Stochastic spin dynamics of superfluid ^{3}He

V. I. Kesaev and A. I. Ugulava

Tbilisi State University

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The magnetization motion induced in superfluid ^{3}He by a periodic sequence of radio-frequency pulses is investigated. It is shown that, under certain conditions and over time periods shorter than the relaxation times, the magnetization component along the direction of the constant field and the phase of the spin precession vary in a stochastic manner; the steady-state distribution function for this component and the correlation function for the phase are found. It is also found that in this case the magnitude of the spin and the angle connected with the rotation of the order parameter execute small oscillations about their equilibrium values.

1. In the present paper we consider the motion of the magnetization of superfluid ^{3}He in the presence of a pair of magnetic fields: a strong constant field (the adiabatic approximation[1]) and the alternating field, directed perpendicularly to the constant field, of a periodic series of radio-frequency (RF) pulses. Here we shall assume that the repeated supply of pulses ceases before the relaxation properties begin to manifest themselves. The preference of a periodic series of RF pulses to the monochromatic field used in the ordinary NMR method is due to the fact that, under conditions of a sufficiently large dynamical precession-frequency shift, the magnetization motion in a monochromatic field turns out to be unstable when the relaxation is neglected. In the case of a periodic series of RF pulses the magnetization motion can become stochastically unstable,[2,3] so that the average magnetization manages to assume its steady-state value before the appearance of the relaxation mechanism.

The stochastic magnetization motion induced in superfluid ^{3}He by a periodic series of RF pulses has been investigated before[3] for the mathematically simpler nonresonance[1] case, which is realized by abruptly changing the strong constant field by an amount of the order of the field strength itself prior to the supply of the periodic series of RF pulses. In the present paper we investigate the stochastic magnetization motion regime for the so-called resonance[1] case, which does not presuppose the realization of the indicated constant-field jump.

2. Let us consider a situation similar to the one proposed by Fomin[1] for the deflection of the magnetization in pulsed NMR experiments, with the only difference that in our case the variable field has the form

$$\mathbf{H}_1(t)=\mathbf{H}_1\sum_{n=-\infty}^{+\infty} f(t/T-n)\cos(\zeta\omega_L t), \quad (1)$$

where $\omega_L=gH_0$ is the Larmor frequency; H_0 is the strength of the constant field, which is directed in the direction opposite to that of the z axis; H_1 and $\zeta\omega_L$ are the amplitude and frequency of the variable magnetic field, oriented along the y axis, at the peak of the pulse; ζ is a numerical coefficient of the order of unity; $f(t)$ is the form function of the individual pulses; and T is the pulse repetition period. Since the number of harmonics in the spectral expansion $\Sigma_n f(t/T-n)$ is, in practice, limited by the interval $1/\tau_p$, where τ_p is the pulse duration, we can, by imposing on the pulse duration the condition $1/\tau_p \ll \omega_L$, ensure that these harmonics vary at a rate that is low compared to ω_L. Then, following Ref. 1, we can derive the system of equations

$$\begin{aligned}\dot S&=-\omega_L\frac{\partial V}{\partial\Phi},\\ \dot\Phi&=\omega_L(S-1)+\omega_L\frac{\partial V}{\partial S}-\frac{\omega_L P}{[-P(2S+P)]^{1/2}}h_1(t)\sin\xi,\\ \dot P&=-\omega_L[-P(2S+P)]^{1/2}h_1(t)\cos\xi,\\ \dot\xi&=\Delta+\omega_L\frac{\partial V}{\partial P}-\frac{\omega_L(P+S)}{[-P(2S+P)]^{1/2}}h_1(t)\sin\xi.\end{aligned} \quad (2)$$

