose axis lies along the coordinate z axis. The vector $\mathbf{r} = (x, y)$ lies in the plane perpendicular to this axis. The refractive index $n = n(\mathbf{r}, z)$ of this medium depends both on the transverse coordinates x, y and the longitudinal coordinate z. The propagation of a monochromatic wave $u(\mathbf{r}, z)$ of frequency ν is described by the Helmholtz wave equation

$$[\Delta + k_0^2 n^2] u(\mathbf{r}, z) = 0, \tag{2.1}$$

where $k_0 = \nu/c$ is the wave number and c is the velocity of the wave in a homogeneous medium.

Suppose that the waveguide channel is excited in the $z = 0$ plane by a spatially coherent wave beam whose lateral distribution is described by the function $u_0(\mathbf{r})$. In the geometric-optics approximation, the wave field $u(\mathbf{r}, z)$ in a $z > 0$ plane in the waveguide channel with the above boundary condition is given by (see, for example, Ref. 13)

$$u(\mathbf{r}, z) = \sum_{l=1}^{N} \frac{u_0(\mathbf{r}_{0l})}{[\mathscr{T}_l(\mathbf{r}, z)]^{1/2}} \exp\left\{ ik_0 S_l(\mathbf{r}, z) + \frac{i\pi\mu_l}{2} \right\}, \tag{2.2}$$

where the integral

 0038-5646/84/090435-07$04.00

$$S_l(\mathbf{r},z)=\int\limits_{\Gamma_l} n(\mathbf{r},z)\,ds$$

is evaluated along the arc Γ_l of the trajectory of the lth ray arriving at the point of observation $(\mathbf{r}, z)$, $\mathbf{r}_{0l}$ is the initial coordinate of the lth arc in the $z=0$ plane, and μ_l is the Morse index for the lth ray. The generalized divergence of the ray on the trajectory of the lth ray is given by

$$\mathscr{T}_l(\mathbf{r},z)=\frac{n(\mathbf{r},z)}{n(\mathbf{r}_{0l},0)}\frac{da}{da_0},$$

where da_0 and da are the transverse cross sections of the ray tube in the $z=0$ and z planes, respectively. The sum in (2.2) is evaluated ove all ray trajectories arriving at the point of observation $(\mathbf{r}, z)$.

Consider the case where the inhomogeneities in the wave channel are such that, in certain regions of the ray phase space, the ray trajectories are stochastic. It has already been noted[3,4] that ray stochastization arises both for periodic disturbances along the direction of propagation of the wave and for disturbances of the lateral cross section of the waveguide channel. The ray stochastization region in the waveguide is determined by the specific refractive index profile of the medium and the inhomogeneity of the latter.

When the waveguide channel is excited by an arbitrary wave beam $u_0(\mathbf{r})$, all modes of the waveguide will, in general, be excited. The wave field (2.2) will therefore consist of two parts, one of which corresponds to regular rays, i.e., those that do not enter the stochastic region, and the other corresponds to those that do enter this region. We shall confine our attention to the latter rays. Their trajectories are stochastically unstable, i.e., two ray trajectories with similar initial coordinates in phase space will diverge exponentially as they propagate in the waveguide channel. We then have $\mathscr{T}_l\sim e^{hz}$ where $h=z_0^{-1}\ln K$ is the instability growth rate (Kolmogorov entropy), z_0 is a characteristic period of the ray, and K is a parameter such that ray trajectories become stochastic for $K\gtrsim 1$ ($K\sim 1$ is the stochastic limit).

The number N of rays arriving at the point of observation is also found to increase exponentially, i.e., $N\sim e^{hz}$. Since the ray trajectories are stochastic, the optical paths $S_l(\mathbf{r}, z)$ traversed by different rays will be random and randomly distributed. It may be expected that the phases $k_0S_l(\mathbf{r}, z)/2\pi(\mathrm{mod}\,1)$ are uniformly or almost uniformly distributed in the interval (0, 1).

The wave field (2.2) is thus the sum of a large number of quasiplane waves with phases that are uniformly distributed in the interval $(0,2\pi)$. Since the waveguide channel is excited by a coherent field, all the quasiplane waves arriving at the point of observation will be mutually coherent. They will therefore interfere. However, the wave phases are random so that the interference pattern will be irregular, and the field amplitude maxima and minima will have a random distribution in space. The picture we have just described corresponds to what is commonly called the speckle structure.[5,14]

Let us now estimate the characteristic size Δl_s of speckle-structure spots in the plane perpendicular to the z axis. We shall do this by representing the wave field by the sum of normal modes of the undisturbed waveguide. For short-wave normal modes, the characteristic separation between successive maxima of the wave function corresponding to a particular mode with longitudinal wave number $k=k_0E$ is of the order of $\Delta l\approx\pi/k_0p(r)$, where $k_0p(r)=k_0[n^2(r)-E^2]^{1/2}$ is the local transverse wave number of this mode. The quantity E is called the mode delay. The characteristic size of the speckle structure spot is determined by the mode with the smallest separation Δl, i.e., the highest mode entering the stochastic region. When the longitudinal wave number of this mode is $k_F=k_0E_F$, the minimum size of the speckle-structure spot is of the order of

$$\Delta l_s\approx\pi/k_0p_F(r),\quad p_F(r)=[n^2(r)-E_F^2]^{1/2}. \tag{2.3}$$

The quantity E_F is determined by the particular form of the problem.

