Damping and fluctuations in coupled quantum oscillator systems

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The damping and equilibrium state due to random fluctuations of the external forces are expressed in terms of the quantum quasiprobability (the P-distribution). A system of N coupled oscillators in a heat bath comprising a system of various heat baths with different temperatures, each of which interacts with its own oscillator, is considered. The correlation-function matrix is calculated exactly for the coupled oscillators.

INTRODUCTION

The origin of irreversibility in the behavior of a quantum oscillator interacting with a heat bath modeled by a set of an infinite number of oscillators was considered previously.[1] The relaxation time of an oscillator located in such a bath was calculated in terms of the parameters of the Hamiltonian of the oscillator. It was shown that the equilibrium state which the oscillator achieves is a thermal equilibrium state set by the temperature of the heat bath. The use of the Weisskopf-Wigner approximation in the solution of the Heisenberg equations of motion for the creation and annihilation operators of the oscillators was an essential element of the analysis in Ref. 1. In this analysis account was taken of the properties of the coherent states of the oscillators, using the function or quantum quasi-probability (the P-distribution) introduced in Ref. 3. We note that different approaches to the consideration of damping of a quantum one-dimensional oscillator have also been developed in Refs. 4–6.

The purpose of the present work is the establishment of the properties of the equilibrium state reached by a set of N coupled oscillators interacting with a heat bath. In this case, the heat bath is modeled by an infinite set of oscillators possessing the feature that each of the systems of N coupled oscillators interacts with its own bath characterized by its own temperature. Thus, the prupose of the present research is to generalize the results, obtained in Ref. 1 for a single quantum oscillator, to the case of a set of N coupled oscillators. We also calculate the correlation functions for the N coupled oscillators. The physical problems (in addition to the consideration of the irreversibility of the behavior of a system of coupled oscillators, which is of general theoretical interest) in which the details of the behavior of a set of oscillators with damping is important, are in particular the problems of the construction of various gravitational detectors, analyzed, for example, in Refs. 7–9.

SET OF QUANTUM OSCILLATORS COUPLED WITH A FORCE

Following Ref. 1, we consider an N-dimensional quantum oscillator, designated in what follows as the A oscillator, interacting in linear fashion with the infinite set of oscillators of the heat bath, which are denoted below as the B oscillators. The A oscillator can model individual modes of the electromagnetic field, for example, the modes in a resonator with reflecting walls. It is known that the amplitudes of these modes can be regarded as the coordinates of harmonic oscillators. If the walls of the resonator are completely reflecting, these oscillators are not coupled with any other dynamical system. Once begun, these oscillators last infinitely long. In actuality, however, the walls of the resonator are never completely reflecting. Usually, a rapid decrease in the amplitude of the field of the considered modes (or damping) takes place, and at the same time, the walls of the resonator heat up slightly, if, on the other hand, the modes of the field are not excited at all, as would be the case if they corresponded to the temperature of absolute zero, while the walls were heated to a finite temperature, they would "heat up" the field also. These are examples of processes that are irreversible in the thermodynamic sense. The appearance of irreversibility can be traced in a mathematically explicit form. We shall trace it through the example of a set of oscillators interacting with the heat bath, while the bath itself is modeled by an infinite set of oscillators.

Thus, we consider the A oscillator in a heat bath. The heat bath is chosen in the form of a set of N independent heat baths, each of which interacts with its own oscillator of the set of N coupled oscillators. Physically, this means that each oscillator has the temperature of its own bath at the initial time, and that we neglect indirect interaction of the oscillators through the heat bath, which is of course much smaller than the direct interaction of the oscillators. We write out the Hamiltonian of the set of oscillators subjected to the action of driving forces:

$$\hat{H}=\sum_{\alpha=1}^{N}\hbar\omega_\alpha\hat{a}_\alpha^+\hat{a}_\alpha+\hbar\sum_{\alpha,\beta=1}^{N}\lambda_{\alpha\beta}\hat{a}_\alpha\hat{a}_\beta^+ +\sum_{\alpha=1}^{N}\sum_{k}\hbar\omega_k\hat{b}_{\alpha k}^+\hat{b}_{\alpha k}$$
$$+\hbar\sum_{\alpha=1}^{N}\sum_{k}(\lambda_k\hat{a}_\alpha\hat{b}_{\alpha k}^+ +\lambda_k^*\hat{a}_\alpha^+\hat{b}_{\alpha k})$$
$$+i\hbar\sum_{\alpha=1}^{N}(F_\alpha(t)\hat{a}_\alpha^+ -F_\alpha^*(t)\hat{a}_\alpha). \qquad (1)$$

The Greek subscripts label oscillators of the set of N coupled oscillators and run from 1 to N. The Latin subscripts vary from 1 to ∞ and they label oscillators in the αth heat

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bath corresponding to the αth oscillator of the set of N interacting oscillators. The operators $\hat{b}_{\alpha k}^{+}$ are the Hermitian adjoints of the operators $\hat{b}_{\alpha k}$, and the numbers λ_k^* are the complex conjugates of the numbers λ_k. Naturally, it is assumed that the interaction of each oscillator $\hat{a}_\alpha$ with the heat bath is symmetric and that the parameters ω_k and λ_k are one and the same for each αth oscillator. The term in (1) with the complex external force $F_\alpha(t)$ describes an effect on the oscillator proportional both to its coordinate and to its momentum. This term can correspond, within the framework of the single-mode model of a gravitation array, to the effect of a gravitational wave on the array. The fact that the A oscillator is coupled only with oscillators may seem to be an unrealistic element of the model. However, large systems, made up not only of oscillators, frequently possess modes of collective excitations whose amplitudes behave dynamically like the amplitudes of harmonic oscillators. The Hamiltonian (1) can describe a broad class of damping mechanisms with participation of collective excitations. The operators $\hat{a}_\alpha$, $\hat{a}_{\alpha k}^{+}$, $b_{\alpha k}$ $\hat{b}_{\alpha k}^{+}$ in (1) ($\alpha = 1,2,\ldots,N$) are boson creation and annihilation operators for the A and B oscillators, respectively. There are no terms in the Hamiltonian of the form $\hat{a}_\alpha\,\hat{b}_{\alpha k}$ and $\hat{a}_\alpha^{+}\,\hat{b}_{\alpha k}^{+}$. The dynamical effect of such antiresonance terms is usually small in connection with their very rapid oscillations, and they can be neglected. This corresponds to the so-called rotating-wave approximation. We shall assume that the subscript k can be continuous and shall understand in this case, throughout the rest of the paper, that the summation over the subscript is actually integration over a continuous exponent k.

We know that there exists a set of normal vibrations for the coupled harmonic oscillators, for which we assume that the corresponding quadratic form is positive definite. Thus, it is known that there exists a unitary matrix U and there exist operators of the normal modes $\hat{A}_\alpha$ ($\alpha = 1,2,\ldots,N$), such that there is a coupling of the operators $\hat{A}_\alpha$ with the operators $\hat{a}_\alpha$ of the following form:

$$\hat{A}_\alpha = \sum_{\beta=1}^{N} U_{\beta\alpha}^{*}\hat{a}_\beta. \tag{2}$$

The asterisk denotes the complex conjugate. The Hamiltonian (1), expressed in terms of the new coordinates $\hat{A}_\alpha$ of the normal modes (without account of the heat bath) would be diagonal.

