Modulation instability in a relativistic isotropic plasma

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Modulation instability is observed in waves heretofore considered to be stable. The excitation growth rates for coupled high- and low-frequency oscillations are found for various polarizations of the secondary waves. When $(ecE/\omega_p) < T$, longitudinal waves perpendicular to the pump are excited because of the relativistic dependence of the electron momentum on the velocity. At temperatures $T \sim mc^2$ and higher, the electron-ion modulation instability enters as a correction.

1. INTRODUCTION

Modulation instability has been well investigated in the nonrelativistic kinetic theory (see e.g., Refs. 1–3). Modulation instability in a collisionless plasma can be accompanied by excitation of quasistatic magnetic fields. Observation of this fact[4,5] prompted a more thorough study of modulation instability. Its study was also stimulated by experiments in which strong magnetic fields were excited in a plasma by intense electromagnetic radiation.[6,7]

The cited theoretical studies started out with nonrelativistic kinetic equations for the collisionless plasma. These equations do not combine with the Lorentz-invariant field equations to form a self-consistent system. Even at nonrelativistic plasma temperatures, certain physical processes can be investigated only on the basis of a self-consistent Lorentz-invariant theory.[8,9] Qualitative changes in the effects predicted by nonrelativistic theory, as well as new quantitative results, can also be expected from a study of modulation instability, since the electromagnetic-field growth rates calculated in Refs. 4, 5, and 10 contain the speed of light in the denominator.

A nonresonant aperiodic growth of low-frequency magnetic fields was obtained via a relativistic approach in Refs. 11–13. No account was taken in these references, however, of the connection between the excited low-frequency and high-frequency fields, so that the validity of the main results is limited.

Modulation instability due to the relativistic dependence of the electron velocity on its momentum was observed[14] in a cold electron plasma. A consistent kinetic theory of this phenomenon was developed in Ref. 15. At relativistic temperatures, the relativistic mechanism is predominant. At nonrelativistic temperatures this effect leads only to small growth-rate corrections to allow for the electron-ion mechanisms. If it is recognized, however, that the range of the wave-vectors k_0 for which the electron-ion mechanism comes into play is quite narrow, $k_0^2 r_D^2 < v_E^2/v_T^2$ ($v_E \approx eE_0/m\omega_0$) (Ref.2), it becomes of interest to study the influence of the relativistic instability mechanism outside the indicated range of wave vectors at nonrelativistic and relativistic temperatures. We solve therefore in the present paper the kinetic equations of a collisionless plasma in the form of a nonlinear stationary longitudinal wave at arbitrary relativistic temperatures and show that periodic modulation instability of such waves takes place at $k_0^2 r_D^2 > v_E^2/v_T^2$.

The resonant high-frequency and stimulated low-frequency transverse and longitudinal waves were investigated in a uniform self-consistent pump field. The new result here was that when longitudinal secondary waves were excited in a direction perpendicular to the pump field $\mathbf{E}_0$ the contribution to the increment on account of the relativistic dependence of the electron momentum on its velocity at $v_E < T/mc$ turned out to be predominant. The nonlinear connection between the high-frequency waves and the low-frequency pulsations of the magnetic field becomes weaker in this case.

We shall show below that coupled high- and low-frequency secondary waves are excited in the pump-field direcion ($k \| E_0$). At sufficiently small k ($k \sim \omega_p v_E v_T/c^3$) the growth rate γ of such waves is of the order of $\omega_p v_E^2 v_T/c^3$ and is c/v_E times larger than the growth rate of the low-frequency transverse field, if no account is taken of the growth rate of the connection of this field with the high-frequency transverse field.

Generation of secondary transverse high-frequency fields makes also possible transverse modulation of the primary-wave field.

When the temperature is raised to $T \sim mc^2$ and higher, the modulation-instability growth rate is of the order of $\omega_p v_E^2 \alpha^{5/2}/c^2$ ($\alpha = mc^2/T$) regardless of the secondary-wave polarization. The electron-ion mechanism is a correction in this case, and the quasistatic magnetic fields excited during the development of the modulation instability have no effect on the nonlinear dispersion relations.

