Effect of a finite Larmor radius on the equilibrium of plasmas

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Planar equilibria of low-pressure plasmas in longitudinal magnetic fields are considered. It is shown that effects associated with finite Larmor radius (which in hydrodynamic language corresponds to the inclusion of the so-called collisionless viscosity of ions and their inertia) are small but, nevertheless, produce a definite restriction on the class of equilibrium configurations allowed by the equations of ideal magnetohydrodynamics. The physical mechanism responsible for the effect is interpreted in terms of the drift theory of motion of charged particles. The implications of this effect for magnetic traps are briefly discussed.

1. INTRODUCTION

Studies of equilibrium plasma configurations are usually performed for single-fluid ideal magnetohydrodynamics (MHD), and are based on the solution of the equation[1]

$$\nabla p=\frac{1}{4\pi}[\mathrm{rot}\,\mathbf{B}\times\mathbf{B}], \tag{1}$$

where p is the plasma pressure and $\mathbf{B}$ is the magnetic induction. Since Shafranov's early work,[2,3] this approach has been widely used to calculate stationary states in tokamaks and stellarators (see, for example, the review given by Freidberg[4]). The corresponding generalization to the case of nonisotropic pressures serves as a starting point for the study of equilibria in open traps.[5]

The validity of the MHD approximation (1) can be rigorously demonstrated by performing a kinetic analysis of the problem.[6] At the same time, kinetic theory enables us to calculate corrections to (1) and to establish its range of validity. Thus, in the case of high-temperature plasmas for which Coulomb collisions can be neglected, and which are of particular interest from the standpoint of controlled thermonuclear fusion, kinetic theory predicts the appearance of anisotropic components in the stress tensor due to the finite Larmor radius (FLR) of the particles. The resulting additional terms in (1) are of the order of $p\rho_{Hi}^2/a^3$, where ρ_{Hi} is the ion Larmor radius and a is a characteristic scale over which the plasma parameters undergo an appreciable variation. Under typical conditions, these corrections are small in comparison with gas-dynamic pressure if the parameter $(\rho_{Hi}/a)^2$ is small.[7]

Nevertheless, the inclusion of formally small FLR terms may, in some cases, turn out to be fundamental, and may affect in an essential way the results based on solutions of (1). This was noted in a recent paper[8] in relation to the equilibrium of plasmas in long open traps, although analogous effects may, in principle, obtain in other plasma-confinement systems. It therefore seems expedient to investigate (without reference to any specific trap geometry) the basic FLR effects in the simplest configuration, i.e., planar equilibrium in a magnetic field with straight lines of force. This problem is examined in the present paper. Our results can be used to investigate more complex equilibria for which the planar model is the zero-order approximation.

Let us now formulate our problem. We shall suppose that the magnetic field is parallel to the z axis

$$\mathbf{B}=\mathbf{e}_z B(x,\ y), \tag{2}$$

and the pressure p is independent of z, i.e., $p=p(x,y)$. It then follows from (1) that the well-known relation

$$p+B^2/8\pi=\mathrm{const}, \tag{3}$$

must be satisfied, which means that, for any function $p(x,y)$, there exists an equilibrium configuration with $B(x,y)$ given by (3). In this sense, the planar problem is highly degenerate within the framework of (1). We shall see below that inclusion of the FLR effect removes this degeneracy to some extent (see Section 4 for further details) and that the main problem is to derive additional restrictions that must be obeyed by FLR equilibria.

One of the possible ways of solving the above problem is to use the equations of two-fluid magnetohydrodynamics,[9] which take into account the so-called collisionless viscosity of ions and their inertia. However, we shall use the kinetic approach which is simpler in some respects and whose main feature is that it enables us to exhibit and readily interpret physically the mechanism responsible for the FLR effect. It will be seen from the ensuing account that the latter is due to drifts of a higher (third) order in ρ_H/a. These drifts are calculated in Section 2. Section 3 formulates the kinetic equation for the particle guiding centers, and Section 4 solves the equilibrium problem. The main results of our work are discussed in Section 5.