It should be emphasized that we can introduce new coordinates of the heat bath $\hat{B}_{\alpha k}$ for which the interaction of the normal modes (the coordinates $\hat{A}_\alpha$) with these coordinates would appear to be the same as the interaction of the initial coordinates of the oscillators $\hat{a}_\alpha$ with the initial coordinates of the heat bath $\hat{b}_{\alpha k}$ without the interaction term $\lambda_{\alpha\beta}\hat{a}_\alpha\hat{a}_\beta^{+}$. Let us introduce such coordinates:

$$\hat{B}_{\alpha k} = \sum_{\beta=1}^{N} U_{\beta\alpha}^{*}\hat{b}_{\beta k}, \tag{3}$$

using the same matrix U as in the transformation (2). We obtain the following expression for the Hamiltonian (1) of the initial system in the new variables $\hat{A}_\alpha$ and $\hat{B}_{\alpha k}$:

$$\hat{H} = \sum_{\alpha=1}^{N}\hbar\Omega_\alpha\hat{A}_\alpha^{+}\hat{A}_\alpha + \sum_{\alpha=1}^{N}\sum_{k}\hbar\omega_k\hat{B}_{\alpha k}^{+}\hat{B}_{\alpha k} + \hbar\sum_{\alpha=1}^{N}\sum_{k}(\lambda_k\hat{A}_\alpha\hat{B}_{\alpha k}^{+} + \lambda_k^{*}\hat{A}_\alpha^{+}\hat{B}_{\alpha k}) + i\hbar\sum_{\alpha=1}^{N}(f_\alpha(t)\hat{A}_\alpha^{+} - f_\alpha^{*}(t)\hat{A}_\alpha). \tag{4}$$

In the Hamiltonian (4) we have introduced the transformed external force

$$f_\alpha(t) = \sum_{\beta=1}^{N} U_{\beta\alpha}^{*}F_\beta(t). \tag{5}$$

We now rewrite the formulas (2), (3) and (5), introducing vector notation, namely, we let the operator $\hat{\mathbf{a}}$ denote the vector (column) with components $\hat{a}_\alpha$, $\alpha = 1,2,\ldots,N$; the operators $\hat{\mathbf{b}}_k$ denote the vectors (columns) with components $\hat{\mathbf{a}}_{\alpha k}$; we write the N-dimensional vector $\hat{A}$ and the N-dimensional vectors $\hat{\mathbf{B}}_k$, $\mathbf{f}(t)$ and $\mathbf{F}(t)$ in corresponding fashion. Then the relations (2), (3) and (5) have the following form

$$\hat{\mathbf{a}} = U\hat{\mathbf{A}}, \quad \hat{\mathbf{b}}_k = U\hat{\mathbf{B}}_k, \quad \mathbf{F}(t) = U\mathbf{f}(t). \tag{6}$$

We shall now assume that after reduction to normal form, all the frequencies Ω_α of the normal oscillations of the A oscillator are different (no degeneracy). Thus, after a canonical transformation of (2), (3) and (5), and after reducing of the Hamiltonian (1) to the form (4), we arrive at the problem of the behavior of a system of independent $\hat{A}_\alpha$ oscillators interacting with their own independent heat baths (the $\hat{B}_{\alpha k}$ oscillators). This allows us to use the method of Ref. 1 with account only of small complications connected with the presence of the driving force. Our aim is to calculate the correlation functions for the initial A oscillator. In order to do this, we first find the Heisenberg creation and annihilation operators $\hat{\mathbf{a}}(t)$ and $\hat{\mathbf{b}}_k(t)$. We shall always write out the Heisenberg operators, noting their time dependencies, with $\hat{\mathbf{a}}(0) \equiv \hat{\mathbf{a}}$, $\hat{\mathbf{b}}_k(0) \equiv \hat{\mathbf{b}}_k$. The Heisenberg equations of motion for the operators $\mathbf{a}(t)$ and $\mathbf{b}_k(t)$ have the form

$$\frac{d}{dt}\hat{a}_\alpha(t) = -i\omega_\alpha\hat{a}_\alpha(t) - i\sum_{k}\lambda_k\hat{b}_{\alpha k}(t) + F_\alpha(t),$$
$$\frac{d}{dt}\hat{b}_{\alpha k}(t) = -i\omega_k\hat{b}_{\alpha k}(t) - i\lambda_k^{*}\hat{a}_\alpha(t). \tag{7}$$

For the solution of these equations, we make the change of variables (2), (3) and (5), and then obtain the equation of motion for the Heisenberg operators $\hat{A}_\alpha(t)$ and $\hat{B}_{\alpha k}(t)$, which we write directly in matrix form:

$$\frac{d\hat{\mathbf{A}}(t)}{dt} = -i\Omega\hat{\mathbf{A}}(t) - i\sum_{k}\lambda_k\hat{\mathbf{B}}_k(t) + \mathbf{f}(t),$$
$$\frac{d\mathbf{B}_k(t)}{dt} = -i\omega_k\hat{\mathbf{B}}_k(t) - i\lambda_k^{*}\hat{\mathbf{A}}(t). \tag{8}$$

Here the N-dimensional matrices Ω, λ_k and ω_k are diagonal matrices, while the matrices λ_k and ω_k are simply unit matrices multiplied by a number, while the diagonal elements of the matrix are the frequencies of the normal vibrations of the A oscillators $\Omega_1, \Omega_2, \ldots, \Omega_N$.

The crucial factor is that the system of unlinked equations (8) (without the term with the driving force) can be solved in the Weisskopf-Wigner approximation, as was done for the one-dimensional A oscillator in Ref. 1. We first write down the solution of this system in the general case in the following form, by virture of its linearity (we first assume no force, i.e., $\mathbf{F}(t)=0$):

$$\hat{\mathbf{A}}(t)=u(t)\hat{\mathbf{A}}+\sum_k v_k(t)\hat{\mathbf{B}}_k,\quad \hat{\mathbf{B}}_k(t)=\sum_{k'} x_{k'k}(t)\hat{\mathbf{B}}_{k'}+y_k(t)\hat{\mathbf{A}}. \tag{9}$$

Here $\hat{\mathbf{A}}=\hat{\mathbf{A}}(t=0)$, $\hat{\mathbf{B}}_k=\hat{\mathbf{B}}_k(t=0)$, and four N-dimensional diagonal matrices are introduced: $u(t)$, $v_k(t)$, $x_{k'k}(t)$ and $y_k(t)$; on the main diagonals of the matrices $u(t)$ and $v_k(t)$ we have the numbers $u_\alpha(t)$ and ($\alpha=1,2,\ldots,N$) while the forms of the matrices $y_k(t)$ and $x_{k'k}(t)$ are not of interest to us, since they do not enter into the final result. The initial values for these matrices are the following:

$$u(0)=E,\quad v_k(0)=0,\quad y_k(0)=0,\quad x_{k'k}(0)=\delta_{k'k} \tag{10}$$

(E is a unit matrix).