2. NONLINEAR STATIONARY WAVE IN A RELATIVISTIC PLASMA

A solution in the form of nonlinear stationary waves was obtained for the equations of a cold relativistic plasma in Ref. 16, and for arbitrary temperatures in Ref. 17. However, the nonlinear terms omitted in Ref. 17 from the expansion in the wave field are of the same order as the included first nonlinear correction. Both the dispersion equation and the nonlinear frequency shift must therefore be revised.

To obtain the solution of the relativistic collisionless kinetic equation and of Maxwell's equations in the form of a longitudinal stationary wave, the distribution function must be expressed in terms of an integral of motion in the field

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$\mathbf{E}=\{0,\ 0,\ E_0(t-k_0z/\omega_0)\}$ and substituted in Maxwell's equations. The resultant ordinary differential equation can be easily solved up to terms cubic in the amplitude E_0, by using, e.g., the Krylov-Bogolyubov method. Assuming that the electron distribution function coincides in the limit as $E_0\to 0$ with the relativistic Maxwellian distribution, we get

$$E_0(\tau)=E_0\sin(\omega_0\tau+\varphi_0)+\frac{M}{3\omega_0}E_0^2\sin 2(\omega_0\tau+\varphi_0)-\frac{3}{32\omega_0^2}\left(N-\frac{2}{3}M^2\right)E_0^3\sin 3(\omega_0\tau+\varphi_0),\qquad(1)$$

where

$$\omega_0^2=\omega_l^2\left[1-\left(\frac{5}{6}M^2+\frac{3}{4}N\right)\frac{E_0^2}{\omega_l^2}\right],\qquad(2)$$

ω_l is the frequency of the Langmuir oscillations, and $\tau=t-k_0z/\omega_0$. We have introduced also the notation $p_0^2=p^2+m^2c^2$,

$$\begin{pmatrix}M\\N\end{pmatrix}=-\frac{2\pi e^3}{m\omega_l^2}\int\begin{pmatrix}B\\2A/3mc\end{pmatrix}\frac{\partial f_M/\partial p_z}{1-k_0v_z/\omega_0}d\mathbf{p}$$

$$A=\frac{3p_0B^2}{2p_{0\perp}^2}\left(p_z-\frac{p_0k_0c}{\omega_0}\right),\qquad(3)$$

$$B=\frac{mcp_{0\perp}^2}{p_0^3(1-k_0v_z/\omega_0)^2};$$

$p_{0\perp}^2=p_0^2-p_z^2$, and f_M is the relativistic Maxwellian distribution.

At nonrelativistic temperatures ($\alpha\gg 1$) the nonlinear frequency shift is determined by the relativistic dependence of the electron pulse on the velocity and by "4-wave interaction" (Ref. 18) in accordance with the equation

$$\omega_0^2=\omega_l^2\left[1-\frac{3}{8}\left(\frac{v_E}{c}\right)^2+\frac{15}{2}\left(\frac{k_0v_T}{\omega_l}\right)^2\left(\frac{k_0v_E}{\omega_l}\right)^2\right].\qquad(4)$$

For an ultrarelativistic plasma this result takes the form

$$\omega_0^2=\omega_l^2\left[1-\frac{3}{40}\left(\frac{v_E}{c}\right)^2\alpha^2-\frac{24}{175}\left(\frac{k_0v_E}{\omega_l}\right)^2\alpha^2\right]\qquad(5)$$

and the frequency shift is proportional to α^2.

3. MODULATION INSTABILITY OF LONGITUDINAL OSCILLATIONS WITH FINITE WAVELENGTH: $(k_0r_D)^2>v_E^2m^2/\langle p_0\rangle^2$

In an isotropic nonmagnetized plasma, and in the approximation quadratic in E_0 used here, modulation instability of a Langmuir wave is due to decay of the second harmonic of the field $E_0(\tau)$ into two normal modes:

$$2\omega_0=\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2);\quad 2\mathbf{k}_0=\mathbf{k}_1+\mathbf{k}_2.\qquad(6)$$

For finite k_0 modulation instability is possible only with respect to excitation of longitudinal modes. In this case $|\mathbf{k}_0-\mathbf{k}_{1,2}|/|\mathbf{k}_0|$ turn out to be of the order of the nonlinearity parameter $m^2v_E^2/\langle p_0^2\rangle$.