Here S is the dimensionless spin per unit volume; $P=S_z-S$, where S_z is the spin component along the z axis; $\Delta=\omega_L(\zeta-1)\ll\omega_L$;

$$h_1(t)=h_1\sum_{n=-\infty}^{+\infty} f(t/T-n),$$

where $h_1=H_1/2H_0\ll 1$; $\xi=\alpha+\zeta\omega_L t$, $\Phi=\alpha+\gamma$, $\cos\beta=P/S+1$, α, β, and γ being the Euler angles determining the orientation of the moving coordinate system (rigidly fixed to the order parameter) relative to the stationary system:

$$V_A=-\frac{\Omega_A^2}{8\omega_L^2}\left[\left(1+\frac{P}{S}\right)^2+\frac{1}{2}\left(2+\frac{P}{S}\right)^2\cos 2\Phi\right],$$

$$V_B=\frac{2\Omega_B^2}{15\omega_L^2}\left[\frac{1}{2}+\frac{P}{S}+\left(2+\frac{P}{S}\right)\cos\Phi\right]^2,$$

where Ω_A and Ω_B are the frequencies of the small longitudinal oscillations of the spin and the angle of orientation of the order parameter in the A and B phases of superfluid ^{3}He.

Further, to simplify the computations, let us assume that the pulse form function has the form of a delta function. Then, in the spirit of the foregoing, the expansion

$$\sum_{n=-\infty}^{+\infty}\delta(t/T-n)=\sum_{n=-\infty}^{+\infty}\cos n\Omega t, \quad (3)$$

where $\Omega=2\pi/T$, actually contains approximately $1/2\,\tau_p\Omega$ harmonics. For prescribed $P(t)$ and $\xi(t)$ functions, the first two equations in (2) form a closed system of equations for S and Φ. These quantities excute forced oscillations about the points $S=S_0$ and $\Phi=\Phi_0$ given by the equations $\partial V/$

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$\partial\Phi_0 = 0$ and $S_0 = 1 - \partial V/\partial S_0$. If these oscillations are sufficiently small, then the second pair of equations in (2) naturally form a closed system of equations of P and ξ. For both the A and the B phases we easily obtain the following equations from the system (2) in the vicinity of the points S_0 and Φ_0:

$$\ddot{X}_{A,B}+\omega_{A,B}^2 X_{A,B}=F_{A,B}^X(\xi,\beta;t); \quad (4)$$

$$\dot{\beta}=\omega_L h_1(t)\cos\xi, \quad (5a)$$

$$\dot{\xi}=\Delta_{A,B}+\omega_{pA\,B}\cos\beta-\omega_L h_1(t)\operatorname{ctg}\beta\sin\xi, \quad (5b)$$

where $X_{A,B} = S_{A,B} - 1$, $\Phi_{A,B}$, while

$$\Delta_A=\Delta+{}^1/_3\omega_{pA},\ \Delta_B=\Delta+{}^1/_4\omega_{pB},\ \omega_A{}^2={}^1/_4\Omega_A{}^2(1+\cos\beta)^2,$$
$$\omega_B{}^2=-{}^8/_{15}\Omega_B{}^2(1+\cos\beta)({}^1/_4+\cos\beta),$$
$$\omega_{pA}=-{}^3/_8\Omega_A{}^2/\omega_L,\ \omega_{pB}={}^{16}/_{15}\Omega_B{}^2/\omega_L, \quad (6)$$
$$F_{A,B}^{\beta}=\omega_{A,B}^2\frac{\cos\beta-1}{\sin\beta}h_1(t)\sin\xi,$$

$$F_{A,B}^{\Phi}=\omega_L\frac{1-\cos\beta}{\sin\beta}(\Delta_{A,B}h_1(t)\cos\xi+\dot{h}_1(t)\sin\xi).$$

Let us note that in this paper we investigate the B phase with a dynamical precession-frequency shift, i.e., for which the condition $-1 \ll \cos\beta \ll -\frac{1}{4}$ is fulfilled. The system (4) was derived under the assumption that $\Delta_{A,B} \sim \Omega_A, \Omega_B$.

3. The equations (5) describe the nonlinear precession of the spin under the action of the periodic series of RF pulses. They coincide with the equations investigated in Ref. 3. It is shown there that if $K \sim |\omega_1\omega_p|T^2 \gg 1$ and $|\omega_1|T \ll 1$, where $\omega_1 = gH_1\tau_p/T$, there develops in the system described by the equations (5) a statistical instability in which β and ξ become random functions of the time with known distribution $\rho_{A,B}(\beta)$ and correlation functions respectively. Using the results of Ref. 3, we can write down the kinetic equations for $\rho(\beta)$:

$$\frac{\partial\rho}{\partial t}=D\frac{\partial}{\partial\beta}\left(\frac{\partial}{\partial\beta}-\operatorname{ctg}\beta\right)\rho, \quad (7)$$

where $D = \omega_1^2 T/2$ is the coefficient of "diffusion." The function $\rho(\beta)$ should satisfy the following normalization conditions:

$$\int_{\beta^0_{A,B}}^{\pi}\rho_{A,B}(\beta)\,d\beta=1, \quad (8)$$

where $\beta^0_A = 0$ and $\beta^0_B = \arccos(-\frac{1}{4})$. Notice that Eq. (7) coincides with the kinetic equation for the polar-angle distribution function for rigid dipoles executing thermal motion in a constant longitudinal electric field, i.e., for particles executing rotational Brownian motion.[4]

On the basis of (7) we can easily show[3] that there get established over a time period $t \gg 1/6D$ the steady-state distribution functions $\rho^0_{A,B} = C_{A,B}\sin\beta$, where the constants $C_{A,B}$ are determined from (8), and are equal to: $C_A = \frac{1}{2}$, $C_B = 4/3$. The mean value of some function $f(\beta)$ in the steady state regime will then have the form

$$\overline{f(\beta)}=\int_{\beta'}^{\pi}\rho^0(\beta)f(\beta)\,d\beta. \quad (9)$$

In particular, in the A phase $\overline{\cos\beta} = 0$, and, consequently, all the spin directions in the steady state regime are equally probable. For the B phase we find that $\overline{\cos\beta} = -5/8$.

Further, to investigate the equations (4), we must know that the correlation functions of the following form for the random quantity $\xi(t)$:

$$R^{\pm}(t)=\frac{1}{2\pi}\int_0^{2\pi}\exp\{i[\xi(t)\pm\xi(t_0)]\}\,d\xi(t_0). \quad (10)$$

Following Zaslavskiĭ,[2] we first determine the correlations $R_m^{\pm} = \langle\exp\{i(\xi_m \pm \xi_0)\}\rangle$, where $\langle...\rangle$ denotes averaging in the sense of (10) and $\xi_m = \xi(t_m)$ is the value of ξ after the mth pulse, and then go over to continuous time by making the substitution $mT \to t$, $m \gg 1$. Let us, in accordance with the foregoing, find $R_1^{\pm}$. Determining with the aid of the formulas of Ref. 3 the jump $\Delta\xi$ after the first pulse:

$$\Delta\xi=\xi_1-\xi_0\approx\Delta_{A,B}T+K_0\cos\xi_0, \quad (11)$$

where $K_0 = -\frac{1}{2}\omega_1\omega_p T^2\sin\beta(t_0)$, we obtain for $R_1^{\pm}$ the expressions

$$R_1^-\approx\exp\{i\Delta_{A,B}T\}J_0(K_0),\quad R_1^+\approx-\exp\{i\Delta_{A,B}T\}J_2(K_0), \quad (12)$$

where $J_0(K_0)$ and $J_2(K_0)$ are the Bessel functions of order 0 and 2, respectively. Now, using (12), we easily find the first-order correlators:

$$\langle\cos\xi_1\cos\xi_0\rangle\approx{}^1/_2\cos\Delta_{A,B}T(J_0(K_0)-J_2(K_0)),$$
$$\langle\cos\xi_1\sin\xi_0\rangle\approx-{}^1/_2\sin\Delta_{A,B}T(J_0(K_0)+J_2(K_0)),$$
$$\langle\sin\xi_1\cos\xi_0\rangle\approx{}^1/_2\sin\Delta_{A,B}T(J_0(K_0)-J_2(K_0)), \quad (13)$$
$$\langle\sin\xi_1\sin\xi_0\rangle\approx{}^1/_2\cos\Delta_{A,B}T(J_0(K_0)+J_2(K_0)).$$

Since in (5) the stochastic regime is realized when $K \gg 1$, we must take in (13) the asymptotic forms of the Bessel functions for $|K_0| \gg 1$. Limiting ourselves to the first terms of the expansion, we can write

$$J_0(K_0)+J_2(K_0)\sim1/|K_0|^{3/2},$$
$$J_0(K_0)-J_2(K_0)\sim1/|K_0|^{1/2}+1/|K_0|^{3/2}. \quad (14)$$