We note in conclusion of this section that although, so far, we have confined our attention to isotropic plasma pressures, there is no difficulty in abandoning this restriction. Equation (3) and the entire formulation of the problem remain unaltered when this is done, and the only difference is that p must be understood to represent the pressure component perpendicular to the magnetic field. The results given below are, in fact, valid in this more general case of anisotropic plasmas.

2. THIRD-ORDER DRIFT THEORY

Suppose that the electric field in plasma can be derived from a potential $\varphi=\varphi(x,y)$. This field does not appear in (1) and there is an infinite set of possible distributions of poten-

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tial for given equilibrium distributions of p and B when FLR is ignored in the theory. It will be seen below that the only requirement is that the equipotentials $\varphi = \text{const}$ must coincide with lines of constant p (and B). Our problem now is to obtain the drift equations of motion for a charged particle in the magnetic field (2) and potential $\varphi(x, y)$ to within terms proportional to the cube of the ratio of the Larmor radius ρ_H to the characteristic scale a.

We shall suppose that the change in the potential φ over the scale length a is of the order of the mean kinetic energy of the particles, i.e., $e\varphi \sim mv^2$. This means that

$$cE/vB \sim \rho_H/a, \tag{4}$$

i.e., the ratio of the electrical drift velocity

$$\mathbf{v}_E = c[\mathbf{E}\times\mathbf{B}]/B^2 \tag{5}$$

to the characteristic velocity v of the particle is small if the parameter ρ_H/a is small.

As for the magnetic field, the assumption that it changes by a factor of two over the scale length a leads to very unwieldy expressions for the drift. We shall therefore confine our analysis to the situation where the plasma parameter β ($\beta = 8\pi p/B^2$) does not exceed the small quantity ρ_H/a, i.e.,

$$B/|\nabla B| \sim a(a/\rho_H). \tag{6}$$

The above assumptions about the orders of magnitude of the various quantities can be formalized by introducing a dimensionless small parameter $\varepsilon \sim \rho_H/a$. The condition (4) that the electric field must be small in comparison with the magnetic field, and the condition (6) that B must vary slowly in space, will be noted by assigning the parameter ε to the coordinates x, y in the function $B(x, y)$ and then multiplying B by ε^{-1}:

$$B(x, y) \to \varepsilon^{-1} B(\varepsilon x, \varepsilon y). \tag{7}$$

This device enables us to monitor the order of magnitude of the various terms in the above equations in the course of computation; in the final equations, we must put $\varepsilon = 1$.

Projecting the equations of motion of the charged particle onto the (x, y) plane, and introducing the complex coordinate $w = x - iy$, we obtain

$$\varepsilon\Omega^{-1}\ddot{w} - i\dot{w} = \varepsilon\mathscr{E}, \tag{8}$$

where

$$\Omega(\varepsilon x, \varepsilon y) = \frac{e}{mc}B(\varepsilon x, \varepsilon y), \quad \mathscr{E} = \frac{c}{eB}(E_x - iE_y), \tag{9}$$

e and m are the charge and mass of the particle, c is the velocity of light, and differentiation with respect to time is represented by a dot.

In accordance with the general theory of drift motion (see, for example, Ref. 10), the solution of (8) will be sought in the form of the asymptotic series

$$w = w_0 + \varepsilon w_1 e^{i\psi} + \varepsilon^3 w_2 e^{2i\psi} + \varepsilon^3 w_{-1} e^{-i\psi} + \ldots, \tag{10}$$

in which W_k ($k = 0, \pm 1, \pm 2, \ldots$) are continuous functions of time, and rapid oscillations with Larmor frequency are described by the phase factors $\exp(ik\psi)$. The rate of change of the phase ψ is determined by the magnitude of the frequency Ω at the point occupied by the guiding center of the particle:

$$\dot{\psi} = \varepsilon^{-1}\Omega(\varepsilon x_0, \varepsilon y_0), \tag{11}$$

and the coordinates x_0, y_0 are determined by the zero-order term in expansion (10):

$$x_0 = \frac{1}{2}(w_0 + w_0^*), \quad y_0 = -\frac{1}{2i}(w_0 - w_0^*). \tag{12}$$

The amplitudes w_k with $k > 1$ and $k < 0$ [whose order of smallness in (10) refers to the particular problem that we are considering and must be verified in the course of subsequent computations] are also asymptotic series in nonnegative powers of ε and can be expressed in terms of w_0, w_1, and the derivatives of the fields at the point occupied by the guiding center. On the other hand, the evolution of w_0 and w_1 in time is described by the following series:

$$\dot{w}_0 = \varepsilon A_0 + \varepsilon^2 B_0 + \varepsilon^3 C_0 + \ldots, \quad \dot{w}_1 = \varepsilon A_1 + \varepsilon^2 B_1 + \ldots, \tag{13}$$

where the terms on the right-hand sides are functions of x_0, y_0, Re w_1, and Im w_1. Our problem is to determine these terms and find the amplitudes w_{-1} and w_2 in (10).