In the investigation of the modulation instability we take the initial wave field to be $\mathbf{E}_0(\tau)$ [Eq. (1)], and assume that the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel to the wave vector $\mathbf{k}_0$ of this wave. Representing the secondary waves of the perturbations in the form

$$\mathbf{E}=\mathrm{Re}\sum_{n=-\infty}^{\infty}\exp[i(\omega+n\omega_0)t-i(k+nk_0)z]\mathbf{E}(n),\qquad(7)$$

where $\mathbf{k}_{1,2}=\mathbf{k}_0\pm\mathbf{k}$ and $|\omega|\ll\omega_0$, we linearize the equations for the distribution function and for the field with respect to the perturbations E and seek solutions accurate to $\sim E_0^2$ for the equations obtained in this manner. According to conditions (6), the harmonics $n=\pm 1$ are resonant while the remaining harmonics are due to interaction of the fields $E_0(\tau)$ and $E(\pm 1)$ and are induced. We arrive thus at a linear homogeneous system of equations in the resonant harmonics:

$$(\varepsilon(\omega\pm\omega_0;\ k\pm k_0)+P(\pm 1))E(\pm 1)+Q(\pm 1)E(\mp 1)=0.\qquad(8)$$

The coefficients of this system are defined as

$$P(\pm 1)=-\frac{v_E^2}{4c^2}\left\{2\int\frac{A_2L\,d\mathbf{p}}{\Delta_{\pm 1}^2}-2kc\int\frac{B^2L\,d\mathbf{p}}{\Delta_{\pm 1}\Delta_0\Delta_{\pm 2}}+\varepsilon^{-1}(\omega\pm 2\omega_0;\ k\pm 2k_0)\times\left[\int\frac{BL\,d\mathbf{p}}{\Delta_{\pm 1}\Delta_{\pm 2}}\right]^2+\varepsilon^{-1}(\omega,k)\left[\int\frac{BL\,d\mathbf{p}}{\Delta_0\Delta_{\pm 1}}\right]^2\right\};$$

$$Q(\pm 1)=\frac{v_E^2}{4c^2}\left\{\int\frac{A_1L\,d\mathbf{p}}{\Delta_1\Delta_{-1}}-kc\int\frac{B^2L\,d\mathbf{p}}{\Delta_0\Delta_1\Delta_{-1}}+\varepsilon^{-1}(\omega,k)\int\frac{BL\,d\mathbf{p}}{\Delta_0\Delta_1}\int\frac{BL\,d\mathbf{p}}{\Delta_0\Delta_{-1}}\right\},\qquad(9)$$

where we have introduced for brevity the notation

$$L=4\pi e^2c(1-k_0v_z/\omega_0)\partial f_M/\partial p_z,$$
$$\Delta_n=\omega+n\omega_0-(k+nk_0)v_z+io,$$
$$A_1=A-mcMB/3e,\quad A_2=A+mcMB/e.$$

The condition that the system (8) be self-consistent leads to the nonlinear dispersion equation

$$[\varepsilon(\omega+\omega_0;\ \mathbf{k}+\mathbf{k}_0)-P(+1)][\varepsilon(\omega-\omega_0;\ \mathbf{k}-\mathbf{k}_0)-P(-1)]=Q(+1)Q(-1).\qquad(10)$$

At nonrelativistic temperatures, the coefficients (9) have the same structure and the same physical meaning as the obtained frequency shift (4):

$$P(\pm 1)={}^3/_4(v_E^2/c^2-14(k_0v_E/\omega_p)^2(k_0v_T/\omega_p)^2),$$
$$Q(\pm 1)={}^3/_8(v_E^2/c^2-8(k_0v_E/\omega_p)^2(k_0v_T/\omega_p)^2).\qquad(11)$$

The nonlinear dispersion equation leads therefore to

$$\mathrm{Re}\,\omega=3kv_T(k_0v_T/\omega_p),$$
$$\mathrm{Im}\,\omega=[\gamma_{max}-(3k^2v_T^2/2\omega_p-\gamma_{max})^2]^{1/2},\qquad(12)$$

where

$$\gamma_{max}=\frac{3\omega_p}{16}\frac{v_E^2}{c^2}\left[1-\frac{8c^2}{v_T^2}(k_0v_T/\omega_p)^4\right].\qquad(13)$$

The ion contribution, which is essential as $k_0 \to 0$, was left out of Eqs. (11) – (13). For finite k_0 ($k_0 v_T/\omega_0 \gg v_E/c$) this approximation is legitimate in the region

$$k_0 v_T/\omega_0 \gg 0.54 (m\alpha/m_i)^{1/2}. \tag{14}$$

For instability to set in, $\gamma_{\max}$ must be positive, hence

$$k_0 v_T/\omega_0 < 0.6\alpha^{-1/4}. \tag{15}$$

Both inequalities (14) and (15) are satisfied at not too low electron temperatures

$$1/\alpha \gg (m/m_i)^{2/3}. \tag{16}$$

For helium and neon plasmas these temperatures are $T \gg mc^2/300$ and $T \gg mc^2/900$, respectively.