Noting that the transformation law for the phase (11) coincides with the one investigated in Ref. 2, we can omit the intermediate computations for $R_m^{\pm}$, and write down the final expression for the correlators:

$$\langle\cos\xi_1\cos\xi_2\rangle\approx{}^1/_2\cos\Delta_{A,B}(t_1-t_2)$$
$$(\exp\{-|t_1-t_2|/\tau\}+\exp\{-3|t_1-t_2|/\tau\}),$$
$$\langle\cos\xi_1\sin\xi_2\rangle\approx-{}^1/_2\sin\Delta_{A,B}(t_1-t_2)\exp\{-3|t_1-t_2|/\tau\},$$
$$\langle\sin\xi_1\sin\xi_2\rangle\approx{}^1/_2\cos\Delta_{A,B}(t_1-t_2)\exp\{-3|t_1-t_2|/\tau\}, \quad (15)$$
$$\langle\sin\xi_1\cos\xi_2\rangle\approx{}^1/_2\sin\Delta_{A,B}(t_1-t_2)$$
$$(\exp\{-|t_1-t_2|/\tau\}+\exp\{-3|t_1-t_2|/\tau\}),$$

where $\tau = 2T/\ln K$, $K \approx |K_0|$.

3. Let us now show that, in the steady-state stochastic regime of motion of β and ξ, the solutions to the system (4) can remain close to the equilibrium values $S_0 = 1$, $\Phi_0 = 0$. Indeed, after the establishment of the steady state β distribution in (5), the equations (4) will assume the form

$$\ddot{X}+\omega^2 X=F^x(\xi;\ t),\tag{16}$$

where $\omega^2 = \overline{\omega^2_{A,B}}, F^X(\xi;t) = \overline{F^X_{A,B}(\xi,\beta;t)}$. Averaging over β, we find that $\omega^2_A = \Omega^2_A/3$, $\omega^2_B = \Omega^2_B/20$, while

$$\begin{aligned} \bar{F}_A{}^S &= {}^1/_{16}\pi\Omega_A{}^2 h_1(t)\sin\xi, \\ \bar{F}_B{}^S &= 3({}^4/_{15})^3\Omega_B{}^2 h_1(t)\sin\xi, \\ \bar{F}_A{}^\Phi &= {}^1/_2\pi\omega(\Delta_A h_1(t)\cos\xi+\dot{h}_1(t)\sin\xi), \\ \bar{F}_B{}^\Phi &\approx 3{,}1\omega_L(\Delta_B h_1(t)\cos\xi+\dot{h}_1(t)\sin\xi). \end{aligned}\tag{17}$$

To analyze (16), let us introduce the function $\eta(t) = \dot{X} + i\omega X$. Then, if, as the initial values, we take $X(0) = 0$ and $\dot{X}(0) = 0$, we can show that

$$\eta(t)=e^{i\omega t}\int_0^t e^{-i\omega t'}F^x(\xi;t')dt'.\tag{18}$$

Since the amplitude of the X oscillations is a slowly varying function of the time ($|\dot{A}/A| \ll \omega$),

$$\omega^2\langle(A^x)^2\rangle \approx \langle|\eta(t)|^2\rangle$$
$$=\int_0^t\int_0^t dt'\,dt''\exp\{i\omega(t''-t')\}\langle F^x(\xi;t')F^{x*}(\xi;t'')\rangle.\tag{19}$$

Substituting (15) and (17) into (19), and performing the simple integrations, we obtain for times $t \gg \tau$:

$$\langle(A^x)^2\rangle \approx \lambda^x\sum_{n,k}\sigma_{nk}{}^x\frac{\sin(n-k)\Omega t}{(n-k)\Omega},\tag{20}$$