Correspondingly, the solution of the equation of motion (7) for the operator $\hat{\mathbf{a}}(t)$ in th case $\mathbf{F}(t)=0$ can be rewritten in the form

$$\hat{\mathbf{a}}(t)=S(t)\hat{\mathbf{a}}+\sum_k V_k(t)\hat{\mathbf{b}}_k. \tag{11}$$

Here we have introduced the matrices $S(t)$ and $v_k(t)$, which are connected with the matrices $u(t)$ and $v_k(t)$ of (9) by the unitary transformation matrix [see (2)]:

$$S(t)=Uu(t)U^+;\quad V_k(t)=Uv_k(t)U^+. \tag{12}$$

We have the following solution for the quantity $\hat{\mathbf{b}}_k(t)$

$$\hat{\mathbf{b}}_k(t)=\sum_{k'} X_{k'k}(t)\hat{\mathbf{b}}_{k'}+Y_k(t)\hat{\mathbf{a}}, \tag{13}$$

where the N-dimensional matrices $x_{k'k}(t)$ and $Y_k(t)$ are connected with $x_{k'k}(t)$ and $y_k(t)$ by formulas analogous to (2):

$$X_{k'k}(t)=Ux_{k'k}(t)U^+;\quad Y_k(t)=Uy_k(t)U^+. \tag{14}$$

The simplicity of the Hamiltonian (1) becomes especially evident when we express the state of the system in terms of coherent states. This Hamiltonian belongs to the class of Hamiltonians to which corresponds an evolution operator that leaves the coherent states coherent.[10] Thus, if we begin the motion from the state of the system in which the A oscillator and the B oscillator are in coherent states, these oscillators will remain in coherent states for all time. We shall assume that the state of the system in the Heisenberg representation is given as a product of coherent states:

$$|\alpha\rangle\prod_{\mu k}|\beta_{\mu k}\rangle\equiv|\alpha,\{\beta_k\}\rangle. \tag{15}$$

Here the conditions

$$\hat{\mathbf{a}}(0)|\alpha,\{\beta_k\}\rangle=\alpha|\alpha,\{\beta_k\}\rangle,\quad \hat{\mathbf{b}}_k(0)|\alpha,\{\beta_k\}\rangle=\beta_k|\alpha,\{\beta_k\}\rangle \tag{16}$$

are satisfied. It follows from the relations (11) and (13) that this state is an eigenstate also for the operators $\hat{\mathbf{a}}(t)$ and $\hat{\mathbf{b}}_k(t)$ over all time t. We define the time-dependent functions $\alpha(t)$, $\beta_k(t)$:

$$\alpha(t)=S(t)\alpha+\sum_k V_k(t)\beta_k, \tag{17}$$

$$\beta_k(t)=\sum_{k'} X_{k'k}(t)\beta_{k'}+Y_k(t)\alpha, \tag{18}$$

which obey the equations

$$\frac{d}{dt}\alpha_\mu(t)=-i\omega_\mu\alpha_\mu(t)-i\sum_k\lambda_k\beta_{\mu k}(t),\quad \mu=1,2,\ldots,N \tag{19}$$

$$\frac{d}{dt}\beta_{\mu k}(t)=-i\omega_k\beta_{\mu k}(t)-i\lambda_k^*\alpha_\mu(t), \tag{20}$$

which are completely analogous to the corresponding operator equations. Then the Heisenberg state of the system satisfies the equations for the eigenvalues

$$\hat{\mathbf{a}}(t)|\alpha,\{\beta_k\}\rangle=\alpha(t)|\alpha,\{\beta_k\}\rangle, \tag{21}$$

$$\hat{\mathbf{b}}_k(t)|\alpha,\{\beta_k\}\rangle=\beta_k(t)|\alpha,\{\beta_k\}\rangle \tag{22}$$

for all times t.

We note that for a transition to the Schrödinger representation it is necessary to use the unitary evolution operator $\hat{R}(t)$, which connects the opeaor $\hat{\mathbf{a}}(t)$ with its initial values

$$\hat{\mathbf{a}}(t)=\hat{R}^{-1}(t)\hat{\mathbf{a}}(0)\hat{R}(t). \tag{23}$$

We then have relations equivalent to (21) and (22):

$$\hat{\mathbf{a}}(0)\hat{R}(t)|\alpha,\{\beta_k\}\rangle=\alpha(t)\hat{R}(t)|\alpha,\{\beta_k\}\rangle, \tag{24}$$

$$\hat{\mathbf{b}}_k(0)\hat{R}(t)|\alpha,\{\beta_k\}\rangle=\beta_k(t)\hat{R}(t)|\alpha,\{\beta_k\}\rangle. \tag{25}$$

The state $\hat{R}(t)|\alpha,\{\beta_k\}\rangle$ is simply a time-dependent state of the system in the Schrödinger representation, and we see that if it is coherent at the initial instant of time ($\hat{R}(0)=\hat{I}$), then it remains coherent throughout all subsequent times.[10] A number of conclusions on the properties of this state can be drawn without solving the Schrödinger equation explicity. The motion of the system is such that it is as close as possible to classical. The system is described in this case by a wave packet for which the product of the indeterminacies of the coordinate and momentum is as small as possible, while the mean values of the coordinate and momentum move in the phase space of the system along a classical trajectory given by the relations (16) and (17); no spreading of the packet occurs. Since the Schrödinger state of the system is a coherent state, we can specify it by means of a time-dependent eigenvalue, writing

$$|\alpha(t),\{\beta_k(t)\}\rangle\equiv\hat{R}(t)|\alpha,\{\beta_k\}\rangle. \tag{26}$$

It is clear from the relations (10), (17) and (18) that $\alpha(0)=\alpha$, $\beta_k(0)=\beta_k$.

We now take into account the effect of the external force $\mathbf{F}(t)$. We shall temporarily denote the solutions of Eqs. (7) in the absence of a force by $\hat{\mathbf{a}}_0(t)$ and $\hat{\mathbf{b}}_{0k}(t)$. In order to solve the equation of motion in the presence of the force, we use the solution of the equation $\mathbf{a}_0(t)$ and $\mathbf{b}_{k0}(t)$ in the absence of the force, which coincide with the $\mathbf{a}(t)$ and $\mathbf{b}_k(t)$ given by Eqs. (11) and (13).

We shall solve Eqs. (7) in such a way as to allow us to make actual use of the solutions (11) and (13) for the case

without the force. To this end we consider first a very particular case, namely, we let

$$\mathbf{F}(t)=\mathbf{F}_0\delta(t-t'). \tag{27}$$

We shall actually construct a Green's function for the linear system of equations (7). Such a character of the force means that up to the instant of time t', the force does not act on the A oscillator; therefore, for earlier times $t<t'$, the solution of the set (7) is simply identical with the solution of the set of equations (11) and (13) without the force.

We thus have

$$t<t',\quad \hat{\mathbf{a}}(t)=S(t)\hat{\mathbf{a}}(0)+\sum_k V_k(t)\hat{\mathbf{b}}_k(0), \tag{28}$$

$$\hat{\mathbf{b}}_k(t)=\sum_{k'} X_{k'k}(t)\hat{\mathbf{b}}_{k'}(0)+Y_k(t)\hat{\mathbf{a}}(0). \tag{29}$$

Since a force determined by a delta function acts at the instant t' on the A oscillator, it follows, in connection with the fact that the first derivative with respect to the time $d\hat{\mathbf{a}}/dt$ is proportional to a delta function, the operator $\hat{\mathbf{a}}(t)$ at the time t' undergoes a jump, such that

$$\hat{\mathbf{a}}(t'+0)-\hat{\mathbf{a}}(t'-0)=\mathbf{F}_0. \tag{30}$$

After this, the time evolution of the solution of Eqs. (7) proceeds again in the same fashion as the evolution of the solutions of the equations without the force, (11) and (13), but with initial conditions that are changed because of the jump (30). We thus have

$$t>t',\quad \hat{\mathbf{a}}(t)=S(t-t')\hat{\mathbf{a}}(t'+0)+\sum_k V_k(t-t')\hat{\mathbf{b}}_k(t'), \tag{31}$$

$$\hat{\mathbf{b}}_k(t)=\sum_{k'} X_{k'k}(t-t')\hat{\mathbf{b}}_{k'}(t')+Y_k(t-t')\mathbf{a}(t'+0). \tag{32}$$