Our procedure from now on will be to substitute the series (10) directly in (8) and to equate terms of the same order in ε. The coordinate functions Ω and $\mathscr{E}$ must then be expanded in Taylor series in the difference $w - w_0$ up to terms of the second order in ε, inclusive. We shall not reproduce the straightforward but laborious derivations,[1] and simply present the final result:

$$A_0 = i\mathscr{E}, \quad B_0 = -\frac{i}{2}|w_1|^2\left(\frac{\partial\Omega}{\partial x} - i\frac{\partial\Omega}{\partial y}\right),$$

$$A_1 = -\frac{i}{2}w_1\left(\frac{\partial\mathscr{E}}{\partial x} + i\frac{\partial\mathscr{E}}{\partial y}\right),$$

$$w_{-1} = -\frac{1}{4}w_1^*\left(\frac{\partial\mathscr{E}}{\partial x} - i\frac{\partial\mathscr{E}}{\partial y}\right), \quad w_2 = \frac{w_1^2}{4\Omega}\left(\frac{\partial\Omega}{\partial x} + i\frac{\partial\Omega}{\partial y}\right),$$

$$C_0 = \frac{i}{4}|w_1|^2\left(\frac{\partial^2\mathscr{E}}{\partial x^2} + \frac{\partial^2\mathscr{E}}{\partial y^2}\right)$$

$$-\frac{i}{2\Omega}\left[(\mathscr{E}^* - \mathscr{E})\frac{\partial\mathscr{E}}{\partial x} - i(\mathscr{E}^* + \mathscr{E})\frac{\partial\mathscr{E}}{\partial y}\right]. \tag{14}$$

The functions $\mathscr{E}$ and Ω in these formulas and their derivatives are evaluated at the point occupied by the guiding center and, to simplify the notation, $\partial\Omega/\partial x$ and $\partial\Omega/\partial y$ represents the derivatives of Ω with respect to the "slow" variables εx and εy, which are of the zero order of smallness (we shall employ this notation throughout).

We must now rewrite the above result in vector form. Let $\mathbf{r}$ represent the two-dimensional position vector (x, y) of the particle and $\mathbf{R}$ the position vector of the guiding center (x_0, y_0). Let us also introduce the real variables u and γ defined by

$$w_1 = ue^{i\gamma}. \tag{15}$$

Equation (10) then determines the position of the particle in terms of the "drift variables" $\mathbf{R}, u, \gamma$ and, when (15) is taken into account, it can be written in the following form:

$$\mathbf{r} = \mathbf{R} + \varepsilon\frac{u}{\Omega}(\mathbf{e}_x\cos\theta - \mathbf{e}_y\sin\theta) - \varepsilon^3\frac{eu}{4m\Omega^3}$$

$$\times\left[\left(\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y}\right)(\mathbf{e}_x\cos\theta - \mathbf{e}_y\sin\theta)\right.$$

$$+\left(\frac{\partial E_y}{\partial x}+\frac{\partial E_x}{\partial y}\right)(-\mathbf{e}_x\sin\theta+\mathbf{e}_y\cos\theta)\Big]$$
$$+\varepsilon^3\frac{u^2}{4\Omega^2}\Big[\frac{\partial\Omega}{\partial x}(\mathbf{e}_x\cos 2\theta-\mathbf{e}_y\sin 2\theta)$$
$$+\frac{\partial\Omega}{\partial y}(-\mathbf{e}_x\sin 2\theta-\mathbf{e}_y\cos 2\theta)\Big]+O(\varepsilon^4), \qquad (16)$$