The distortion of the wave front in the course of propagation along the channel can be described in greater detail as follows. The exponential development of stochastic instability ensures that the phases of ray oscillations in the x, y plane will mix over the shortest scale along the z axis. This gives rise to a loss of information about the detailed structure of the wavefront of the original field. The slow diffusion of rays leading to the random distribution of intensity in the x, y plane at right angles to the channel axis occurs over much longer distances along this axis. However, a uniform intensity distribution is not established on an arbitrary $z=\mathrm{const}$ plane. This is prevented by the non-zero field correlation radius $r_c\sim\Delta l_s$, determined by (2.3) (The relationship between r_c and Δl_s will be examined in greater detail in Section 4.)

We note that the appearance of the same correlation scale in quantum mechanics is determined by the uncertainty principle[10] where Δl_s must be interpreted as the minimum de Broglie wavelength.

3. STOCHASTIC PROPERTIES OF THE SPECKLE STRUCTURE

We shall now describe the wave field in the plane perpendicular to the z axis of the waveguide channel when the rays exhibit stochastic instability. We shall consider that an individual speckle structure constitutes one of the realizations of the wave field in the $z=\mathrm{const}$ plane. Owing to stochastic instability of the rays, the wave-field distribution is random in space. The properties of the speckle structure can be determined by studying the statistical properties of the field distribution $U(\mathbf{r}, z)$.

For simplicity, we shall examine the case where the wave propagates at small angles to the z axis. If we then substitute

$$u(\mathbf{r},z)={}^1/_2\{v(\mathbf{r},z)\exp(ik_0n_0z)+v^*(\mathbf{r},z)\exp(-ik_0n_0z)\},$$
$$n_0=\max n(\mathbf{r},z), \tag{3.1}$$

the Helmholtz equation (2.1) can be replaced by the approximate parabolic equation (see, for example, Ref. 15)

$$2ik_0n_0\partial v/\partial z+\Delta_\perp v+k_0^2[n^2(\mathbf{r},z)-n_0^2]v=0,$$
$$\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2,\quad v|_{z=0}=u_0(\mathbf{r}). \tag{3.2}$$

It will be convenient to write the refractive index $n(\mathbf{r}, z)$ of the medium in the form

$$n^2(\mathbf{r}, z) = n^2(\mathbf{r}) + \varepsilon f(\mathbf{r}, z), \tag{3.3}$$

where $n(\mathbf{r})$ is the refractive index of the wave channel that is homogeneous (undisturbed) along the z axis, $\varepsilon f(\mathbf{r}, z)$ is the perturbation that is due either to the in homogeneity of the medium in the z direction, or the inhomogeneity of the lateral cross section, and ε is a small dimensionless parameter.

Suppose that the orthonormal functions $\varphi_m(\mathbf{r})$ are the normal modes of the undisturbed waveguide, i.e.,

$$[\Delta_\perp + k_0^2 n^2(\mathbf{r})]\varphi_m(\mathbf{r}) = k_m^2 \varphi_m(\mathbf{r}), \tag{3.4}$$

where $k_m = k_0 E_m$ are the longitudinal wave numbers of the modes. The wave field $v(\mathbf{r}, z)$ in the disturbed waveguide can be written in the form of the expansion

$$v(\mathbf{r}, z) = \sum_m a_m(z)\varphi_m(\mathbf{r}). \tag{3.5}$$

The expansion coefficients $a_m(z)$ satisfy the coupled-wave equation (see, for example, Ref. 16):

$$i\frac{da_m(z)}{dz} = \tilde{E}_m a_m(z) + \sum_{m'} V_{mm'}(z) a_{m'}(z), \qquad \tilde{E}_m = (k_m^2 - k_0^2)/2k_0 n_0, \tag{3.6}$$

$$V_{mm'}(z) = -\frac{k_0 \varepsilon}{2n_0}\int d^2 r f(\mathbf{r}, z)\varphi_m^*(\mathbf{r})\varphi_{m'}(\mathbf{r}).$$

Let us investigate the solution of (3.6) subject to the boundary condition $a_m(z) = a_m(0)$ for $z = 0$. Suppose that $G(\mathbf{r}, z; \mathbf{r}', 0)$ is the Green function for the parabolic equation (3.2). Using the orthogonality of the functions $\varphi_m(\mathbf{r})$ and expressing the solution $v(\mathbf{r}, z)$ in terms of the Green function, we obtain

$$a_m(z) = \sum_{m'} G_{mm'}(z, 0) a_{m'}(0), \tag{3.7}$$

$$G_{mm'}(z, 0) = \int dr^2 \int dr_0^2 \varphi_m(\mathbf{r}) G(\mathbf{r}, z; \mathbf{r}_0, 0)\varphi_{m'}(\mathbf{r}_0).$$

To simplify our calculations, we shall confine our attention to the case of a plane-layered waveguide channel. The refractive index is then independent of one of the transverse coordinates, say, y. Moreover, we shall suppose, as in Ref. 3, that the perturbation $\varepsilon f(x, z)$ is periodic along z.