The evolution of the operators $\hat{\mathbf{a}}(t)$ and $\hat{\mathbf{b}}(t)$ from the time $t'+0$ takes place as a function of these same functions $S(t)$, $V_k(t)x_{k'k}(t)$, $Y_k(t)$, but with an argument shifted by a quantity since the equations of motion for the oscillator without the force possess symmetry relative to time shifts. Thus, substituting the value of the jump (30) in (31) and (32), we have

$$\hat{\mathbf{a}}(t)=S(t-t')\hat{\mathbf{a}}(t'-0)+\sum_k V_k(t-t')\hat{\mathbf{b}}_k(t')+S(t-t')\mathbf{F}_0, \tag{33}$$

$$\hat{\mathbf{b}}_k(t)=\sum_{k'} X_{k'k}(t-t')\hat{\mathbf{b}}_{k'}(t')+Y_k(t-t')\hat{\mathbf{a}}(t'-0)+Y_k(t-t')\mathbf{F}_0. \tag{34}$$

The operators $\hat{\mathbf{b}}_k(t)$ at the time t', naturally, do not undergo a jump, as is obvious from (34), since $Y_k(t-t')|_{t=t'}=0$. The terms on the right sides of Eqs. (33) and (34), which do not contain the quantity $\mathbf{F}_0$, are simply rewritten with account taken of the shift of the time origin by t' in the solution of the Heisenberg equation for the oscillator without the force. If, however, the force is a smeared out function of time, $F(t)$, then it is necessary, as usual, to take the superposition of the effects on the solution of each instant of time t', writing the solution in the following form:

$$\hat{\mathbf{a}}(t)=\mathbf{a}_0(t)+\int_0^t S(t-t')\mathbf{F}(t')\,dt', \tag{35}$$

$$\hat{\mathbf{b}}_k(t)=\hat{\mathbf{b}}_{k0}(t)+\int_0^t Y_k(t-t')\mathbf{F}(t')\,dt'. \tag{36}$$

Thus, the presence of the external driving force $\mathbf{F}(t)$ shifts the annihilation operators $\hat{\mathbf{a}}_0(t)$ and $\hat{\mathbf{b}}_{k0}(t)$—the solutions of the Heisenberg equations of motion for the A oscillator without the force—by c-number vectors. We write down the final solution $\hat{\mathbf{a}}(t)$ of the initial equations of motion for the A oscillator with the force, expressing them in terms of the matrices $S(t)$ and $V_k(t)$. We have

$$\hat{\mathbf{a}}(t)=S(t)\hat{\mathbf{a}}+\sum_k V_k(t)\hat{\mathbf{b}}_k+\int_0^t S(t-t')\mathbf{F}(t')\,dt'. \tag{37}$$

We take into account the presence of the force by the additional term in (17), introducing the complex vector

$$\alpha(t)=S(t)\alpha+\sum_k V_k(t)\beta_k+\int_0^t S(t-t')\mathbf{F}(t')\,dt'. \tag{38}$$

THE DENSITY MATRIX OF A SET OF OSCILLATORS

We now consider the initial state of the A oscillator and the oscillators of the heat bath. We assume that the initial state $|\alpha\rangle$ of the A oscillator is coherent while the oscillators of the heat bath are in completely random states. This means that the initial density matrix of the entire system $\hat{\rho}(0)$ has the factorized form

$$\hat{\rho}(0)=|\alpha\rangle\langle\alpha|\prod_{\mu k}\hat{\rho}_{\mu k}. \tag{39}$$

Here $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_N)$ is a complex vector, the density matrix $\hat{\rho}_{\mu k}$ describes the random state of the k th oscillator in the αth heat bath. The product in (39) is over all the indices of the heat baths from 1 to N, and over all possible indices of the oscillators k in the specified μth heat bath. In accord with Ref. 1, the matrix $\hat{\rho}_{\mu k}$ in the P-representation[3] is described by the formula

$$\hat{\rho}_{\mu k}=\frac{1}{\pi\langle n_{\mu k}\rangle}\int\exp\left\{-\frac{|\beta_{\mu k}|^2}{\langle n_{\mu k}\rangle}\right\}|\beta_{\mu k}\rangle\langle\beta_{\mu k}|\,d^2\beta_{\mu k}. \tag{40}$$

Here $\langle n_{\mu k}\rangle$ is the mean number of quanta and is described, at the temperature of the μth bath T_μ, by the Planck distribution

$$\langle n_{\mu k}\rangle=(e^{\hbar\omega_k/T_\mu}-1)^{-1}. \tag{41}$$

Then, as can be proved, the solution for the quantum mechanical Liouville equation for the density operator $\hat{\rho}(t)$

$$i\hbar\frac{\partial\hat{\rho}(t)}{\partial t}=[\hat{H},\hat{\rho}(t)] \tag{42}$$

is the following operator:

$$\hat{\rho}(t)=\int\prod_{\mu k}|\alpha(t),\{\beta_k(t)\}\rangle\langle\alpha(t),\{\beta_k(t)\}|\exp\left\{-\frac{|\beta_{\mu k}|^2}{\langle n_{\mu k}\rangle}\right\}\frac{d^2\beta_{\mu k}}{\pi\langle n_{\mu k}\rangle}. \tag{43}$$

Here the numerical vector $\alpha(t)$ is given by Eq. (38), the vector $\beta_k(t)$ by the equation

$$\beta_k(t)=\sum_{k'} X_{k'k}(t)\beta_{k'}+Y_k(t)\alpha+\int_0^t Y_k(t-t')\mathbf{F}(t')\,dt', \tag{44}$$

and the product over the indices of the components of the complex vector $\beta_k = (\beta_{1k}, \beta_{2k}, \ldots, \beta_{Nk})$ and over all possible values of the index k. Our purposes is the calculation of the abbreviated density matrix $\hat{\rho}_A(t)$ for the A oscillator, obtained by averaging over all the coordinates of the B oscillators of the heat bath:

$$\hat{\rho}_A(t) = \mathrm{Tr}_B \hat{\rho}(t). \tag{45}$$

We shall seek here the density operator $\hat{\rho}(t)$ in the P-representation:

$$\hat{\rho}_A(t) = \int P(\alpha, 0|\gamma, t)\, |\gamma\rangle\langle\gamma|\, d^2\gamma. \tag{46}$$

The integration in (45) is carried out over the N-dimensional complex density, and the quantity $P(\alpha, 0|\gamma, t)$, called the quasi-probability, can be found from the following considerations. If we know that the function P is Gaussian, then it is determined by the following set of parameters: the mean values $\langle\hat{\mathbf{a}}(t)\rangle$ and the matrix of second moments σ, i.e.,

$$P = \frac{1}{\pi^N \det\sigma} \exp\{-(\gamma - \langle\hat{\mathbf{a}}(t)\rangle)^+ \sigma^{-1} (\gamma - \langle\hat{\mathbf{a}}(t)\rangle)\}. \tag{47}$$

Therefore, in order to calculate the function P, if we are sure that it is Gaussian, it suffices to calculate the averages and the variances, and it is not necessary to carry out the integration over the coordinates of the B oscillators explicitly, using formulas (38) and (44). We calculate the matrix of second moments σ by using its connection with the correlation functions, computing first the correlation functions themselves.