where $\theta=\psi+\gamma$ and $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors in the direction of the x and y axes. The time variation of $\mathbf{R}$ and u and of the variable θ introduced above is determined by (11), (13), and (14). In vector form, this variation is described by

$$\dot{\mathbf{R}}=\varepsilon\mathbf{v}_E+\varepsilon^2\frac{u^2}{2\Omega^2}[\mathbf{e}_z\times\nabla\Omega]+\varepsilon^3\frac{eu^2}{4m\Omega^3}[\Delta\mathbf{E}\times\mathbf{e}_z]$$
$$-\frac{\varepsilon^3}{\Omega}[(\mathbf{v}_E\nabla)\mathbf{v}_E\times\mathbf{e}_z]+O(\varepsilon^4), \qquad (17)$$

$$\dot{u}=\varepsilon^2\frac{u}{2\Omega}\mathbf{v}_E\nabla\Omega+O(\varepsilon^3), \qquad (18)$$

$$\dot{\theta}=\frac{1}{\varepsilon}\Omega-\varepsilon\frac{e}{2m\Omega}\operatorname{div}\mathbf{E}+O(\varepsilon^3), \qquad (19)$$

where v_E is the electrical drift velocity (5). Finally, we shall need the following expression for the particle velocity $v_\perp$, obtained by differentiating (16) with respect to time:

$$\mathbf{v}_\perp=\dot{\mathbf{R}}-u(\mathbf{e}_x\sin\theta+\mathbf{e}_y\cos\theta)$$
$$+\varepsilon^2\frac{eu}{2m\Omega^2}\operatorname{div}\mathbf{E}(\mathbf{e}_x\sin\theta+\mathbf{e}_y\cos\theta)$$
$$-\varepsilon^2\frac{eu}{4m\Omega^2}\Big[\left(\frac{\partial E_x}{\partial x}-\frac{\partial E_y}{\partial y}\right)(-\mathbf{e}_x\sin\theta+\mathbf{e}_y\cos\theta)$$
$$+\left(\frac{\partial E_y}{\partial x}+\frac{\partial E_x}{\partial y}\right)(-\mathbf{e}_x\cos\theta-\mathbf{e}_y\sin\theta)\Big]$$
$$+\varepsilon^2\frac{u^2}{2\Omega}\Big[\frac{\partial\Omega}{\partial x}(-\mathbf{e}_x\sin 2\theta-\mathbf{e}_y\cos 2\theta)$$
$$+\frac{\partial\Omega}{\partial y}(-\mathbf{e}_x\cos 2\theta+\mathbf{e}_y\sin 2\theta)\Big]+O(\varepsilon^3). \qquad (20)$$

It is clear from (20) that u has the significance of the smoothed transverse velocity of the particle (it differs from $v_\perp$ by an amount of the order of ε).

A well-known result[11] is obtained by discarding terms of the order of ε^3 in (16)–(20). The drift velocity (17) then consists of the sum of the electric and magnetic gradient drifts (the latter is small in comparison with the former because of our assumption that $\beta\ll 1$), and the rate of change of u given by (18) corresponds to the conservation of the magnetic moment of the particle, i.e.,

$$\frac{d}{dt}\frac{u^2}{\Omega}=0.$$

The new terms in (17), which are of the order of ε^3, describe drifts beyond the limits of ordinary drift theory. They can be interpreted as follows. The first term, which is linear in the electric field, appears when, in more accurate calculations of the electric drift, the field $\mathbf{E}$ in (5) is replaced with the electric field $\langle\mathbf{E}\rangle$ averaged of the Larmor circle of the particle and not the field $\mathbf{E}(\mathbf{R})$ at the guiding center [this interpretation of FLR effects was previously noted in Ref. 12 (p. 173) in relation to plasma oscillations]. Elementary calculation then shows that

$$\langle\mathbf{E}\rangle-\mathbf{E}(\mathbf{R})=\varepsilon^2\frac{v_\perp^2}{4\Omega^2}\Delta\mathbf{E},$$

which means that this difference gives the third term in (17). The second term, on the other hand, which is proportional to ε^3 [the last term in (17) that is quadratic in E] can be seen to have the significance of drift under the influence of the force of inertia due to the variation in the velocity v_E.

