Dynamics of superfluid ^{3}He in ^{3}He–^{4}He solutions

A. É. Meĭerovich

S. I. Vavilov Institute of Physical Problems, Academy of Sciences of the USSR, Moscow

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The dynamics of the superfluid ^{3}He–^{4}He solution containing two condensates (^{3}He and ^{4}He) is investigated. Although the hydrodynamics of the system is of the three-velocity type (two superfluid and one normal velocity), all the thermodynamic and hydrodynamic functions are determined by only one linear combination of the velocities. The thermodynamic quantities as functions of the velocities and the critical velocities are readily calculated for both the homogeneous and inhomogeneous phases of the solution by using the analogy between a moving solution and a BCS system with pairing and nonzero momentum. In a magnetic field, temperature oscillations (the analog of second sound in a superfluid solution) are accompanied by oscillations of the magnetic moment. The velocity and damping of such spin temperature waves are determined. The orienting effect of the mass flux on the inhomogeneous phases in the solution are discussed. It is shown that, owing to drag effects, the energy and size of vorticies in the superfluid are oscillating functions of the effective mass of the ^{3}He quasiparticles (of the pressure). At pressures of the order of 10 atm, a first-order transition should occur in a vortex line, and should be accompanied by an abrupt change in the circulation of the superfluid velocity of ^{3}He at fixed circulation of the ^{4}He velocity.

1. INTRODUCTION

Advances in the production of infralow temperatures have stimulated many new attempts to detect the superfluid transition of ^{3}He in the ^{3}He–^{4}He solution. Such experiments have been unsuccessful so far,[1–4] but the situation does not seem hopeless. Existing low-temperature experimental data on ^{3}He–^{4}He solutions show that, in contrast to pure ^{3}He, the interaction in the s-channel is attractive for ^{3}He quasiparticles in solutions (see, for example, Ref. 5). This means that the temperature T_c of the superfluid transition of ^{3}He in solution may be estimated from BCS theory and, since there are no grounds for expecting a substantial reduction in attraction (the s-scattering amplitude is small in comparison with the atomic size), the only fundamental reason for the reduction in the superfluid transition temperature is the low concentration of ^{3}He.

There are also several possible additional reasons for the absence of ^{3}He superfluidity in the temperature region in which BCS estimates[5,6] show that superfluidity should have been observed. They include the possible relatively slow relaxation of temperature or magnetization, insufficiently precise data on the s-scattering amplitude, measurements at concentrations too high for the BCS theory to provide reliable results, and so on. For example, if the degree of residual spin polarization of ^{3}He in solution is of the order of $P \sim T_c/T_F$, i.e., in practice, at $P \gtrsim 10^{-4}$–10^{-3} (T_F is the ^{3}He degeneracy temperature in solution) s-pairing cannot, in general, take place, and the transition temperature falls by several orders of magnitude.[5,7] The last factor may be very important because the longitudinal relaxation times due to this exceedingly weak dipole-dipole interaction or to collisions between ^{3}He atoms and the walls are quite long, and strong magnetic fields are used to produce low temperatures. This means that both the temperature of the ^{3}He quasiparticle system and the degree of residual magnetization must be monitored in experimental attempts to detect superfluidity.

Although superfluid ^{3}He in ^{3}He-^{4}He solutions is adequately described, at least at low ^{3}He concentrations, within the framework of BCS theory, the properties of superfluid ^{3}He are very different from those of ordinary superconducting systems. This is so because ^{3}He is uncharged (the Meissner effect is absent, second sound can propagate, and the magnetic field has a much simpler effect), there is practically no impurity in the system, and a second (^{4}He) condensate is present (drag effects). Some of the differences between superfluid ^{3}He–^{4}He solutions and superconducting BCS systems are examined below. The essential analogy between ^{3}He–^{4}He solutions and superconductors is nevertheless preserved. For example, the effect of the motion of ^{4}He on superfluid ^{3}He is, to some extent, analogous to the appearance of the vector potential in superconductors, with gauge invariance replaced by the requirement of Galilean invariance. As a result, all the thermodynamic functions are determined by a single linear combination of the velocities, despite the fact that the solution is described by three-velocity hydrodynamics.[8] This will, in the next section, help us to determine quite readily the critical velocities and the thermodynamics of the moving superfluid solution. Morover, since the magnetic field affects only the spin populations of ^{3}He, all the calculations are readily generalized to a nonzero external magnetic field or a residual magnetization of the system.