The stability conditions (14)–(16) do not depoend on E_0. Instability occurs therefore also at small v_E^2/v_T^2, such that

$$(k_0 r_D)^2 > v_E^2/v_T^2,$$

i.e., in the region where it was assumed to be stable in accordance ith the nonrelativistic electron-ion mechanism. The lower limit of v_E is set by the condition that the threshold connected with the Landau damping must be exceeded. For superluminal waves $\omega_0/k_0 c \gtrsim 1$, however, there is no such limitation.

At relativistic temperatures we get for the coefficients (9) of the dispersion equation in the long-wave approximation

$$P(\pm 1) = \frac{\alpha^2 v_E^2}{4c^2}\left(\frac{16}{15} + \frac{62}{35}\frac{k_0^2 c^2}{\omega_0^2}\right),$$
$$Q(\pm 1) = -\frac{\alpha^2 v_E^2}{4c^2}\left(\frac{23}{30} + \frac{214}{175}\frac{k_0^2 c^2}{\omega_0^2}\right). \tag{17}$$

From the dispersion equation (10) we obtain in this case

$$\mathrm{Re}\,\omega = (3kc/5)(k_0 c/\tilde{\omega}_p),$$
$$\mathrm{Im}\,\omega = [\gamma_{max}^2 - (3k^2c^2/5\tilde{\omega}_p^2 - \gamma_{max})^2]^{1/2}, \tag{18}$$

where

$$\gamma_{max} = \frac{\alpha^2 v_E^2 \tilde{\omega}_p}{8c^2}\left(\frac{23}{30} + \frac{214}{175}\frac{k_0^2 c^2}{\omega_0^2}\right). \tag{19}$$

The ion contribution at relativistic temperatures can be neglected under the condition

$$k_0^2 c^2/\omega_0^2 \gg T/m_i c^2, \tag{20}$$

which is equivalent to the inequality $\omega/k \gg v_s$ and is meaningful at $v_s \ll c$.

Propagation of a Langmuir wave of finite amplitude and of frequency ω_0 in a plasma is accompanied in all cases by excitation of two Langmuir waves at frequencies $\omega_0 \pm \omega$. In this case the primary wave increases exponentially during the modulation, both in space (with a characteristic scale $2\pi/k$) and in time (with scale $2\pi/\omega$).

4. MODULATION INSTABILITY IN A UNIFORM SELF-CONSISTENT PUMP FIELD

In the limit as $k_0 \to 0$ the nonlinear frequency shift (2) is purely relativistic. The field polarization $E_0(\tau)$ becomes indeterminate and both longitudinal and tranverse secondary natural plasma oscillations can be excited in the course of the modulation instability.

In a coordinate system in which $\mathbf{k} = \{k, 0, 0\}$, $\mathbf{E}_0 = \{E_0 \times \cos\psi, E_0 \sin\psi, 0\}$ the equations (7) for the resonant perturbations $E(\pm 1)$ [Eq. (7)] break up into two independent sets:

$$\begin{cases} [\Lambda_{33}(+1) + P_{33}(+1)]E_3(+1) + Q_{33}(+1)E_3(-1) = 0, \\ [\Lambda_{33}(-1) + P_{33}(-1)]E_3(-1) + Q_{33}(-1)E_3(+1) = 0 \end{cases} \tag{21}$$

and in addition

$$\hat{\Pi}\begin{pmatrix} E_1(+1) \\ E_1(-1) \\ E_2(+1) \\ E_2(-1) \end{pmatrix} = 0, \tag{22}$$