where

$$\lambda_A{}^S=\frac{18\pi^2}{(16)^3}\left(\frac{\Omega_A}{\omega_L}\right)^2\frac{\omega_1{}^2}{\tau},\quad \lambda_B{}^S=18\left(\frac{4}{15}\right)^5\left(\frac{\Omega_B}{\omega_L}\right)^2\frac{\omega_1{}^2}{\tau},$$
$$\lambda_A{}^\Phi=\frac{3\pi^2}{32}\left(\frac{\omega_1}{\Omega_A}\right)^2\frac{1}{\tau},\quad \lambda_B{}^\Phi=\frac{5}{2}(3{,}1)^2\left(\frac{\omega_1}{\Omega_B}\right)^2\frac{1}{\tau},\tag{21}$$
$$(\sigma^S_{A,B})_{nk}=\frac{1}{(n\Omega+\Delta_{A,B}-\omega)^2+9/\tau^2}+\frac{1}{(n\Omega-\Delta_{A,B}-\omega)^2+9/\tau^2}.$$
$$(\sigma^\Phi_{A,B})_{nk}=\frac{\Delta_{A,B}(\Delta_{A,B}+n\Omega)}{(n\Omega+\Delta_{A,B}-\omega)^2+1/\tau^2}+\frac{\Delta_{A,B}(\Delta_{A,B}-n\Omega)}{(n\Omega-\Delta_{A,B}-\omega)^2+1/\tau^2}$$
$$+\frac{3(n\Omega+\Delta_{A,B})(k\Omega+\Delta_{A,B})}{(n\Omega+\Delta_{A,B}-\omega)^2+9/\tau^2}+\frac{3(n\Omega-\Delta_{A,B})(k\Omega-\Delta_{A,B})}{(n\Omega-\Delta_{A,B}-\omega)^2+9/\tau^2}.$$

The dominant contribution to the sum (20)–(21) is made by the so-called resonance (secular) terms, which increase in time. Assuming that the condition $\Delta_{A,B} = \omega$ is fulfilled, and limiting ourselves to the summation of the secular terms, we obtain

$$\langle(A^x)^2\rangle\sim\varkappa^x t,\tag{22}$$

where

$$\varkappa_A{}^S=0{,}03(\Omega_A/\omega_L)^2\omega_1{}^2T,\quad \varkappa_A{}^\Phi={}^1/_2\omega_1{}^2T,$$
$$\varkappa_B{}^S=0{,}16(\Omega_B/\omega_L)^2\omega_1{}^2T,\quad \varkappa_B{}^\Phi=1{,}9\omega_1{}^2T.$$

Notice that in (22) the time is measured from the moment when the steady state distribution is established in (5). It follows from (22) that, when we use a periodic series of RF pulses with $T\sim 10^{-3}$ sec, $\tau_p \sim 10^{-5}$ sec and $|\omega_1|T \ll 1$, since we are using the adiabatic approximation, i.e., since Ω_A, $\Omega_B \ll \omega_L$, and since the duration of the steady state regime is of the order of the duration of two or three score pulses, the $S_{A,B} - 1$ oscillations remain small even in the presence of the resonance terms in (20)–(21). In this case the Φ oscillations are, generally speaking, no longer small. For example, in the A phase we find that, for the characteristic parameter values $T\sim 2\times 10^{-3}$ sec, $\Omega_A \sim 2\times 10^5$ sec^{-1}, $\omega_L \sim 8\times 10^5$ sec^{-1}, and $|\omega_1| \sim 2\times 10^2$ sec^{-1}, the quantity $(\langle(A^\Phi)^2\rangle)^{1/2}$ is of the order of unity. But if we detune the resonance $\Delta_{A,B} = \omega$, and choose a periodic series of RF pulses with such a pulse duration that resonance denominators do not arise in the coefficients (21) of the double series (20) (this requires that we choose $1/\tau_p < \omega$), then the coefficients in the relation (22) for Φ will be much smaller:

$$\varkappa_A{}^\Phi\sim\frac{3\pi}{16}\left(\frac{\omega_1}{2\Omega_A}\right)^2\frac{\ln K_A}{\tau_p},$$
$$\varkappa_B{}^\Phi\sim\frac{20\pi}{16}\left(\frac{\omega_1}{\Omega_B}\right)^2\frac{\ln K_B}{\tau_p},$$

and, consequently, the Φ oscillations will be small oscillations. For example, for the A phase, using the parameter values given above, we obtain $(\langle(A^\Phi)^2\rangle)^{1/2} \sim 0.03$. Thus, the stationary stochastic spin state described by the distribution function $\rho^0(\beta)$ can be realized when the periodic series of RF pulses is appropriately chosen.

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