In Sec. 3 we shall investigate the propagation of temperature waves (second sound in a superfluid Fermi gas) in the ^{3}He–^{4}He solution. In an external magnetic field, temperature oscillations are accompanied by magnetization oscillations and constitute coupled spin temperature waves; this may facilitate the corresponding experiments. However,

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the detection of spin temperature waves encounters a basic difficulty, namely, the high damping of the oscillations.

In Sec. 4 we shall analyze vortices in the superfluid solution, and show that at least one first-order transition occurs in the system for which a change in pressure should be accompanied by an abrupt change in ^{3}He circulation around a vortex. The radius and energy of the vortex line are oscillating functions of pressure (of the effective quasiparticle mass).

One of the interesting properties of superfluid ^{3}He in the ^{3}He–^{4}He solution is the possible appearance, in an external magnetic field, of spatially inhomogeneous states in which the system behaves as a peculiar superfluid liquid crystal. The orienting effect of a current on inhomogeneous systems and the propagation of spin-temperature oscillations in such phases are discussed in the last section.

2. THERMODYNAMICS OF A MOVING SUPERFLUID SOLUTION

The procedure used to derive the equations of the three-velocity hydrodynamics of superfluid ^{3}He–^{4}He solutions and to determine superfluid densities are now well established[8–10] (see also the review[5]). Thus, the superfluid motion of ^{4}He with velocity v_4 leads to the appearance in the ^{3}He Hamiltonian of the additional term

$$\mathbf{p}\mathbf{v}_4(1-m_3/M), \tag{1}$$

where $\mathbf{p}$ and M are the momentum and effective mass of ^{3}He quasiparticles and m_3 is the mass of the ^{3}He atom (for zero pressure, $M \simeq 2.3m_3$). The precision of the description based on the Hamiltonian (1) corresponds to an approximation linear in $\mathbf{v}_4$, i.e., to neglect of terms of the order of $m_3v_4^2$ compared with p_0v_4 ($p_0 = \hbar(3\pi^2N_3)^{1/3}$ is the Fermi momentum of the ^{3}He quasipartcles and N_3 is the number of ^{3}He atoms per unit volume of solution). Since it is quite clear that the critical velocities are $v_c \sim T_c/p_0$, and the superlfuid transition temperature T_c of ^{3}He is exponentially small in the ^{3}He concentration compared with the degeneracy temperature $T_F = p_0^2/2M$, the linear approximation ensures good accuracy for velocities up to v_c. This means that, when the effect of superfluid motion on the properties of the solution is described, the shift of the Green-function poles must be taken into account by allowing for contributions such as (1), and terms of the form Mv^2 are neglected in the denominator. In this approximation, the appearance of the superluid velocity $\mathbf{v}_4$ is analogous to the behavior of the vector potential A in superconductors, apart from the replacement $(M-m_3)\mathbf{v}_4 \rightarrow (e/c)\mathbf{A}$. We emphasize that, in this case, the linear approximation implies not the linearization of the final results with respect to $\mathbf{v}$, but the fact that even quadratic (and higher-order) corrections to the Hamiltonian (1), which are of the form $(p_0v/T_c)^2$, can be retained as compared with the contribution Mv^2/T_c that is linear in the discarded terms. In the absence of ^{3}He superfluidity, when the characteristic scale T_c is lost, the use of the approximate Hamiltonian (1) could actually signify the necessity for linearization of the results with respect to $\mathbf{v}$.