where the matrix $\hat{\Pi}$ is of the form

$$\hat{\Pi} = \begin{pmatrix} \Lambda_{11}(1) + P_{11}(1) & Q_{11}(1) & P_{12}(1) & Q_{12}(1) \\ Q_{11}(-1) & \Lambda_{11}(-1) + P_{11}(-1) & Q_{12}(-1) & P_{12}(-1) \\ P_{21}(1) & Q_{21}(1) & \Lambda_{22}(1) + P_{22}(1) & Q_{22}(1) \\ Q_{21}(-1) & P_{21}(-1) & Q_{22}(-1) & \Lambda_{22}(-1) + P_{22}(-1) \end{pmatrix}$$

and the matrix elements are designated as

$$\Lambda_{ij}(n) = \varepsilon_{ij}(\omega + n\omega_0, k) - (\delta_{ij} - k_i k_j/k^2)k^2c^2/(\omega + n\omega_0)^2,$$
$$P_{ij}(\pm 1) = {}^3/_2 \overset{(3)}{\varkappa}_{ij}(\pm 1, 1, -1) - \overset{(2)}{\varkappa}_{ik}(0, \pm 1)\overset{(2)}{\varkappa}_{kj}(\pm 1, \mp 1)/\Lambda_{kk}(0)$$
$$- \varkappa_{ik}(\pm 2, \mp 1)\overset{(2)}{\varkappa}_{kj}(\pm 1, \pm 1)/\Lambda_{kk}(\pm 2),$$
$$Q_{ij}(\pm 1) = {}^3/_4 \varkappa_{ij}(\mp 1, \pm 1; \pm 1) \tag{23}$$
$$- \varkappa_{ik}(0, \pm 1)\overset{(2)}{\varkappa}_{kj}(\mp 1, \pm 1)/\Lambda_{kk}(0).$$

The equations

$$\overset{(3)}{\varkappa}_{ij}(n, m, p) = \overset{(3)}{\varkappa}_{ijkl}(\omega + n\omega_0, k; m\omega_0, 0; p\omega_0, 0)E_{0k}E_{0l},$$

$$\varkappa_{ij}(n, m) = \varkappa_{ijk}(\omega + n\omega_0, k; m\omega_0, 0)E_{0k}$$

represent the dependences of the coefficients of Eqs. (21) and (22) on the tensor nonlinear susceptibilities $\varkappa_{ijk}^{(2)}$ and $\varkappa_{ijkl}^{(3)}$ (Ref. 18), in which allowance is made for the relativitic dependence of the electron momentum on its velocity.

The interaction of the field E_0 with the fields $E(\pm 1)$ can excite also low-frequency oscillations of frequency ω and wave vector $\mathbf{k}$:

$$E_i(0) = -\sum_{\pm} \varkappa_{ij}^{(2)} (\pm 1, \mp 1) E_j(\pm 1)/\Lambda_{ii}(0). \quad (24)$$

If these oscillations are transverse, a quasistatic magnetic field $\mathbf{B}(0) = (c/\omega_0)\mathbf{k}\times\mathbf{E}(0)$ is also excited. This always occurs if $\mathbf{k}\times[\mathbf{E}(\pm 1)\times\mathbf{E}_0]\neq 0$. For a transverse high-frequency field polarized along the 3-axis, we have according to (21) the dispersion equation

$$[\Lambda_{33}(1)+P_{33}(1)]\,[\Lambda_{33}(-1)+P_{33}(-1)] = Q_{33}(1)Q_{33}(-1), \quad (25)$$

from which it follows that instability does indeed take place and envelops with growth rates $\gamma \lesssim \gamma_{\max}$, where

$$\gamma_{max} = \begin{cases} \omega_p v_E^2 v_T/4\sqrt{2}c^3, & T \ll mc^2 \\ \tilde{\omega}_p v_E^2 \alpha^2/40c^2, & T \gg mc^2 \end{cases}. \quad (26)$$

This is accompanied by excitation of transverse high-frequency fields with wave vectors k such that $\hat{k} < |\mathbf{k}| \lesssim \tilde{k}$, where

$$\hat{k} = \begin{cases} \sim \omega_p v_E^2 v_T^2/c^5, & T \ll mc^2 \\ 0{,}013\tilde{\omega}_p v_E^2 \alpha^2/c^3, & T \gg mc^2 \end{cases},$$

$$\tilde{k} = \begin{cases} \omega_p v_E v_T/\sqrt{2}c^3, & T \ll mc^2 \\ \tilde{\omega}_p v_E \alpha/2\sqrt{6}c^2, & T \gg mc^2 \end{cases}. \quad (27)$$