To determine the coefficients $a_m(z)$ in accordance with (3.7), we shall use the short-wave approximation[17] for the Green function of the parabolic equation (3.2):

$$G(x, z; x_0, 0) = \sum_{l=1}^{N} \frac{k_0 n_0}{(2\pi i)^{1/2}} \left| \frac{\partial^2 S_l(x, z; x_0, 0)}{\partial x \partial x_0} \right|^{1/2} \times \exp\{ik_0 n_0 S_l(x, z; x_0, 0) + i\pi\mu_l/2\}, \tag{3.8}$$

where $S_l(x, z; x_0, 0)$ is the optical path along the lth ray joining the points (x, z) and $(x_0, 0)$. The ray trajectories $x(z)$ are the solutions of the Hamilton equation

$$\dot{x} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial x,$$
$$\dot{x} = dx/dz, \quad \dot{p} = dp/dz, \quad p = n(x, z)\dot{x}, \tag{3.9}$$
$$H = p^2/2n_0 - [n^2(x, z) - n_0^2]/2n_0.$$

The Hamiltonian H corresponds to the parabolic approximation (3.2). The sum over l in (3.8) is evaluated over all rays joining the point (x, z) and $(x_0, 0)$.

We can now use (3.3) to write H in the form

$$H = H_0(x, p) + \varepsilon V(x, p, z),$$
$$H_0(x, p) = p^2/2n_0 - [n^2(x) - n_0^2]/2n_0, \tag{3.10}$$
$$V(x, p; z) = -f(x, z)/2n_0.$$

The coefficients $a_m(z)$ are conveniently determined in terms of the action and angle variables[18] (I, ϑ). These variables are introduced for the undisturbed ray trajectories determined by the Hamiltonian $H_0(x, p)$:

$$I = \frac{1}{2\pi}\oint p\,dx, \quad \vartheta = \frac{\partial S(x, I)}{\partial I}, \quad S(x, I) = \int^x p\,dx;$$

$$p = [n^2(x) - n_0^2 - 2n_0 E^{(0)}]^{1/2}, \quad E^{(0)} = -H_0(x, p) = -H_0(I), \tag{3.11}$$

where $E^{(0)}$ is the ray integral.

In terms of the action and angle variables, the normal modes have the form

$$\varphi_m(\vartheta) = (2\pi)^{-1/2}\exp(im\vartheta), \quad \hat{I}\varphi_m(\vartheta) = I_m\varphi_m(\vartheta), \tag{3.12}$$

$$\hat{I} = -ik_0 n_0 \partial/\partial\vartheta, \quad I_m = m/k_0 n_0,$$
$$H_0(\hat{I})\varphi_m(\vartheta) = E^{(0)}(I_m)\varphi_m(\vartheta).$$

We note that $E^{(0)}(I_m)$ is related to the mode delay $E_m = k_m/k_0$ by $E_m = -E^{(0)}(I_m) + n_0$.

For an inhomogeneous wave channel, the ray trajectories are described by the functions $I(z)$, $\vartheta(z)$ which, in turn, are determined by Hamilton dynamics with $H = H(I, \vartheta; z)$ in accordance with (3.10).

Transforming to the variables (I, ϑ) in (3.7) and (3.8), we obtain the following expression after some simple algebra[19]:

$$a_m(z) = \sum_l \sum_{m'} a_{m'}(0) D_l^{mm'} \exp\{-im\vartheta + im'\vartheta_0^l + ik_0 n_0 S_{mm'}^l\}, \tag{3.13}$$

where $S_{mm'}^l = S_{mm'}^l(\vartheta, z; \vartheta_0^l, 0)$ is the optical path along the lth ray with initial coordinates $(I_{m'} = m'/k_0 n_0, \vartheta_0^l)$ in the $z = 0$ plane and final coordinates $(I_m = m/k_0 n_0, \vartheta)$ in the $z = \text{const}$ plane. We note that $\vartheta = \vartheta(\vartheta_0^l, z)$, $\vartheta|_{z=0} = \vartheta_0^l$ for the lth ray. The sum over l in (3.13) is evaluated over all rays with final coordinates (I_m, ϑ). The exact expression for the pre-exponential factor $D_l^{mm'}$ will be unimportant in our subsequent analysis.

The main statistical characteristics of the speckle structure of the field are the spatial correlation functions. They are defined as the average product of the values of the field $v(\mathbf{r}, z)$ taken at different points of space. We shall examine the correlation functions on one of the z planes. The spatial correlation function of order M is defined by

$$\Gamma^{(M)}(x_1, \ldots, x_{2M}; z) = \left\langle \prod_{k=1}^{M} v^*(x_k, z) v(x_{k+M}, z) \right\rangle \quad (M = 1, 2, \ldots), \tag{3.14}$$

where the angle brackets represent averaging over the ensemble of realizations of the speckle structure in the $z = \text{const}$ plane. Substituting the expansion (3.5) in (3.14), we obtain

$$\Gamma^{(M)}(x_1,\ldots,x_{2M};z) = \sum_{m_1,\ldots,m_{2M}} P_{m_1\ldots m_{2M}}(z)\varphi_{m_1}(x_1)\ldots\varphi_{m_{2M}}(x_{2M});$$

$$P_{m_1\ldots m_{2M}}(z)=\left\langle\prod_{k=1}^{M} a_{m_k}^*(z)a_{m_{k+M}}(z)\right\rangle. \tag{3.15}$$

The average will be evaluated by taking the expansion coefficients in the form (3.13) where the sum over l can be replaced by integration with respect to the initial angle variables ϑ_0^l in the interval $(0, 2\pi)$.