The velocity $\mathbf{v}_3$ of the superfluid motion of ^{3}He in solution can be expressed in terms of the gradient of the phase Φ_3 of the ^{3}He condensate: $\mathbf{v}_3 = (\hbar/2M_3)\nabla\Phi_3$. The wave-function replacement $\psi \rightarrow \psi\exp(i\Phi_3/2)$ in the Gor'kov equations is then equivalent in the approximation that is linear in $\mathbf{v}_3$ to the addition of the following term to the Hamiltonian:

$$(m_3/M)\mathbf{p}\mathbf{v}_3. \tag{2}$$

As a result, the Gor'kov equations for a superfluid solution moving in the direction of the z axis in an external magnetic field assume the form

$$\begin{gathered}\hat{L}_{\mathbf{Q}}{}^{\pm}G_{12}{}^{\pm}+\Delta(\mathbf{r}_1)\tilde{F}_{12}=\delta(\mathbf{r}_1-\mathbf{r}_2),\\ \hat{L}_{-\mathbf{Q}}{}^{*\mp}\tilde{F}_{12}-\Delta^*(\mathbf{r}_1)G_{12}{}^{\pm}=0,\\ \Delta^*(\mathbf{r}_1)=gT\sum_s\tilde{F}_{11},\quad g=4\pi|a|\hbar^2/M,\end{gathered} \tag{3}$$

where

$$G_{12}{}^{\pm}=G^{\pm}(\zeta_s,\mathbf{r}_1,\mathbf{r}_2),\quad \tilde{F}_{12}=\tilde{F}(\zeta_s,\mathbf{r}_1,\mathbf{r}_2)$$

are the temperature (Matsubara) Green functions, Δ is the order parameter, g is the coupling constant, $a<0$ is the s-scattering length, $\zeta_s = \pi T(2s+1)$ is the Matsubara frequency, the operators are

$$\hat{L}_{\mathbf{Q}}{}^{\pm}=i\zeta_s-\frac{1}{2M}\left(\hat{\mathbf{p}}+\frac{\mathbf{Q}}{2}\right)^2+\mu_3\pm\beta H,$$

and $\beta \approx 0.08$ mK/kOe is the magnetic moment of the ^{3}He nucleus. According to (1) and (2),

$$\mathbf{Q}=2m_3\mathbf{v}_3+2(M-m_3)\mathbf{v}_4. \tag{4}$$

The very form of the Gor'kov equations (3) shows that the properties of a moving solution correspond to the usual BCS scheme with pairing and momentum $\mathbf{Q}$. However, $\mathbf{Q}$ as given by (4) is not a Galilean invariant. This is so because we have not taken into account the normal motion. The requirement of Galilean invariance shows that the momentum $\mathbf{Q}$ of the pairs in the moving solution must have the form (this is readily confirmed also microscopically; cf., e.g., Refs. 9 and 10).

$$\mathbf{Q}=2m_3\mathbf{v}_3+2(M-m_3)\mathbf{v}_4-2M\mathbf{v}_n. \tag{5}$$

As a result, although the system contains the three independent velocities $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_n$, the thermo- and hydrodynamics of a moving solution are determined by the single quantity $\mathbf{Q}$ [Eq. (5)]. This analogy between a moving superfluid solution and a BCS system with pairing and nonzero momentum enables us to determine readily the critical velocities and the dependence of thermo- and hydrodynamic quantities on the velocities. For example, it is quite clear that the ground state of the solution is the state with $\mathbf{Q}=0$. We note that the condition $\mathbf{Q}=0$ imposed on (5) can also be readily obtained by minimizing the energy of the moving solution[5,9] for fixed ^{3}He and ^{4}He mass fluxed $\mathbf{j}_3$ and $\mathbf{j}_4$. Of course, the reason why the thermodydnamic functions of the moving solution are determined exclusively by $\mathbf{Q}$ is that the normal component of the solution at temperatures $T \lesssim T_c$ is entirely due to the unpaired ^{3}He quasiparticles, while the contribution of Bose excitations (phonons, rotons) is negligible.

BCS paring with nonzero momentum $\mathbf{Q}$ in an external magnetic field has frequently been examined in the literature (see, for example, Refs. 5 and 11–16). Thus, the consistency equation has the form

$$1=gT\sum_{s=-\infty}^{\infty}\int\frac{d^3p}{(2\pi\hbar)^3}\{L_{\mathbf{Q}}{}^{\pm}L_{-\mathbf{Q}}{}^{*\mp}+|\Delta|^2\}^{-1}, \tag{6}$$

and the transition T_{QH} as a function of the magnetic field H and of the quantity **Q** is given by