3. KINETIC DRIFT EQUATION

To formulate the equilibrium problem, we shall use the kinetic drift equation for the guiding-center distribution function f_c which, in general, depends on the variables $\mathbf{R},u,\theta$ and time t:

$$f_c=f_c(\mathbf{R},u,\theta,t).$$

This function is defined so that the number dN of guiding centers on an elementary area d^2R with coordinates u,θ within the intervals $du,d\theta$ is

$$dN=f_c(\mathbf{R},u,\theta,t)\,d^2R\,d\theta\,u\,du. \qquad (21)$$

This function is, of course, different from the true particle distribution function $f_p(\mathbf{r},\mathbf{v}_\perp,t)$, but the important point is that f_c together with (16) and (20) contain all the information on the particle distribution. In fact, to find $f_p(\mathbf{r},\mathbf{v}_\perp,t)$, we must invert (16) and (20) and use them to express R,u,θ (in the form of asymptotic series) in terms of $\mathbf{r}$ and $\mathbf{v}_\perp$, and then substitute these expressions in f_c. At the same time, we must transform the elementary phase-space element $d^2R\,du\,d\theta$ to the variables $\mathbf{r},\mathbf{v}_\perp$. We note that, in principle, this procedure can be implemented in any order of the drift theory.

The calculations become much simpler if we recall that what we require is not the particle distribution function, but merely two of its moments, namely, the particle density n and the current density $\mathbf{j}_\perp$. It is readily seen that the latter can be found directly by integrating f_c:

$$n(\mathbf{r}')=\int d^2R\,u\,du\,d\theta\,f_c\delta(\mathbf{r}'-\mathbf{r}(\mathbf{R},u,\theta)), \qquad (22)$$

$$\mathbf{j}_\perp(\mathbf{r}')=\int d^2R\,u\,du\,d\theta\,f_c\mathbf{v}_\perp(\mathbf{R},u,\theta)\delta(\mathbf{r}'-\mathbf{r}(\mathbf{R},u,\theta)), \qquad (23)$$

where the δ-function represents the difference between the positions in space of the particle and its guiding center. Substituting the series (16) and (20) for $\mathbf{r}$ and $\mathbf{v}_\perp$ in (22) and (23), and using the δ-function expansion

$$\delta(\mathbf{r}+\mathbf{a})=\delta(\mathbf{r})+(\mathbf{a}\nabla)\delta(\mathbf{r})+{}^1/_2(\mathbf{a}\nabla)^2\delta(\mathbf{r})+\ldots,$$

we can evaluate the above integrals and obtain the result again in the form of series in powers of ε. The result of this integration will be given at the end of this section. For the moment, let us consider the kinetic equation of which f_c is a solution.

In general, this equation takes the form of the continuity equation in four-dimensional space whose points are defined by the vector $\mathbf{R}$ and the parameters u and θ:

$$\frac{\partial f_c}{\partial t}+\frac{\partial}{\partial\mathbf{R}}\dot{\mathbf{R}}f_c+\frac{1}{u}\frac{\partial}{\partial u}\dot{u}uf_c+\frac{\partial}{\partial\theta}\dot{\theta}f_c=0, \qquad (24)$$

where $\mathbf{R},\dot{u}$ and $\dot{\theta}$ are given by (17)–(19). The important advantage of (24), which distinguishes it from the kinetic equation for the distribution function f_p, is that its coefficients do not contain the phase θ. This means that, if f_c was initially a function of θ, complete mixing of phases will occur in a time $\sim \Omega^{-1}$ after which we may consider (with exponential precision) that f_c is a function of only the variables $\mathbf{R}, u$, and t. The equation describing the evolution of f_c over times that are long in comparison with the cyclotron period is obtained by averaging (24) over θ. As a result, the last term in (24) vanishes, and the stationary distribution function in which we will be interested below is given by

$$\frac{\partial}{\partial \mathbf{R}} \dot{\mathbf{R}} f_c + \frac{1}{u} \frac{\partial}{\partial u} \dot{u} u f_c = 0. \tag{25}$$