Le us begin by examining the first-order correlation function $\Gamma^{(1)}(x_1, x_2; z)$. If we use (3.13), we obtain the following expression for the expansion coefficients $P_{m_1m_2}(z)$:

$$P_{m_1m_2}(z)=\sum_{m_1',m_2'} a_{m_1'}^*(0)a_{m_2'}(0)(D^{m_1m_1'})^*D^{m_2m_2'}$$

$$\times\frac{1}{(2\pi)^2}\int_0^{2\pi}d\vartheta_0^{(1)}\int_0^{2\pi}d\vartheta_0^{(2)}\exp\{i(m_1\vartheta^{(1)}-m_2\vartheta^{(2)}-m_1'\vartheta_0^{(1)}$$

$$+m_2\vartheta_0^{(2)})+ik_0n_0(S_{m_2m_2'}-S_{m_1m_1'})\}, \tag{3.16}$$

where the integral in this expression was examined in Ref. 19. Using the expression obtained in Ref. 19, we find that

$$P_{m_1m_2}(z)=\sum_{m_1',m_2'} a_{m_1'}^*(0)a_{m_2}'(0)$$
$$\times(D^{m_1m_1'})^*D^{m_2m_2'}|\mathcal{R}(m_1-m_2,z|m_1'-m_2',0)|^2, \tag{3.17}$$

where

$$\mathcal{R}(q,z|s,0)=\frac{1}{2\pi}\int_0^{2\pi}d\vartheta_0\exp(-iq\vartheta+is\vartheta_0) \tag{3.18}$$

is the correlator for the angle variables $\vartheta=\vartheta(\vartheta_0,z)$ and ϑ_0.

The asymptotic form for moderate correlator times of dynamic systems with stochastic instability is relatively well known:[8,19]

$$\mathcal{R}(q,\ z|s,\ 0)\sim\exp(-z/z_R)\qquad(q\neq0,\ s\neq0), \tag{3.19}$$

where $z_R = 1/h$ is the uncoupling length for correlations between ray oscillation phases along the x axis and h is the Kolmogorov entropy.

Thus, for distances $z>z_R$, the off-diagonal expansion coefficients $P_{m_1m_2}(z)$ $(m_1\neq m_2)$ that characterize intermode correlation are seen to decrease exponentially and can be neglected in the correlation function $\Gamma^{(1)}(x_1, x_2; z)$. We then have

$$\Gamma^{(1)}(x_1,x_2;z)=\sum_m P_{mm}(z)\varphi_m^*(x_1)\varphi_m(x_2). \tag{3.20}$$

It can also be shown[19] that the diagonal expansion coefficients $P_{mm}(z)$ describing the average power in the individual modes satisfy the transport equation

$$\frac{dP_{mm}}{dz}=\sum_{m'}W_{mm'}(P_{m'm'}-P_{mm});$$

$$W_{mm'}=\frac{1}{\Delta z}\left|\int_z^{z+\Delta z}dz'U_{mm'}(z')\right|^2, \tag{3.21}$$

$$U_{mm'}(z)=V_{mm'}\exp\{-i(\bar{E}_m-\bar{E}_{m'})z\}.$$

This equation is the analog of the quantum-mechanical Pauli transport equation.

Consider the behavior of the coefficients $P_{m_1\ldots m_{2M}}(z)$ in (3.15) for the correlation function of order M. Using (3.13), we obtain

$$P_{m_1\ldots m_{2M}}(z)$$

$$=\sum_{m_1',\ldots,m_{2M}'}\prod_{k=1}^{M}a_{m_k'}^*(0)a_{m_{k+M}'}(0)(D^{m_km_k'})^*D^{m_{k+M}m_{k+M}'}$$

$$\times\left(\prod_{k=1}^{M}\frac{1}{2\pi}\int_0^{2\pi}d\vartheta_0^{(k)}\right)$$

$$\times\exp\left\{i\sum_{k=1}^{M}[(m_k\vartheta^{(k)}-m_{k+M}\vartheta^{(k+M)}-m_k'\vartheta_0^{(k)}\right.$$

$$\left.+m_{k+M}'\vartheta_0^{(k+M)})+k_0n_0(S_{m_km_k'}-S_{m_{k+M}m_{k+M}'})]\right\}. \tag{3.22}$$

In the short-wave approximation, the fields are determined by the two-point optical paths $S_{mm'}(\vartheta,\vartheta_0)$, so that (3.22) can be split into factors of the form $P_{m_1m_2}(z)$ [see (3.16)]. When $m_1\neq m_2$, they decrease exponentially for $z>z_R$. Hence the expression for $P_{m_1\ldots m_{2M}}(z)$ in (3.22) will differ appreciably from zero for $z>z_R$ only when the last M indices in $P_{m_1\ldots m_{2M}}(z)$ can be obtained by some permutation of the first M indices. For example,

$$P_{m_1\ldots m_Mm_1\ldots m_M}(z)=P_{m_1m_1}(z)\ldots P_{m_Mm_M}(z). \tag{3.23}$$