$$\begin{aligned}0&=\ln\frac{T_c}{T_{QH}}-2\ln 2-C\\&+\frac{i}{2q}\ln\frac{\Gamma([1+iq+ih]/2)\Gamma([1+iq-ih]/2)}{\Gamma([1-iq+ih]/2)\Gamma([1-iq-ih]/2)}\\&\equiv\ln\frac{T_c}{T_{QH}}-2\ln 2+\sum_{\nu=1}^{\infty}\Big\{\frac{1}{q}\operatorname{arctg}(q_\nu+h_\nu)\\&\qquad+\frac{1}{q}\operatorname{arctg}(q_\nu-h_\nu)-\frac{1}{\nu}\Big\},\end{aligned} \tag{7}$$

where $C=0.5772$ is the Euler constant, T_c is the transition temperature for $Q=H=0$,

$$q=\frac{Qv_0}{2\pi T_{QH}},\quad h=\frac{\beta H}{\pi T_{QH}},\quad q_\nu=\frac{q}{2\nu-1},\quad h_\nu=\frac{h}{2\nu-1}.$$

For slow motion, $q\ll 1$ we have

$$\begin{gathered}\frac{T_H-T_{QH}}{T_H}=q^2\varphi_0(H),\quad \ln\frac{T_H}{T_c}=\sum\frac{2}{2\nu-1}\Big[\frac{1}{1+h_\nu^2}-1\Big],\\ \varphi_0(H)=\frac{2}{3}\sum\frac{1}{(2\nu-1)^3}\frac{1-3h_\nu^2}{(1+h_\nu^2)^3}.\end{gathered} \tag{8}$$

The dependence of the order parameter Δ (**Q**, H, T) on **Q** and H is very complicated in the general case, but it becomes much simpler in the region $\tau_{QH}=(T_{QH}-T)/T_{QH}\ll 1$ near the transition:

$$\Delta^2=2\pi^2T_{QH}^2\tau_{QH}(1+\varphi_1)/\varphi_2, \tag{9}$$

where the functions $\varphi_{1,2}(\mathbf{Q},H)$ are given by

$$\begin{aligned}\varphi_1&=\frac{1}{q}\sum\Big\{\frac{q_\nu+h_\nu}{1+(q_\nu+h_\nu)^2}+\frac{q_\nu-h_\nu}{1+(q_\nu-h_\nu)^2}\\&\qquad-\operatorname{arctg}(q_\nu+h_\nu)-\operatorname{arctg}(q_\nu-h_\nu)\Big\},\\ \varphi_2&=\frac{1}{q}\sum\frac{1}{(2\nu-1)^2}\Big\{\frac{q_\nu+h_\nu}{[1+(q_\nu+h_\nu)^2]^2}\\&\qquad+\frac{q_\nu-h_\nu}{[1+(q_\nu-h_\nu)^2]^2}\Big\}.\end{aligned} \tag{10}$$

When $q\ll 1$ and $\tau_H=(T_H-T)/T_H\ll 1$, we have

$$\begin{gathered}\Delta^2=2\pi^2T_H^2(\tau_H-q^2\varphi_0)[1+\varphi_1(0,H)]/\varphi_2(0,H),\\ \varphi_1(0,H)=-4\sum\frac{1}{2\nu-1}\frac{h_\nu^2}{(1+h_\nu^2)^2},\quad \varphi_2(0,H)=3\varphi_0(H).\end{gathered} \tag{11}$$

The maximum admissible value $\mathbf{Q}_M$ for which Eq. (7) for the transition temperature has a solution is determined by

$$\begin{gathered}\{(q_M+\bar h)\ln(q_M+\bar h)+(q_M-\bar h)\ln(q_M-\bar h)\}/2q_M=1-C-2\ln 2,\\ q_M=Q_Mv_0/2\pi T_c,\quad \bar h=\beta H/\pi T_c.\end{gathered} \tag{12}$$