Let us now evaluate the integrals (22) and (23) for f_c that is independent of θ. The results are

$$n(\mathbf{r}) = n_c + \varepsilon^2 \frac{1}{2m\Omega^2} \Delta p_c + O(\varepsilon^4), \tag{26}$$

$$\frac{1}{e} \mathbf{j}_\perp(\mathbf{r}) = \varepsilon n_c \mathbf{v}_E - \varepsilon \frac{1}{m\Omega} [\nabla p_c \times \mathbf{e}_z] + \varepsilon^3 \frac{e}{2m^2\Omega} \operatorname{rot}(\mathbf{e}_z p_c \operatorname{div} \mathbf{E})$$
$$- \varepsilon^3 \frac{1}{16\Omega^3} \operatorname{rot}(\mathbf{e}_z \Delta q_c) - \varepsilon^3 \frac{n_e}{\Omega} [(\mathbf{v}_E \nabla) \mathbf{v}_E \times \mathbf{e}_z]$$
$$+ \varepsilon^3 \frac{e p_c}{2m^2\Omega^3} [\Delta \mathbf{E} \times \mathbf{e}_z] + \varepsilon^3 \frac{e}{2m^2\Omega^3} \Delta p_c [\mathbf{E} \times \mathbf{e}_z] + O(\varepsilon^4), \tag{27}$$

where n_c, p_c, and q_c denote the moments of the guiding-center distribution functions

$$\begin{Bmatrix} n_c(\mathbf{r}) \\ p_c(\mathbf{r}) \\ q_c(\mathbf{r}) \end{Bmatrix} = 2\pi \int_0^\infty u \, du \begin{Bmatrix} 1 \\ mu^2/2 \\ u^4 \end{Bmatrix} f_c(\mathbf{r}, u). \tag{28}$$

4. DETERMINATION OF EQUILIBRIUM CONFIGURATIONS

Let us now turn to the equilibrium problem. The most direct way of constructing the equilibrium configuration is to find the solution of (25) for the electron and ion guiding-center distribution functions, to determine the particle densities n_e and n_c, and to demand that the quasineutrality condition $n_e = n_i$ be satisfied (we assume, for simplicity, that the ions are singly charged). Unfortunately, this approach encounters considerable difficulties because, above all, the quantity $\dot{u}$ in (25) is known only to within terms of the order of ε^2 [see (18)], whereas the effects in which we are interested are described by terms of the order of ε^3 in the drift equations. The determination of the particle density to the required precision would therefore require laborious calculations that would raise the order of precision of (18).

There is, however, a different approach which we shall use below to obtain the required equilibrium conditions. It involves the solution of the kinetic equation in the zero-order approximation alone.

Let us retain only the leading term in the equation for $\dot{\mathbf{R}}$ in (17). This enables us to write (25) in the form

$$\mathbf{v}_E \frac{\partial f_c}{\partial \mathbf{R}} + O(\varepsilon) = 0. \tag{29}$$

The general solution of this is

$$f_c(\mathbf{R}, u) = F(\varphi(\mathbf{R}), u) + O(\varepsilon), \tag{30}$$

where F is an arbitrary function of its arguments. Substituting (30) in (28), we obtain the moments n_c and p_c:

$$n_c = N(\varphi) + O(\varepsilon), \quad p_c = P(\varphi) + O(\varepsilon), \tag{31}$$

where

$$\begin{Bmatrix} N \\ P \end{Bmatrix} = 2\pi \int_0^\infty u \, du \begin{Bmatrix} 1 \\ mu^2/2 \end{Bmatrix} F(\varphi, u).$$

All the expressions given above apply equally to ions and electrons. However, since the electron Larmor radius is much smaller than the ion radius (at comparable temperatures), we shall neglect FLR effects for electrons and will not distinguish between the guiding center and the electron itself, identifying n_{ce} and p_{ce} with n_e and p_e, respectively (here and below, variables referring to electrons and ions are indicated by the additional subscripts e and i, and ε will henceforth represent ρ_{Hi}/a. In particular, we may rewrite (27) in the form

$$\mathbf{j}_{e\perp} = -\varepsilon e n_e \mathbf{v}_E + \varepsilon \frac{c}{B} [\mathbf{e}_z \times \nabla p_e]. \tag{32}$$

In this expression the electron charge is explicitly $-e$.