The solutions of the Heisenberg equations of motion for the operators $\hat{\mathbf{a}}(t), \hat{\mathbf{b}}_k(t)$ or $\hat{\mathbf{A}}(t), \hat{\mathbf{B}}_k(t)$ have so far been regarded as exact, expressed in terms of the matrices $S(t), V_k(t)$ or $u(t), v_k(t)$. If we now substiute the linear form of the solutions of Eq. (8) ($\mathbf{f} = 0$) in these equations, then a set of equations is obtained for the quantities $u(t)$ and $v_k(t)$. As noted in Ref. 1, the obtained set is formally identical with the set of equations solved by Wigner and Weisskopf[2] for another physical problem. In the theory of radiation of atoms, in Ref. 2, these same equations were considered as approximate equations, describing the radiation damping and coupling the amplitude of the atom, which it has in the excited state, with the amplitudes of the atom in the ground state in the presence of a radiation field of photons in various mode states. Using this analogy, we can show that $\mathbf{A}(t)$ plays in Eqs. (8) with $\mathbf{f} = 0$ and in Eq. (5) the role of the amplitude of the excited atom, $\mathbf{B}_k(t)$ plays there the role of the amplitudes of the single-photon states, and the λ_k play the role of matrix elements of the transition. If the assumption that the spectrum of frequencies ω_k of the oscillators of the heat bath is continuous near the frequencies Ω_α, then the mathematical problem of solving Eqs. (8) is identical with that solved by Weisskopf and Wigner in Ref. 2 by the radiation-damping theory. Using the Weisskopf-Wigner approximation, we can write down the solutions for the function $u_\alpha(t)$ and $v_{\alpha k}(t)$ in the form

$$u_\alpha(t) = \exp\{-[\varkappa_\alpha + i(\Omega_\alpha + \delta\Omega_\alpha)]t\}, \tag{48}$$

$$v_{\alpha k}(t) = \frac{-i\lambda_k}{\varkappa_\alpha + i(\Omega_\alpha + \delta\Omega_\alpha - \omega_k)} \times \{\exp(-i\omega_k t) - \exp[-(\varkappa_\alpha + i(\Omega_\alpha + \delta\Omega_\alpha))t]\}, \tag{49}$$

where the constants $\varkappa_\alpha$ and $\sigma\Omega_\alpha$ are determined by the relation

$$\delta\Omega_\alpha - i\varkappa_\alpha = \lim_{\varepsilon\to 0} \sum_k \frac{|\lambda_k|^2}{(\Omega_\alpha - \omega_k + i\varepsilon)}. \tag{50}$$

It is clear from (48) and (49) that the $\varkappa_\alpha$ play the role of damping constants and $\delta\Omega_\alpha$ are the frequency shifts. The Weisskopf-Wigner approximation is valid if these constants are small in comparison with the frequencies Ω_α, and in comparison with the frequency intervals over which the coupling parameters λ_k and the parameters describing the state of the heat bath change significantly. The irreversible character of the behavior of the A oscillator appears in our equations in terms of the Weisskopf-Wigner approximation. In fact, however, it is inferred in one of the assumptions on which this approximation is based, namely, the continuity of the frequency spectrum of the oscillators is assumed implicitly, which means that there is an infinite number of them. In any real example, of course, there exists only a finite number of oscillators of the heat bath, and their spectrum is not continuous but discrete, although very dense. Thus, the Weisskopf-Wigner approximation deals with the asymptotically limiting behavior of a system in which the number of oscillators of the heat bath becomes infinite. In this and only in this limit is an irreversible behavior of the system actually expected. The irreversible solutions (48) and (49) are idealizations, but they model solutions for systems with very large but finite numbers of degrees of freedom with great accuracy over large intermediate times. We shall assume that we can represent the frequency spectrum $\{\omega_k\}$ of the oscillators of the heat bath by means of the functions $g(\omega_k)$—the spectral density. We then get from (50)

$$\varkappa_\alpha = \pi \int |\lambda_k|^2 g(\omega_k)\, \delta(\Omega_\alpha - \omega_k)\, d\omega_k = \pi |\lambda_{\Omega_\alpha}|^2 g(\Omega_\alpha), \tag{51}$$

where λ_{Ω_α} is the value of the coupling constant at $\omega_k = \Omega_\alpha$. The functions $v_{\alpha k}(t)$ describe the contribution to the excitation of the A oscillator by the individual B oscillators. It is seen from (49) that the functions $|v_{\alpha k}(t)|^2$ are proportional to the expression $[(\Omega_\alpha + \delta\Omega_\alpha - \omega_k)^2 + \varkappa_\alpha^2]^{-1}$. Thus, they have a sharp peak at $\omega_k = \Omega_\alpha + \delta\Omega_\alpha$ and fall off rapidly to small values outside a frequency band of width $\Delta\omega \sim 2\varkappa_\alpha$ away from this peak. The correlation function of first order, the N-dimensional matrix $\tilde{G}^{(1)}(t_1, t_2)$, is determined in the following fashion:

$$\tilde{G}^{(1)}_{\mu\nu}(t_1, t_2) = \langle\hat{a}_\mu^+(t_1)\hat{a}_\nu(t_2)\rangle - \langle\hat{a}_\mu^+(t_1)\rangle\langle\hat{a}_\nu(t_2)\rangle. \tag{52}$$

The average in (52) is understood as averaging with the whole initial density matrix of the system $\hat{\rho}(0)$ [(39)]. At $t_1 = t_2 = t$ we obtain the matrix of second moments, which enters into the quasiprobability function (46):

$$\tilde{G}^{(1)}(t, t) = \sigma(t). \tag{53}$$

We define the correlation function of higher order by the formula

$$\tilde{G}^{(n)}_{i_1, i_2, \ldots, i_{2n}}(t_1, t_2, \ldots, t_{2n}) = \left\langle \prod_{\mu=1}^{n} (\hat{a}_{i_\mu}^+ - \langle\hat{a}_{i_\mu}^+(t_\mu)\rangle) \prod_{\nu=1}^{n} (\hat{a}_{i_\nu}(t_\nu) - \langle\hat{a}_{i_\nu}(t_\nu)\rangle) \right\rangle. \tag{54}$$

In (54), the average is understood as taking the trace with the initial density matrix of the entire system $\hat{\rho}(0)$. We shall show that the higher correlation functions $\tilde{G}^{(n)}$ (54) can be expressed in terms of the functions (52). For this purpose, we calculate the generating functional Ξ for the correlation functions $\tilde{G}^{(n)}$. The generating function $\Xi(\zeta,t)$, $\eta(t')$ for the correlation functions $\tilde{G}^{(n)}$ can be found, as in the one-dimensional case,[11] by direct calculation with averaging by means of the matrix $\hat{\rho}(0)$ [Eq. (39)]:

$$\Xi=\left\langle \exp\left[\int \zeta(t)\,(\hat{\mathbf{a}}^{+}(t)-\langle\hat{\mathbf{a}}^{+}(t)\rangle)\,dt\right] \exp\left[\int \eta(t')\,(\hat{\mathbf{a}}(t')-\langle\hat{\mathbf{a}}(t')\rangle)\,dt'\right]\right\rangle . \tag{55}$$

In (55), the integral is Gaussian and we have the result

$$\Xi(\zeta(t),\eta(t'))=\exp\left[\int dt\,dt'\,\zeta(t)\tilde{G}^{(1)}(t,t')\eta(t')\right]. \tag{56}$$

Here the matrix $\tilde{G}^{(n)}$ has the form of the sum of products with seven matrices:

$$\tilde{G}^{(1)}(t_1,t_2)=\sum_k U^{*}v_k^{*}(t_1)\,U^{T}N_kU^{*}v_k(t_2)\,U^{T}. \tag{57}$$