Since the values $\gamma \sim \gamma_{\max}$ are reached at $k \lesssim \tilde{k}$, the constraint on k_0 is in this case actually the inequality

$$k_0 r_D \ll \tilde{k} r_D \sim \begin{cases} v_E v_T^2/c^3, & T \ll mc^2 \\ v_E \alpha/c, & T \gg mc^2 \end{cases}.$$

The quasistatic magnetic field is excited with the same growth rate (26). The coupling coefficient d of the two fields $|\mathbf{B}(0)| = d\cos\psi E_3(\pm 1)$ is of the order of

$$d \sim \begin{cases} v_E^2 v_T^2/c^4, & T \ll mc^2 \\ v_E^2 \alpha^2/c^2, & T \gg mc^2 \end{cases}. \quad (28)$$

The dependence of $\tilde{k}$ and $\gamma_{\max}$ on the temperature is expressed in both cases in terms of modified Bessel functions of imaginary argument and is shown in Figs. 1 and 2. The ion motion plays a negligible role in the development of this instability.

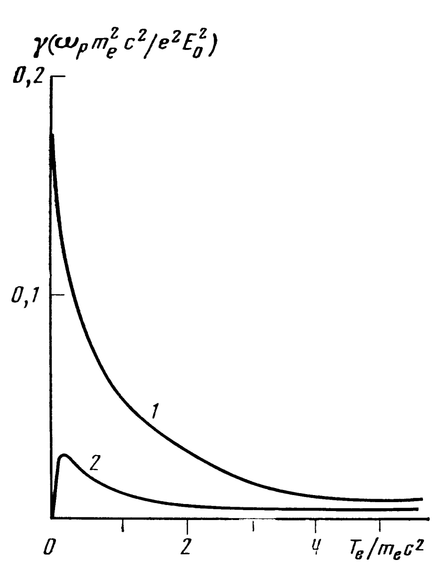

FIG. 1. Plots of $\gamma_{\max}(T_e)$ at $\mathbf{E}(\pm 1)\|\mathbf{E}_0$ (curve 1) and $\mathbf{E}(\pm 1)\perp\mathbf{E}_0$ (curve 2).

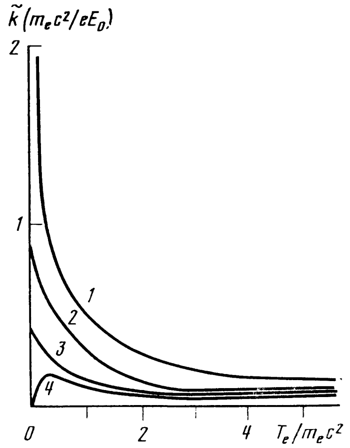

FIG. 2. Plots of $\tilde{k}(T_e)$ in the following cases: 1—at $\mathbf{E}(\pm 1)\|\mathbf{E}_0\|\mathbf{k}$, 2—at $\mathbf{E}(\pm 1)\|\mathbf{E}_0\perp\mathbf{k}$, 3—at $\mathbf{E}(\pm 1)\perp\mathbf{E}_0$, $\mathbf{E}(\pm 1)\|\mathbf{k}$ and 4—at $\mathbf{E}(\pm 1)\perp\mathbf{E}_0$, $\mathbf{E}(\pm 1)\perp\mathbf{k}$.

Thus, in all cases when secondary transverse high-frequency field are excited the growth rate of the low-frequency transverse pulsations is determined by Eqs. (26) and is c/v_E times larger than the value obtained in Ref. 13 as a result of the action of an external field E_0 on the dispersion properties of such pulsations. The influence of the high-frequency field can be neglected either if the high- and low-frequency fields become decoupled as, e.g., in an electron-positron plasma, or if the resonance conditions (6) for the excitation of high-frequency fields are not satisfied, as is the case, for example, at $(k_0 r_D)^2 > m^2 v_E^2/\langle p_0^2\rangle$.