It follows from this property of the coefficients $P_{m_1\ldots m_{2M}}(z)$ that the correlation function of order M in (3.10), where $M>1$, can be expressed in terms of the correlation functions of order 1. Hence, substituting (3.22) in (3.15), and using (3.20), we obtain

$$\Gamma^{(M)}(x_1,\ldots x_{2M};z)=\sum_{(P)}\Gamma^{(1)}(x_1,x_{P1};z)\ldots\Gamma^{(1)}(x_M,x_{PM};z), \tag{3.24}$$

where the sum over (P) is evaluated over all the $M!$ permutations of the numbers $1, 2, \ldots, M$.

The above property of the correlation function (3.24) means that the field that has traversed a distance $z>z_R$ in the waveguide channel in which stochastic ray instability develops is described by near-Gaussian statistics. The quantity z_R can also be referred to as the critical length for the formation of the speckle structure.

4. SPATIAL CORRELATORS AND THE DIMENSIONS OF SPOTS IN THE SPECKLE STRUCTURE

One of the important characteristics of the speckle structure is the spot size. This is determined by the width of the intensity autocorrelation function[5]

$$\Gamma_I(x_1,\ x_2;\ z)=\langle|v(x_1,\ z)|^2|v(x_2,\ z)|^2\rangle \tag{4.1}$$

looked upon as a function of the difference $\Delta x = |x_2 - x_1|$. According to the definition given by (3.14), the function (4.1) is equal to the second-order correlation function $\Gamma^{(2)}(x_1, x_2, x_2, x_1; z)$. According to (3.24), for $z>z_R$, the second-order correlation function can be expressed in terms of the first-order correlation functions, i.e.,

$$\Gamma^{(2)}(x_1, x_2, x_2, x_1; z) = \Gamma^{(1)}(x_1, x_1; z)\Gamma^{(1)}(x_2, x_2; z) + |\Gamma^{(1)}(x_1, x_2; z)|^2. \quad (4.2)$$

Thus, the spot dimensions can be estimated by evaluating only the first-order correlator $\Gamma^{(1)}(x_1, x_2; z)$.

It was shown in the last section that this correlator assumes the form (3.20) for $z > z_R$ and is determined exclusively by the mode power $P_{mm}(z)$ satisfying the transport equation (3.21). Once we know the solutions of this equation and use (3.12) for $\varphi_m(x)$, we can calculate $\Gamma^{(1)}(x_1, x_2; z)$.

Let us consider the case where the ray stochastization region in phase space lies in the interior of the region of all the waveguide rays. Rays in the stochastic region do not then enter the external medium. The total power $\mathscr{P}$ transported by a wave is therefore distributed among all the stochastic ray modes. Let the number of these modes be N. The equilibrium solution of (3.21) is then

$$P_{mm}(z) = \text{const} = \mathscr{P}/N. \quad (4.3)$$

This solution is reached for

$$z > z_D \sim 1/W, \quad (4.4)$$

where W is the characteristic value of the transition probabilities $W_{mm'}$ in (3.21), and z_D is the characteristic length for field-energy diffusion among the modes.

If we now substitute (4.3) in (3.20), we reduce the determination of $\Gamma^{(1)}(x_1, x_2; z)$ to the evaluation of finite trigonometric sums. Let the small parameter of the short-wave approximation be denoted by

$$\xi = (k_0 a p_F(x))^{-1} \ll 1, \quad (4.5)$$

where a is the width of the waveguide channel, $p_F(x)$ is given by (2.3), and $k_0 p_F$ has the significance of the local transverse wave number of the highest mode. The longitudinal wave number of the same mode is $k_F = k_0 E_F$ (see Section 2). For this case, the corresponding sums in (3.20), subject to (4.3), were estimated in Ref. 20. According to these estimates, the correlation radius x_c for $\Gamma^{(1)}(x_1, x_2; z)$ in the $z = \text{cost} \gg z_D$ plane is

$$x_c = \pi / k_0 p_F(x) \quad (4.6)$$

and, in addition, we have for $|x_1 - x_2| \gg x_c$

$$|\Gamma^{(1)}(x_1, x_2; z)| / |\Gamma^{(1)}(x_1, x_1; z)| \sim \xi. \quad (4.7)$$

Thus, according to (4.2), (4.6), and (4.7), the characteristic spot dimension in the speckle structure is $\Delta l_s = x_c$ for relative intensities of the order of ξ, which is in agreement with (2.3).

The order of magnitude of the number of spots in the speckle structure in the lateral cross section of the waveguide channel is given by $N_s \sim a/\Delta l_s \sim 1/\xi \gg 1$, i.e., it is proportional to the reciprocal of the quasiclassical parameter.

It is important to note that the speckle structure in the lateral cross section of the waveguide channel is statistically inhomogeneous, which contrasts with speckle structure obtained by reflection from a rough surface. The correlation radius x_c and the spot size Δl_s are then slowly-varying functions of the lateral coordinate x over scales of the order of the spot size.