Thus, all the higher correlation functions $\tilde{G}^{(n)}$ are expressed only in terms of the matrix function $\tilde{G}^{(1)}(t_1,t_2)$. The generating functional for the functions

$$G^{(n)}_{i_1,i_2,\ldots,i_{2n}}(t_1,t_2,\ldots,t_{2n})=\left\langle\prod_{\mu=1}^{n}\hat{a}_{i_\mu}{}^{+}(t_\mu)\prod_{\nu=1}^{n}\hat{a}_{i_{n+\nu}}(t_{n+\nu})\right\rangle \tag{58}$$

is of the form

$$Z=\left\langle\exp\left[\int\zeta(t)\hat{\mathbf{a}}^{+}(t)\,dt\right]\exp\left[\int\eta(t')\hat{\mathbf{a}}(t')\,dt'\right]\right\rangle , \tag{59}$$

and differs from the functional Ξ by a factor, i.e.,

$$Z=\Xi\exp\left[\int\zeta(t)\langle\hat{\mathbf{a}}^{+}(t)\rangle\,dt\right]\exp\left[\int\eta(t')\langle\hat{\mathbf{a}}(t')\rangle\,dt'\right]. \tag{60}$$

Thus, all the higher correlation functions $G^{(n)}$ are expressed in terms of the average $\langle a_\mu(t)\rangle$ and the functions $G^{1}_{(\hat{\mu}\nu)}(t_1,t_2)$. In order to find $\tilde{G}^{(1)}$ (57) explicitly, we calculate the following sums. The sum rule for the diagonal matrices $u(t)$ and $v_k(t)$ with the diagonal elements (48) and (48) follows from the conservation of the commutation relations fo the creation and annihilation operators under a shift of the time origin. That is,

$$\sum_k v_k^{*}(t_1)v_k(t_2)=\begin{cases}u^{*}(t_1-t_2)-u^{*}(t_1)\,u(t_2), & t_1>t_2;\\ u(t_2-t_1)-u^{*}(t_1)\,u(t_2), & t_2>t_1.\end{cases} \tag{61}$$

With account of this sum rule, we calculate the matrix elements $(U^{T}\tilde{G}^{(1)}U^{*})_{\mu\rho}$, which are expressed in terms of the matrix elements of the matrix

$$K(\mathbf{n}_k)=U^{T}N_kU^{*},\qquad \mathbf{n}_k=(\langle n_{1k}\rangle,\ \langle n_{2k}\rangle,\ \ldots,\ \langle n_{Nk}\rangle). \tag{62}$$

It is easy to see that (the summation here is not carried out over the indices μ and ρ)

$$(U^{T}\tilde{G}^{(1)}U^{*})_{\mu\rho}=\sum_k K(\mathbf{n}_k)_{\mu\rho}\,v_{\mu k}^{*}(t_1)\,v_{\rho k}(t_2). \tag{63}$$

We can calculate the sums over k in the matrix elements (63) either by substituting them in the integrals over the frequencies and estimating the integrals, or by using the unitarity condition of the evolution operator of the entire system [the sum rule (61)] and the slow change in the quantities $\mathbf{n}_k$ [and, thus, in $K(\mathbf{n}_k)$] near the frequencies that minimize the denominators of (49). Then the basic contribution to the sum is obtained from terms on the diagonal of the matrix $U^{T}\tilde{G}^{(1)}U^{*}$ in (63); the nondiagonal terms have the next order of smallness. Recognizing that the diagonal matrix element of the matrix K (62) is written in the form

$$K_{\mu\mu}=|U_{1\mu}|^2\langle n_{1k}\rangle+|U_{2\mu}|^2\langle n_{2k}\rangle+\ldots+|U_{N\mu}|^2\langle n_{Nk}\rangle, \tag{64}$$

where $\langle n_{\mu k}\rangle$ are given by Planck's formula (41), we obtain the following approximate equation for the matrix element of the matrix $\tilde{G}^{(1)}_{\alpha\beta}(t_1,t_2)$ in (63):

$$\tilde{G}^{(1)}_{\alpha\beta}(t_1,t_2)=\sum_{\rho,\mu=1}^{N}|U_{\rho\mu}|^2(e^{\hbar\Omega_\mu'/T_\rho}-1)^{-1}f_\mu(t_1,t_2)\,U_{\alpha\mu}{}^{*}U_{\beta\mu}. \tag{65}$$

Here, with account of (61),

$$f_\mu(t_1,t_2)=\begin{cases}u_\mu^{*}(t_1-t_2)-u_\mu^{*}(t_1)\,u_\mu(t_2), & t_1>t_2,\\ u_\mu(t_2-t_1)-u_\mu^{*}(t_1)\,u_\mu(t_2), & t_2>t_1,\end{cases} \tag{66}$$

$$\Omega_\mu'=\Omega_\mu+\delta\Omega_\mu, \tag{67}$$

where $\delta\Omega_k$ is the frequency shift of the normal oscillation and is given by (50). Substituting the expression for $u_\mu(t)$ given by Eq. (48) in explicit form in (65) and (66), we obtain the following expression for the matrix element $(U^{T}\tilde{G}^{(1)}(t_1,t_2)U^{*})_{\nu\nu}$ in the case of long times $t_{1,2}\gg\varkappa_\nu^{-1}$:

$$(U^{T}\tilde{G}^{(1)}(t_1,t_2)\,U^{*})_{\nu\nu}=\sum_{\mu=1}^{N}|U_{\mu\nu}|^2[e^{\hbar\Omega_\nu'/T_\mu}-1]^{-1} \times\{\exp[-\varkappa_\nu|t_1-t_2|+i\Omega_\nu'(t_1-t_2)]\}. \tag{68}$$

Thus, the desired matrix $\tilde{G}^{(1)}(t_1,t_2)$ is obtained from a diagonal matrix with the matrix elements of (68) by using the expression

$$\tilde{G}^{(1)}_{\alpha\beta}(t_1,t_2)=\sum_{\mu,\rho=1}\{U_{\alpha\mu}{}^{*}U_{\beta\mu}|U_{\rho\mu}|^2[e^{\hbar\Omega_\mu'/T_\rho}-1]^{-1} \times\exp[-\varkappa_\mu|t_1-t_2|+i\Omega_\mu'(t_1-t_2)]\}. \tag{69}$$

All the matrix elements of the matrix $\tilde{G}^{(1)}_{\alpha\beta}(t_1,t_2)$ are linear combinations of expressions that are exponentially damped in the parameter $|t_1-t_2|$ with expressions that oscillating in this case. We consider two special cases: $N=1$ and $N=2$, i.e., one-dimensional and two-dimensional A oscillators. For $N=1$, we have the following expression for the correlation function:

$$\tilde{G}^{(1)}(t_1,t_2)=\langle\hat{a}^{+}(t_1)\hat{a}(t_2)\rangle-\langle\hat{a}^{+}(t_1)\rangle\langle\hat{a}(t_2)\rangle, \tag{70}$$

which, with account taken in (65) of the fact that there is only a single term from the sum and $U_{\alpha\beta}=1$, gives the following for not too long times,

$$\tilde{G}^{(1)}(t_1,t_2)=[e^{-\hbar\Omega'/T}-1]^{-1}[e^{-\varkappa|t_1-t_2|}-e^{-\varkappa(t_1+t_2)}]\,e^{i\Omega'(t_1-t_2)}. \tag{71}$$