We note further that p_{ci} and n_{ce} are respectively equal to the particle pressures p_i and p_e apart from corrections of the order of the parameter ε. The relationship given by (31) thus proves the proposition formulated at the beginning of Section 2, namely, that the $\varphi = \text{const}$ lines coincide with lines of constant p (and also p_i, p_e, and n). Of course, this is true to within the adopted precision in the parameter ε.

We must now find the resultant current $\mathbf{J}$ of electrons and ions that flows in the plasma. We shall do this by adding $\mathbf{j}_{e\perp}$ to the ion current (17), using the quasineutrality condition which, in view of (26), can be written in the form

$$n_{ci} = n_e - \varepsilon^2 \frac{1}{2m_i\Omega_i^2} \Delta p_{ci} + O(\varepsilon^4). \tag{33}$$

The result is

$$\mathbf{J} = \varepsilon \frac{e}{m_i\Omega_i} [\mathbf{e}_z \times \nabla (p_{ci} + p_e)] + \varepsilon^3 \mathbf{J}_i' + O(\varepsilon^4), \tag{34}$$

where $\varepsilon^3 \mathbf{J}_i'$ represents all the terms in $\mathbf{j}_{i\perp}$ that are cubic in ε. The magnetic field distribution in this equilibrium state is described by

$$\operatorname{rot} \mathbf{B} = [\nabla B \times \mathbf{e}_z] = \frac{4\pi}{c} \mathbf{J}, \tag{35}$$

so that

$$\frac{1}{4\pi} B \nabla B = -\varepsilon \nabla (p_{ci} + p_e) + \varepsilon^3 \frac{B}{4\pi} [\mathbf{e}_z, \mathbf{J}_i'] + O(\varepsilon^4). \tag{36}$$

We must now use the above relationships to determine the divergence of the resultant current (34) and equate it to zero. Formula (36) will help us to establish that the divergence of the first term on the right of (34) is zero to within terms $\sim O(\varepsilon^4)$, so that we must have

$$\operatorname{div} \mathbf{J}_i' = 0. \tag{37}$$

We now isolate the current $\mathbf{J}_i'$ in (27) [the contribution to it of the first term on the right of (27), which arises from the difference between the guiding-center density n_{ci} and the particle density n_i, must be taken into account in accordance with (26)] and evaluate its divergence. We can then readily show that

$$A[\nabla\varphi\times\nabla\Delta\varphi]+\frac{1}{2}\frac{dA}{d\varphi}[\nabla\varphi\times\nabla(\nabla\varphi)^2]=0, \tag{38}$$

where A represents the sum

$$A(\varphi)=eN+dP/d\varphi$$

(we recall that N and P have the significance of the density and pressure of ions in the zero-order approximation in ε). In the special case of Maxwellian ion and electron distribution functions and $\beta=0$ (the latter corresponds to $\nabla B=0$, the expression given by (38) can readily be obtained from formula (7.44) in Ref. 12.

Turning now to the analysis of (38), we note first of all that, for given distributions N and P on the (x, y) plane [by virtue of (31), we can always assume that $N=N(P)$], this equation can be formally satisfied by taking A identically equal to zero. This will occur when the potential distribution is defined by

$$\varphi=-\frac{1}{e}\int N\,dP, \tag{39}$$

in which case the initial degeneracy of the planar problem discussed in the Introduction will remain. However, in real magnetic traps, to which the above problem is the zero-order approximation, the distribution of the potential φ is usually determined by other physical factors (for example, by the requirement of ambipolar diffusion), so that (39) can be satisfied only in exceptional cases. The most interesting solutions of (38) are therefore those for which the function A does not vanish identically.

Thus, we shall suppose that A and $dA/d\varphi$ are not zero, and will begin by considering the special case of axially symmetric equilibria $\varphi=\varphi(r)$. It is readily seen that the two vector products on the left of (38) will then vanish, so that (38) will be automatically satisfied for arbitrary $A(\varphi)$. The reverse can also be demonstrated: the fact that the two vector products in (38) vanish identically means that the equipotential lines form a set of concentric circles or parallel straight lines (see the Appendix).