Excitation of longitudinally-tranverse waves in the plane of the vectors $\mathbf{E}_0$ and $\mathbf{k}$ is described by the set of equations (22) and by the nonlinear dispersion equation det $\hat{\Pi} = 0$. We consider first nonrelativistic temperatures. At $\psi = 0$ the system (22) describes independently longitudinal $E_1(\pm 1)$ and transverse $E_2(\pm 1)$ fields, and the system of equations for $E_2(\pm 1)$ does not differ from the system investigated above for transverse fields $E_3(\pm 1)$. The longitudinal field $E_1(\pm 1)$ field is excited if $|k| < \sqrt{2/3}\,\omega_p v_E/v_T^2$. The dependence of the growth rate $\gamma = \gamma(k)$ for certain values of the parameters α, m/m_i, and v_E/v_T is shown in Fig. 3. The same figure shows for comparison a plot of $\gamma(k)$ without allowance for the relativistic dependence of the electron momentum on velocity. From the equation for the maximum growth rate of the excitation of the longitudinal field along E_0 we have

$$\gamma_{max} = \omega_p [2mv_E^2/3m_i v_T^2 + 9v_E^4/256c^4]^{1/2} \quad (29)$$

and it is seen from Fig. 3 that the relativistic corrections are small.

Longitudinally-transverse waves can be excited at an angle $\psi(0 < \psi < \pi/2)$ to $\mathbf{E}_0$. Th nonlinear dispersion relation for this case is

$$1 + P\sin^2\psi[(\varepsilon^{tr}(\omega+\omega_0, k) - k^2c^2/(\omega+\omega_0)^2)^{-1} + (\varepsilon^{tr}(\omega-\omega_0, k)$$

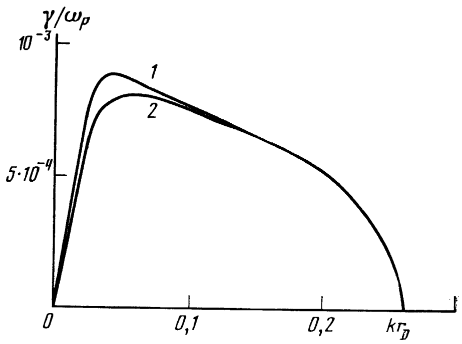

FIG. 3. Plot of $\gamma(k)$ for the parameter values: 1—$v_E^2/v_T^2 = 0, 1$, $T_e/m_ec^2 = 1/60$, $m_i/m_e = 10^5$; 2—$c \to \infty$, but with the same m_i, T_e, and v_E.

$$-k^2c^2/(\omega-\omega_0)^2)^{-1}]$$

$$+P\cos^2\psi[\varepsilon_l^{-1}(\omega+\omega_0, k)+\varepsilon_l^{-1}(\omega-\omega_0, k)]=0, \quad (30)$$

where $P = v_E^2\omega_s^2/(\omega_s^2-\omega^2)v_T^2$, and ω_s is the ion-sound frequency.

It follows from (30) that in the vicinity of the wave vectors

$$k^2 \sim \omega_p^2 v_E |\cos\psi| (m/m_i)^{1/2}/v_T^3$$

there is excited mainly a longitudinal field $E_1(+1) \sim E_1(-1)$ with a growth rate

$$i\omega=\gamma\approx\omega_p(mv_E^2\cos^2\psi/m_iv_T^2)^{1/2}. \quad (31)$$

The transverse component is in this case $E_2(\pm 1) \sim (v_T^2/c^2)E_1(\pm 1)$. At

$$k^2 \sim \omega_p^2 v_E |\sin\psi| (m/m_i)^{1/2}/c^3,$$

conversely, the field is mainly transverse: $E_2(+1) \sim E_2(-1)$. The longitudinal component is $E_1(\pm 1) \sim (v_T^2/c^2)E_2(\pm 1)$. The excitation growth rate

$$i\omega=\gamma\approx\omega_p(mv_E^2\sin^2\psi/m_ic^2)^{1/2} \quad (32)$$

coincides with the result of Ref. 19. The possibility of exciting low-frequency magnetic pulsations with a growth rate (31) was demonstrated in Ref. 5.

At $\psi = \pi/2$ the set of equations (22) again breaks up into two independent sets, one describing the transverse field $E_2(\pm 1)$ and the other the longitudinal field $E_1(\pm 1)$. The transverse field is excited in this direction in accordance with the dispersion relation (30) at $\psi = \pi/2$ and is not accompanied by generation of a quasistatic magnetic field.