5. UNDERWATER ACOUSTIC CHANNEL WITH PERIODIC SURFACE INHOMOGENEITY

To estimate the parameters of the speckle structure of the wave field and the critical length z_R for its formation, introduced in the last section, let us investigate the special case of the underwater acoustic channel with a bottom that is homogeneous along the z direction, is perfectly reflecting, and has a periodic surface. For simplicity, we shall suppose that the velocity of sound is independent of the transverse coordinate x. The refractive index of the medium can then be described by the function

$$n(x, z) = \begin{cases} n_0 & \text{for } 0 < x < a + \varepsilon f(z) \\ 0 & \text{for } x < 0, \quad x > a + \varepsilon f(z) \end{cases}, \quad (5.1)$$

where a is the average width of the acoustic channel in the x direction and $f(z) = f(z + L)$ is the periodic perturbation of the surface of period L.

For the sake of generality, we shall obtain the ray trajectories without using the approximation corresponding to the parabolic equation. The ray trajectories $x = x(z)$ are described by the Hamilton equations

$$\dot{x} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial x;$$
$$H = -[n^2(x, z) - p^2]^{1/2}, \quad p = n(x, z)\dot{x}/(1 + \dot{x}^2)^{1/2}. \quad (5.2)$$

For small perturbations $\varepsilon |f(z)| \ll a$ and the Hamiltonian H can be written in the form

$$H = H_0(x, p) + \varepsilon V(x, p; z);$$
$$H_0(x, p) = -(n_0^2 - p^2)^{1/2}, \quad V(x, p; z) = -f(z)\dot{p}. \quad (5.3)$$

In the absence of the perturbation ($\varepsilon f(z) \equiv 0$), the constant of motion E corresponding to the mode delay assumes values in the range $0 < E < n_0$. In terms of the action and angle variables (I, ϑ), the ray trajectories are described by

$$x = 2a \begin{cases} \{\vartheta/2\pi\} & \text{for } 0 < \{\vartheta/2\pi\} < 1/2 \\ (1 - \{\vartheta/2\pi\}) & \text{for } 1/2 < \{\vartheta/2\pi\} < 1 \end{cases}$$
$$p = \begin{cases} I/I_0 & \text{for } 0 < \{\vartheta/2\pi\} < 1/2 \\ -I/I_0 & \text{for } 1/2 < \{\vartheta/2\pi\} < 1 \end{cases}$$
$$\vartheta = \omega(I) z + \vartheta_0, \quad \omega(I) = \frac{dH_0(I)}{dI} = \frac{I}{(I_0^2 |H_0(I)|)},$$
$$H_0(I) = -[n_0^2 - (I/I_0)^2]^{1/2}, \quad I_0 = a/\pi;$$
$$E = -H_0(I), \quad 0 < I < n_0 I_0, \quad (5.4)$$

where curly brackets represent the fractional part of the argument. The quantity $\omega(I)$ has the significance of the spatial frequency of oscillation of the ray trajectory about the axis of the waveguide channel, and $2\pi/\omega(I)$ is the spatial period of the ray trajectory. The tangent to the trajectory makes the angle $\theta = \arcsin(I/n_0 I_0)$ with the z axis.

Suppose that the perturbation $\varepsilon f(z)$ is

$$\varepsilon f(z) = b \cos(2\pi z/L), \quad (5.5)$$

where b is the maximum perturbation. Expanding p into a Fourier series in ϑ, we can write the perturbation $\varepsilon V(x,p;z)$ in the form

$$\varepsilon V(x,p;z)=\frac{\varepsilon}{2}\sum_{q=-\infty}^{+\infty} V_q \exp\left\{i\left[(2q+1)\vartheta-\frac{2\pi z}{L}\right]\right\} \tag{5.6}$$

$$\varepsilon V_q=(2\pi^2 I^2 b)/(a^3 E).$$

The perturbation has the strongest effect on the ray trajectories when the following nonlinear resonance condition is satisfied:

$$(2q+1)\,\omega(I^{(q)})=2\pi/L. \tag{5.7}$$

Rays with parameters close to this resonance execute additional phase oscillations about the undisturbed trajectory that is determined by the resonance value of the action $I^{(q)}$. The maximum widths of the nonlinear resonance in terms of action I and frequency $\omega(I)$ are respectively given by[8]

$$\Delta\omega=\left(\frac{4\varepsilon V_q d\omega}{dI}\right)^{1/2}=\frac{2^{5/2}\pi^2 n_0 I(b/a)^{1/2}}{a^2E^2}, \qquad \Delta I=\frac{\Delta\omega}{d\omega/dI}. \tag{5.8}$$

The separation between neighboring resonances in these two variables is respectively given by

$$\delta\omega=|\omega(I^{(q+1)})-\omega(I^{(q)})|\approx\pi^3 L I^2/a^4E^2, \qquad \delta I=|I^{(q+1)}-I^{(q)}|\approx\delta\omega/(d\omega/dI). \tag{5.9}$$

It is clear from (5. 8) and (5.9) that, for low values of I, the separation $\delta\omega$ between the resonances decreases more rapidly than $\Delta\omega$. It follows that the nonlinear resonances will overlap for $I<I_c$, where I_c is the critical action. The overlap condition is given by the inequality

$$K=(\Delta\omega/\delta\omega)^2=[2^{5/2}n_0(ab/L^2)^{1/2}(I_0/I)]^2>1. \tag{5.10}$$

It is well known[8] that the trajectories of a dynamic system are stochastic when the nonlinear resonances overlap. The quantity K is the parameter of the dynamic system that appears in the formula for the Kolmogorov entropy $h\sim\ln K$ (see Sections 2 and 3).