(We have omitted the indices, assuming that the temperatures of the baths are the same and equal to T, the frequencies of the normal vibrations are the same and equal to Ω, the frequency shift $\delta\Omega$ and the relaxation time are equal to $\varkappa^{-1}$.) For long times $t_{1,2}\gg\varkappa^{-1}$ only the dependence on the time difference $t_1-t_2=\tau$ remains:

$$\bar{G}^{(1)}(t_1-t_2)=[e^{\hbar\Omega'/T}-1]^{-1}\exp[-\varkappa|t_1-t_2|+i\Omega'(t_1-t_2)]. \quad (72)$$

With account taken of the action of the external force, we have the following expression for the correlation function of a one-dimensional A oscillator:

$$\begin{aligned}G^{(1)}(t_1,t_2)&=\bar{G}^{(1)}(t_1,t_2)\\&+\alpha\left\{\int_0^{t_1}[\exp[-(\varkappa-i\Omega')(t_1-t')]]F^*(t')\,dt'\right\}\\&\times\exp[-(\varkappa+i\Omega')t_2]\\&+\alpha^*\left\{\int_0^{t_2}[\exp[-(\varkappa+i\Omega')(t_2-t')]]F(t')\,dt'\right\}\\&\times\exp[-(\varkappa-i\Omega')t_1]+\int_0^{t_1}\int_0^{t_2}dt'\,dt''\,F^*(t')F(t'')\\&\times\exp[-(\varkappa+i\Omega')(t'-t_2)-(\varkappa-i\Omega')(t''-t_1)].\end{aligned} \quad (73)$$

The higher correlation functions for the one-dimensional A oscillator $\tilde{G}^{(n)}$ are expressed in terms of the function $\tilde{G}^{(1)}$ in the following fashion:

$$\tilde{G}^{(n)}(t_1\ldots t_n,t_{n+1}\ldots t_{2n})=\sum_P\prod_{j=1}^{n}\tilde{G}^{(1)}(t_j,t_{P(n+j)}). \quad (74)$$

Here the summation over P is carried out over all $n!$ permutations of the numbers j. For example, for the function of 2nd order we have

$$\begin{aligned}&\tilde{G}^{(2)}(t_1,t_2,t_3,t_4)\\&=\tilde{G}^{(1)}(t_1,t_3)\tilde{G}^{(1)}(t_2,t_4)+\tilde{G}^{(1)}(t_1,t_4)\tilde{G}^{(1)}(t_2,t_3).\end{aligned} \quad (75)$$

For an N-dimensional A oscillator, the formula connecting $\tilde{G}^{(n)}$ with $\tilde{G}^{(1)}$ is analogous.

We now give as an example $\tilde{G}^{(1)}(t_1,t_2)$ for the problem of a two-dimensinoial symmetric oscillator with the Hamiltonian

$$\begin{aligned}\hat{H}=\hbar[&\omega(\hat{a}_1^+\hat{a}_1+\hat{a}_2^+\hat{a}_2)+\lambda(\hat{a}_1\hat{a}_2^++\hat{a}_2\hat{a}_1^+)\\&+i(\hat{a}_1^+F_1(t)+\hat{a}_2^+F_2(t)-\hat{a}_1F_1^*(t)-\hat{a}_2F_2^*(t))].\end{aligned} \quad (76)$$

In this case, the matrix elements of the matrix U are the following $U_{11}=U_{12}=U_{21}=-U_{22}=1/\sqrt{2}$; $\Omega_1=\omega+\lambda$, $\Omega_2=\omega-\lambda$, while the parameter λ is sufficiently large, so that $\varkappa_1$, $\delta\Omega_1$, $\varkappa_2$, $\delta\Omega_2$ but also much smaller than $\Omega_{1,2}$ and $\Omega_1-\Omega_2$. This assumption means that the mixing of the oscillations, because of the interactions of the two oscillators, takes place rapidly in comparison with the relaxation processes connected with the interaction with the heat baths. In the other limiting case, in which the parameter λ is very small in comparison with $\varkappa_1$, $\varkappa_2$, $\delta\Omega_1$, and $\delta\Omega_2$, this means in practice that $\varkappa_1\approx\varkappa_2$, $\delta\Omega_1\approx\delta\Omega_2$ and $u_1(t)=u_2(t)$. In the language of zeros in the denominators of the functions $v_{1k}(t)$ and $v_{2k}(t)$, this means equality of the zeros of the denominators of these functions and the equality of the functions themselves $v_{1k}(t)\approx v_{2k}(t)$. As is seen from (37), the damping of each of the two oscillators in this case takes place independently, as was to have been expected. Since $N=2$, we have the following for the diagonal elements of the matrix of correlation functions:

$$\begin{aligned}\tilde{G}_{11}^{(1)}(t_1,t_2)&=\tilde{G}_{22}^{(1)}(t_1,t_2)\\&=\frac{1}{4}\{[[e^{\hbar\Omega_1'/T_1}-1]^{-1}+[e^{\hbar\Omega_1'/T_2}-1]^{-1}]\\&\times\exp[-\varkappa_1|\tau|+i\Omega_1'(t_1-t_2)]+[[e^{\hbar\Omega_2'/T_1}-1]^{-1}\\&+[e^{\hbar\Omega_2'/T_2}-1]^{-1}]\exp[-\varkappa_2|\tau|+i\Omega_2'(t_1-t_2)]\}\end{aligned} \quad (77)$$

and for the nondiagonal elements,

$$\begin{aligned}\tilde{G}_{12}^{(1)}(t_1,t_2)&=\tilde{G}_{21}^{(1)}(t_1,t_2)\\&=\frac{1}{4}\{[[e^{\hbar\Omega_1'/T_1}-1]^{-1}+[e^{\hbar\Omega_1'/T_2}-1]^{-1}]\\&\times\exp[-\varkappa_1|\tau|+i\Omega_1'(t_1-t_2)]\\&-[[e^{\hbar\Omega_2'/T_1}-1]^{-1}+[e^{\hbar\Omega_2'/T_2}-1]^{-1}]\\&\times\exp[-\varkappa_2|\tau|+i\Omega_2'(t_1-t_2)]\}.\end{aligned} \quad (78)$$

The expressions for the mean $\langle\hat{a}_\beta(t_2)\rangle$, $\langle\hat{a}_\alpha^+(t_1)\rangle$ are given by the formulas

$$\begin{aligned}\langle\hat{a}_1(t_2)\rangle&=\tfrac{1}{2}\Big\{(\alpha_1+\alpha_2)\exp[-\varkappa_1t_2+i\Omega_1't_2]\\&+(\alpha_1-\alpha_2)\exp[-\varkappa_2t_2+i\Omega_2't_2]\\&+\int_0^{t_2}[[F_1(t')+F_2(t')]\exp[-\varkappa_1(t_2-t')+i\Omega_1'(t_2-t')]\\&+[F_1(t')-F_2(t')]\exp[-\varkappa_2(t_2-t')+i\Omega_2'(t_2-t')]]dt'\Big\}\\\langle\hat{a}_2(t_2)\rangle&=\frac{1}{2}\Big\{(\alpha_1+\alpha_2)\exp[-\varkappa_1t_2+i\Omega_1't_2]\\&-(\alpha_1-\alpha_2)\exp[-\varkappa_2t_2+i\Omega_2't_2]\\&+\int_0^{t_2}[[F_1(t')+F_2(t')]\exp[-\varkappa_1(t_2-t')+i\Omega_1'(t_2-t')]\\&-[F_1(t')-F_2(t')]\exp[-\varkappa_2(t_2-t')+i\Omega_2'(t_2-t')]]dt'\Big\},\end{aligned} \quad (79)$$

$$\langle\hat{a}_1^+(t_1)\rangle=\langle\hat{a}_1(t_2\to t_1)\rangle^*,\quad \langle\hat{a}_2^+(t_1)\rangle=\langle\hat{a}_2(t_2\to t_1)\rangle^*. \quad (80)$$