Turning now to the general case and substituting $\chi=\chi(\varphi)$, which ensures that the equipotential lines remain unaltered, we can readily verify that, if we take

$$\chi(\varphi)=\int^{\varphi}|A(\varphi')|^{1/2}d\varphi' \tag{40}$$

equation (38) will assume the form

$$[\nabla\chi\times\nabla\Delta\chi]=0,$$

which can be satisfied by any function $\chi(x,y)$ that is a solution of the equation

$$\Delta\chi=G(\chi), \tag{41}$$

where $G(\chi)$ is an arbitrary function of χ. We thus arrive at the conclusion that admissible distributions of the potential $\varphi(x,y)$ can be deduced from a given (nonzero) function $A(\varphi)$ and any chosen function $G(\chi)$ by solving (41) (subject to the required boundary conditions) and inverting (40). Although this procedure yields a wide class of axially nonsymmetric solutions, it removes the initial arbitrariness in the equilibrium theory without FLR, i.e., an equilibrium configuration can no longer be constructed for an arbitrary distribution of potential $\varphi(x,y)$.

5. CONCLUSION

We have used the example of planar equilibrium in a magnetic field with straight lines of force to show that the inclusion of effects due to the finite Larmor radii of ions results in a considerable reduction in the size of the class of equilibrium configurations admitted by ideal MHD theories. The microscopic mechanism underlying the effect is the drift of ions that appears in third-order drift theory, the velocity $v_{\rm LFR}$ of which is of the order of $v_{Ti}(\rho_{\rm Hi}/a)^3$, where $v_{\rm Ti}$ is the thermal velocity of ions. We have considered the problem corresponding to small values of β, but the main results remain valid even when $\beta\sim 1$ although quantitative analysis is then much more complicated.

Interpretation of the FLR effect in the language of drift theory enables us to understand qualitatively what happens when a magnetic field with curved line of force is introduced. In the latter case, we must take into account the so-called centrifugal drift v_c, which is of the order of $v_{Ti}(\rho_{\rm Hi}/\mathscr{R})$, where $\mathscr{R}$ is the radius of curvature of the line of force. We know that inclusion of this curvature removes the degeneracy of the planar problem (it ensures that the $p=$ const surfaces are determined by the constancy of the integral $\int dl/B$) and, in this sense, competes with FLR effects. The latter will play the dominant role when $v_{\rm FLR}\gtrsim v_c$, i.e.,

$$(\rho_{Hi}/a)^2\gtrsim a/\mathscr{R}. \tag{42}$$

FLR can be neglected when the reverse inequality is valid.

The inequality given by (42) is satisfied in many modern experiments with open traps. The corresponding equilibria will be examined in a future paper.

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APPENDIX

Suppose that the following equations are satisfied simultaneously:

$$[\nabla\varphi\times\nabla\Delta\varphi]=0, \tag{A1}$$

$$[\nabla\varphi\times\nabla(\nabla\varphi)^2]=0. \tag{A2}$$

It is readily seen that (A1) and (A2) imply that the level lines of the functions φ,$\Delta\varphi$ and $(\nabla\varphi)^2$ are identical or, in other words, $\Delta\varphi$ and $(\Delta\varphi)^2$ depend on the coordinates x, y through the function φ:

$$\Delta\varphi=G(\varphi), \tag{A3}$$

$$(\nabla\varphi)^2=H(\varphi). \tag{A4}$$

We shall now use the following expression for the radius of curvature of the $\mathscr{R}$-line defined by $\varphi(x,y)=$ const (see, for example, Ref. 13)

$$\mathscr{R}^{-1}=|\nabla\varphi|^{-3}(\tfrac{1}{2}\nabla\varphi\nabla(\nabla\varphi)^2-(\nabla\varphi)^2\Delta\varphi). \tag{A5}$$

Together with (A3) and (A4), this means that $\mathscr{R}$ is also a function of φ alone, i.e., it is a constant along an equipotential. The equipotential lines are therefore circles. Since $|\nabla\varphi|^{-1}$ is proportional to the distance between neighboring equipotentials, and, by virtue of (A4), it is also constant along the $\varphi = \text{const}$ line, we may conclude that the equipotentials form a family of concentric circles (which degenerates to a set of parallel lines in the limit as $\mathscr{R} \to \infty$).

[1)]The orders of magnitude of the different terms in (10) are checked in the course of this derivation.

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