The condition $K=1$ yields the critical value I_c:

$$I_c/I_0=2^{5/2}n_0(ab/L^2)^{1/2}. \tag{5.11}$$

In terms of the variables E and I, the stochastic region of rays is defined by

$$E_F=n_0(1-8ab/L^2)^{1/2}<E<n_0, \quad 0<I<I_c. \tag{5.12}$$

According to (2.3), the minimum spot size Δl_s in the speckle structure of the wave field in the plane perpendicular to the z axis is determined by the highest mode entering the stochastic region with minimum longitudinal wave number k_F or minimum delay $E_F=k_F/k_0$. It follows from (2.3) and (5.12) that

$$\Delta l_s\approx\lambda(8ab/L^2)^{-1/2}, \quad \lambda=2\pi/k_0n_0. \tag{5.13}$$

For small values of the combination $8ab/L^2\ll1$, consisting of the waveguide and perturbation parameters, the spot size Δl_s is much greater than the wavelength, i.e., $\Delta l_s\gg\lambda$. This is connected with the result $\Delta l_s\approx1/k_0p_F$ since, for $8ab/L^2\ll1$, the stochastic region intercepts modes with transverse wave numbers $k_0p_F\ll k_0n_0$. For $8ab/L^2\approx1$, the stochastic region contains all the waveguide modes. Since, in this case, $k_0p_F\approx k_0n_0$, the spot size is of the order of wavelength, i.e., $\Delta l_s\approx\lambda$. This result also follows from (5.13).

According to (5.4), (5.8), (5.9), and (5.10), the critical length for the formation of the speckle structure $z_R=z_0/\ln K=2\pi/\omega(I)\ln K$ is given by

$$z_R=a^2E\{\pi I\ln[n_0(8ab/L^2)^{1/2}(I_0/I)]\}^{-1}, \quad 0<I<I_c. \tag{5.14}$$

When $8ab/L^2\ll1$, the minimum value of z_R is $z_R\approx a(8ab/L^2)^{-1/2}\gg a$.

When all the waveguide rays are in the stochastic region, i.e., when $8ab/L^2\approx1$, the speckle structure is formed for relatively small values of z_R of the order of the waveguide width a, i.e., $z_R\sim a$.

6. CONCLUSIONS

The foregoing simple examples of wave propagation in regularly inhomogeneous waveguide channel show that dynamic ray stochasticity over lengths corresponding to the uncoupling of correlations between the phases of ray oscillations leads to the formation of a speckle structure of a specific type. In particular, any weak partial incoherence of the initial wave in space or time must lead to an exponentially rapid smearing out of the speckle structure. This is in contrast to the slow linear smearing out of the interference pattern in regular channels with waveguide dispersion (see, for example, Refs. 21 and 22). This, in turn, produces an irreversible loss of information about the structure of the initial wave front in the direction of the z axis. The speckle structure of the field becomes stationary in the z direction over distances exceeding the ray diffusion length.

The exponentially rapid smearing out of the speckle structure of the field when the initial radiation is weakly noncoherent is important in systems in which data are transmitted along waveguide channels, including fiber-optics communication lines. One of the possible applications of stochastic instability is the suppression of noise due to the speckle structure of the field. On the other hand, dynamic stochasticity of rays together with regular dislocations of the wave front of the speckle structure[6,7] prevents the complete reversal of the wave front by adaptive systems.

A more detailed analysis of the development of stochastic instability would include allowance for islands of regular motion of rays and nonexponential asymptotic forms of the correlators $\mathscr{R}$. This type of analysis would yield more accurate information about the speckle structure due to dynamic stochasticity of rays in waveguide channels.

Finally, let us examine the difference between the speckle structure produced in the homogeneous wave channel with the inhomogeneous case in which stochastic ray instability is known to arise. Consider a waveguide that is homogeneous in r and z and has a "good" profile (separable variables). Suppose that a field u_0 is introduced into the waveguide and contains a sufficiently large number of modes u_m with eigenvalues (delays) E_m that are not equidistant. The latter condition means that the quantity

$$\omega'(I_m)=\frac{d^2H_0(I_m)}{dI_m^2}=\frac{d^2E_m}{dm^2}$$

is not zero. It is known[23,24] that, here again, the speckle structure will appear. It is readily seen that the characteristic length along the z axis for the formation ot this structure is

$$z_0\sim(d^2E_m/dm^2)^{-1}=1/\omega'(I).$$

Comparison of this with the length z_R for the formation of the speckle structure in the case of stochastic instability [see (5.14) and the formula preceding it] shows that

$$z_R\sim\omega' z_0/\omega \ln K.$$

Since the stochastic parameter K satisfies the condition $K\gg1$, it follows that $z_R \ll z_0$ and, as noted at the end of Section 5, the value $z_R \sim a$ can be reached, where a is the minimum transverse size of the waveguide.