(81)

At long times, much longer than the times of action of the force $F(t)$, and such that $t\gg\tilde{\varkappa}_{1,2}^{-1}$, and not too large $|\alpha_{1,2}|$ (the initial amplitude of the vibrations is not too large and is already undergoing damping), we have

$$\langle\hat{a}_\alpha(t)\rangle=0,\quad \alpha=1,2 \quad (82)$$

and, consequently,

$$G_{\alpha\beta}^{(1)}(t_1,t_2)\approx\bar{G}_{\alpha\beta}^{(1)}(t_1,t_2). \quad (83)$$

If we take the value of $\tilde{G}_{\alpha\beta}^{(1)}(t_1,t_2)$ at equal times $t_1=t_2=t$, the expression for $\tilde{G}_{\alpha\beta}^{(1)}(t,t)$ is identical with the matrix of second moments $\sigma(t)$. All the higher correlation functions $G^{(n)}$ are expressed in terms of the correlation function $G^{(1)}$ and the mean values $\langle\hat{a}_{1,2}^+(t)\rangle$ and $\langle\hat{a}_{1,2}(t)\rangle$. Substituting $t_1=t_2=t$ in (65), we obtain the following expression for the diagonal matrices of the elements of the dispersion matrix:

$$\begin{aligned}\sigma_{11}(t)=\sigma_{22}(t)=&\tfrac{1}{4}\{[\langle n_{\Omega_1'}(T_1)\rangle+\langle n_{\Omega_1'}(T_2)\rangle](1-e^{-2\varkappa_1t})\\&+[\langle n_{\Omega_2'}(T_1)\rangle+\langle n_{\Omega_2'}(T_2)\rangle](1-e^{-2\varkappa_2t})\},\end{aligned} \quad (84)$$

and for the nondiagonal elements, we have

$$\sigma_{12}(t)=\sigma_{21}(t)=\tfrac{1}{4}\{[\langle n_{\Omega_1'}(T_1)\rangle+\langle n_{\Omega_1'}(T_2)\rangle](1-e^{-2\varkappa_1 t}) -[\langle n_{\Omega_2'}(T_1)\rangle+\langle n_{\Omega_2'}(T_2)\rangle](1-e^{-2\varkappa_2 t})\}. \quad (85)$$

Here $\langle n_{\Omega_\mu'}(T_\nu)\rangle$, $\nu=1,2$ are the Planck distributions at the temperature T_ν for oscillators with frequencies Ω_μ'. The determinant of the matrix $\sigma(t)$, which determines the factor before the exponential in the expression for the quasi-probability P in Eq. (47) for the case $N=2$, has the form

$$\det\sigma(t)=\tfrac{1}{4}[n_{\Omega_1'}(T_1)\rangle+\langle n_{\Omega_1'}(T_2)\rangle] \times[\langle n\Omega_{2'}(T_1)\rangle+\langle n_{\Omega_2'}(T_2)\rangle](1-e^{-2\varkappa_1 t})(1-e^{-2\varkappa_2 t}). \quad (86)$$

The matrix elements of the reciprocal matrix σ^{-1} that enters in the exponential in the expression for the quasi-probability P in Eq. (4) are easily found from (84), (85) and (86):

$$(\sigma^{-1})_{11}=(\sigma^{-1})_{22}=\sigma_{11}/\det\sigma, \quad (\sigma^{-1})_{12}=(\sigma^{-1})_{21}=-\sigma_{12}/\det\sigma. \quad (87)$$

Thus, we have calculated explicitly the quasi-probability function for a two-dimensional symmetric oscillator: the mean values $\langle\hat{a}_1(t)\rangle$ and $\langle\hat{a}_2(t)\rangle$ that enter in the exponential in Eq. (47) are given by Eqs. (79) and (80). The center of the Gaussian distribution (47) moves along the trajectories (79) and (80), while the amplitudes of the first and second oscillator are damped out, the damping law being the superposition of the two descending exponentials with the respective relaxation times $\varkappa_1^{-1}$ and $\varkappa_2^{-1}$. The width of the distribution does not depend on the effect of the external force and for times that are much greater than the relaxation time, we have the following for the matrix elements of the matrices of second moment:

$$\sigma_{11}(\infty)=\sigma_{22}(\infty)=\tfrac{1}{4}\{\langle n_{\Omega_1'}(T_1)\rangle+\langle n_{\Omega_1'}(T_2)\rangle +\langle n_{\Omega_2'}(T_1)\rangle+\langle n_{\Omega_2'}(T_2)\rangle\}, \quad (88)$$

$$\sigma_{12}(\infty)=\sigma_{21}(\infty)=\tfrac{1}{4}\{\langle n_{\Omega_1'}(T_1)\rangle+\langle n_{\Omega_1'}(T_2)\rangle -\langle n_{\Omega_2'}(T_1)\rangle-\langle n_{\Omega_2'}(T_2)\rangle\}. \quad (89)$$

Similar to the case of a two-dimensional A oscillator from Eq. (69), we obtain a formula for the matrix elements of the dispersion matrix at long times for an N-dimensional A oscillator:

$$\sigma_{\alpha\beta}(\infty)=\sum_{\rho,\mu=1}^{N} U_{\alpha\mu}^{*}U_{\beta\mu}|U_{\rho\mu}|^2(e^{\hbar\Omega_\mu'/T_\rho}-1)^{-1}. \quad (90)$$

Thus, we arrive at the state of equilibrium of the N-dimensional A oscillator, and the obtained equilibrium state is a non-Gibbsian equilibrium distribution; we define it as an intermediate equilibrium state, or temporal state of equilibrium. This state of equilibrium is stable. If the system of N interacting oscillators is excited, it returns to the described equilibrium state. Thus, we have demonstrated that non-Gibbsian equilibrium states exist in a set of N oscillators; upon the introduction of damping (interaction with the heat bath), the system comes to the specificed intermediate equilibrium state.

The quasi-probability equilibrium function P for the equilibrium state, to which the N-dimensional A oscillator comes after interaction of the external force over long times, is given by Eq. (47), where the mean values $\langle\hat{a}_k(t)\rangle$ are equal to zero, while the variance matrix is given by Eq. (90).

In conclusion, we note that the basic result of this research is the clarification of the details of the approach to the equilibrium state of a system of coupled oscillators. In the case of a heat bath with several temperatures, the equilibrium state of the system of coupled oscillators turns out to be non-Gibbsian (for a one-dimensional system it is Gibbsian). The process of damping of the amplitude of vibrations of the system of oscillators is determined by the superposition of the exponentials, which are decaying with time. The intermediate (non-Gibbsian) equilibrium state of the system of oscillators is stable. We were able to derive the explicit expressions obtained in the work for the quasi-probability function [Eqs. (47), (79), (80), (88), (89)], and also for the correlation functions of a system of coupled oscillators [Eqs. (65), (69), (73)] by use of the Weisskopf-Wigner approximation, which is one of the essential elements of the analysis. We hope that the analysis we have developed will be useful in different physical problems, modeled by a set of coupled oscillators.

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