The nonrelativistic mechanism connected with the ion motion makes no contribution to the dispersion equation of the longitudinal field $E_1(\pm 1)$. A consistent allowance for the relativistic motion of the electron may therefore be important even at nonrelativistic temperatures. In particular, the relativistic terms make the main contribution to the dispersion relation in the region $v_E/c \ll T/mc^2 \ll 1$. The contribution due to generation of a quasistatic magnetic field can be neglected.

Instability develops at wave vector values

$$k \leqslant \tilde{k} = \omega_p v_E/\sqrt{6}c^2 \quad (33)$$

and has a growth rate $\gamma(k) \lesssim \gamma_{\max}$, where

$$\gamma_{max}=\omega_p v_E^2 v_T/4\sqrt{2}c^3. \quad (34)$$

In this case $B(0) \sim (v_E/c)^2 E_1(\pm 1)$.

In the region $T/mc^2 \ll v_E/c$, the contributions due to generation of the magnetic field and due to the relativistic velocity dependence of the electron momentum turn out to be of the same order.[20] Instability sets in at

$$k \leqslant \tilde{k} = \omega_p v_E/\sqrt{24} v_T c \quad (35)$$

and grows at a rate

$$\gamma(k) \leqslant \gamma_{max} = \omega_p v_E^2/16c^2. \quad (36)$$

The latter instability was first considered in Ref. 10 using nonrelativistic particle dynamics. The values of $\tilde{k}$ and $\gamma_{\max}$ obtained in these studies are $\sqrt{2}$ and 2 times larger than (35) and (36).

We have so far considered nonrelativistic plasma temperatures. At relativistic temperatures $T \gtrsim mc^2$ the excitations of $E_1(\pm 1)$ and $E_2(\pm 1)$ at $\psi = 0$ and 2π are also described independently by Eqs. (22). The growth rates and wave vectors at which a longitudinal field is excited in the $\psi = 0$ direction are given by the inequalities ($\alpha \ll 1$)

$$k \leqslant \tilde{k}_1 = \sqrt{\frac{23}{36}}\frac{\alpha\tilde{\omega}_p v_E}{c^2}, \quad \gamma \leqslant \gamma_{max} = \frac{23}{240}\tilde{\omega}_p\frac{v_E^2\alpha^2}{c^2}. \quad (37)$$

For a transverse field we have under the same conditions

$$k \leqslant \tilde{k}_2 = \frac{1}{2\sqrt{6}}\tilde{\omega}_p\frac{\alpha v_E}{c^2}, \quad \gamma \leqslant \gamma_{max} = \frac{1}{40}\tilde{\omega}_p\frac{v_E^2\alpha^2}{c^2}. \quad (38)$$

In the $\psi = \pi/2$ direction, on the contrary, the maximum growth rates for longitudinal and transverse fields are given respectively by (38) and (37). The temperature dependences of the growth rates and of the characteristic wave vectors are shown in Figs. 1 and 2.

At relativistic temperatures, longitudinal and transverse perturbations are simultaneously excited in the directions $\psi \neq 0, \pi/2$, with $E_1(\pm 1) \sim E_2(\pm 2)$. This can be easily deduced by recognizing that in this case the dependences of the longitudinal and transverse waves on the wave vector differ only by a coefficient of order unity. The maximum value of the growth rate is in this case $\gamma_{\max} \sim \tilde{\omega}_p \alpha^2 v_E^2/c^2$, the modulation scale is $2\pi/\hat{k} \sim 2\pi c^2/\alpha\hat{\omega}_p v_E$, and the quasistatic magnetic field is $B(0) \lesssim E(\pm 1) \times \alpha^2 v_E^2/c^2$.

An important feature of relativistic plasma temperatures is that the contribution of the quasistatic magnetic field to the nonlinear dispersion relation is never decisive, and leads only to higher-order corrections in the expansion in power of E_0. When terms due to ion motion are taken into account in this relation, the growth rate is increased by not more than 30% at $\mathbf{E}(\pm 1) \| \mathbf{E}_0$. The ion component plays no role at all in the excitation of the field $\mathbf{E} \perp E_0$.

The decisive role in the development of modulation instability at relativistic temperatures is thus played by the relativistic electronic mechanism due to the nonlinear dependence of the electron momentum on its velocity.

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