Thus, as expected, stochastic instability leads to much shorter lengths for the formation of the speckle structure.

The other difference between the stochastic and regular cases is that, in the former, the speckle structure is irregular (random) whereas, in the latter, it is regular. As already noted, this difference is particularly appreciable when the incident field has some initial noncoherence because an irreversible loss of information about its structure then takes place.

[1)]Special cases such as, for example, the case of separable variables are, of course, exceptions to this.

[1]A. V. Gurevich and E. E. Tsedilina, Sverkhdal'nee rasprostranenie korotkikh radiovoln (Ultralong-range Propagation of Short Radio Waves), Nauka, Moscow, 1979, Chap. 1.

[2]Akustika okeana (Ocean Acoustics), ed. by L. M. Brekhovskikh Nauka, Moscow, 1974, Part 1.

[3]S. S. Abdullaev and G. M. Zaslavskiĭ, Zh. Eksp. Teor. Fiz. **80**, 524 (1981) [Sov. Phys. JETP **53**, 265 (1981)].

[4]S. S. Abdullaev and G. M. Zaslavskiĭ, Zh. Eksp. Teor. Fiz. **85**, 1573 (1983) [Sov. Phys. JETP **58**, 915 (1983)].

[5]J. W. Goodman, in: Laser Speckle and Related Phenomena, Springer-Verlag, Berlin-New York, 1975, Part 1.

[6]N. B. Baranova and B. Ya. Zel'dovich, Zh. Eksp. Teor. Fiz. **80**, 1789 (1981) [Sov. Phys. JETP **53**, 925 (1981)].

[7]N. B. Baranova, B. Ya. Zel'dovich, A. V. Mamaev, N. F. Pilipetskiĭ, and V. V. Shkunov, Zh. Eksp. Teor. Fiz. **83**, 1702 (1982) [Sov. Phys. JETP **56**, 983 (1982)].

[8]G. M. Zaslavskiĭ and B. V. Chirikov, Usp. Fiz. Nauk **105**, 3 (1971) [Sov. Phys. Usp. **14**, 549 (1972)].

[9]G. P. Berman and G. M. Zaslavsky, Physica (Utrecht) *A* **91**, 450 (1978).

[10]G. M. Zaslavsky, Phys. Rep. **80**, 157 (1982).

[11]S. W. McDonald and A. N. Kaufman, Phys. Rev. Lett. **42**, 1189 (1979).

[12]D. W. Noid, M. L. Koszykowski, M. Tabor, and R. A. Marcus, J. Chem. Phys. **72**, 6169 (1980).

[13]Yu. A. Kravtsov and Yu. I. Orlov, Geometricheskaya optika neodnorodnykh sred (Geometric Optics of Inhomogeneous Media), Nauka, Moscow, 1980, Chap. 1.

[14]J. W. Goodman, J. Opt. Soc. Am. **66**, 1145 (1976).

[15]S. M. Rytov, Yu. A. Kravtsov, and V. I. Tatarskiĭ, Vvedenie v statisticheskuyu radiofiziku (Introduction to Statistical Radiophysics), Nauka, Moscow, 1978, Part 2.

[16]B. Z. Katsenelenbaum, Teoriya neregulyarnykh volnovodov s medlenno menyayushchimisya parametrami (Theory of Irregular Waveguides with Slowly Varying Parameters), Academy of Sciences of the USSR, Moscow, 1961.

[17]V. P. Maslov and M. V. Fedoryuk, Kvaziklassicheskoe priblizhenie dlya uravneniĭ kvantovoĭ mekhaniki (Quasiclassical Approximation for the Equations of Quantum Mechanics), Nauka, Moscow, 1976.

[18]G. Goldstein, Classical Mechanics, Addison-Wesley, 1950 [Russian transl., Nauka, Moscow, 1976].

[19]G. P. Berman and G. M. Zaslavsky, Physica (Utrecht) A **97**, 367 (1979).

[20]S. S. Abdullaev, Zh. Tekh. Fiz. **51**, 697 (1981) [Sov. Phys. Tech. Phys. **26**, 415 (1981)].

[21]I. A. Deryugin, S. S. Abdullaev, and A. T. Mirzaev, Kvantovaya Elektron. (Moscow) **4**, 2173 (1977) [Sov. J. Quantum Electron. **7**, 1243 (1977)].

[22]B. Crosignani and P. J. Di Porto, J. Appl. Phys. **44**, 4616 (1973).

[23]V. I. Popovichev, V. V. Ragul'skiĭ, and F. S. Faĭzullov, Pis'ma Zh. Eksp. Teor. Fiz. **19**, 350 (1974) [JETP Lett. **19**, 196 (1974)].

[24]B. Ya. Zel'dovich, N. F. Pilipetskiĭ, and V. V. Shkunov, Usp. Fiz. Nauk **138**, 249 (1982) [Sov. Phys. Usp. **25**, 713 (1982)].

Translated by S. Chomet