At $\mathbf{H}=0$ the solution (12) has the form

$$q_M={}^1/_2\exp(1-C).$$

The hydrodynamics of a moving solution is determined by the dependence of the superfluid and normal densities on **Q**. The ^{3}He mass flux is given by

$$\begin{aligned}\mathbf{j}(\mathbf{r}_1)&=\frac{i}{2}\frac{m_3}{M}\hbar T\sum_s\Big\{\Big(\frac{\partial}{\partial\mathbf{r}_2}-\frac{\partial}{\partial\mathbf{r}_1}\Big)(G_{12}{}^{+}+G_{12}{}^{-})\Big\}_{\mathbf{r}_2=\mathbf{r}_1}\\&\qquad+N_3\frac{m_3}{M}[m_3\mathbf{v}_3+(M-m_3)\mathbf{v}_4],\end{aligned} \tag{13}$$

and, by virtue of (3), can be reduced to

$$\mathbf{j}_3=\rho_s\mathbf{Q}/2m_3+m_3N_3\mathbf{v}_n, \tag{14}$$

where, near the transition point, the superfluid density ρ_s has the form

$$\begin{aligned}\rho_s&=\frac{3}{2\pi^2}\frac{m_3^2}{M}N_3\frac{\Delta^2}{T_{QH}^2}\varphi_3,\\ \varphi_3&=\frac{1}{q^3}\sum\Big\{\operatorname{arctg}(q_\nu+h_\nu)+\operatorname{arctg}(q_\nu-h_\nu)\\&\qquad-\frac{q_\nu}{1+(q_\nu+h_\nu)^2}-\frac{q_\nu}{1+(q_\nu-h_\nu)^2}\Big\}.\end{aligned} \tag{15}$$

The ^{4}He mass flux $\mathbf{j}_4$ is readily expressed in terms of $\mathbf{j}_3$ by the method proposed in Refs. 5 and 9:

$$\mathbf{j}_4=(M/m_3-1)\mathbf{j}_3+[\rho_4-(M-m_3)N_3]\mathbf{v}_4, \tag{16}$$

where $\rho_4=m_4N_4$ is the density of ^{4}He in solution. Expressions (14)–(16) determine the required dependence of all the superfluid and normal densities on **Q**. The critical velocity, i.e., in this case, the critical value $\mathbf{Q}_c$, is determined by the condition for maximum superfluid current, and according to (11), (14), and (15) is given by

$$q_c^2=\tau/3\varphi_0.$$

When $\rho_s=0$ (or $\mathbf{Q}=0$), the fluxes (14) and (16) become identical with the usual formulas for the ^{3}He and ^{4}He mass fluxes in the absence of superfluid ^{3}He (Ref. 5):

$$\begin{gathered}\mathbf{j}_3=m_3N_3\mathbf{v}_n,\quad \mathbf{j}_4=(M-m_3)N_3\mathbf{v}_n+[\rho_4-(M-m_3)N_3]\mathbf{v}_4,\\ \mathbf{j}=\mathbf{j}_3+\mathbf{j}_4=MN_3\mathbf{v}_n+[\rho_4-(M-m_3)N_3]\mathbf{v}_4.\end{gathered}$$

3. SPIN TEMPERATURE WAVES IN A SUPERFLUID SOLUTION

The spin polarization of the solution gives rise to properties of temperature-wave propagation that are analogous to second sound in ^{3}He–He II with two condensates. Temperature oscillations are then accompanied by oscillations in magnetization and this, in turn, leads to a substantial change in the wave propagation velocity and to new possibilities of their experimental detection.

Superfluid motion does not cause transport of magnetic moment and, in the case of a polarized solution, the three-velocity-hydrodynamics equations[5,8–10] must be supplemented with

$$\dot M_z+\operatorname{div}(M_z\mathbf{v}_n)=0, \tag{17}$$

where M_z is the longitudinal component of the magnetic moment **M**. Equation (17) has the form of a conservation law and is valid only in the exchange approximation for very long longitudinal relaxation times $\tau \to \infty$. Inclusion of longitudinal relaxation leads to a frequency dependence of the propagation velocity of spin temperature waves.[17] The expression for the differential of the pressure P (Refs. 5 and 9) must also be modified by taking into account the magnetic component, which has the form (in a constant and uniform external magnetic field)

$$\delta P = (M_z/\chi)\,\delta M_z, \tag{18}$$

where $\chi = M/H$ is the susceptibility of the solution. Comparison of (17) and (18) with the analogous expressions for the entropy S and temperature, namely,

$$\dot{S} + S\,\operatorname{div}\mathbf{v}_n = 0, \quad \delta P = \frac{TS}{C}\,\delta S,$$

where C is the heat capacity, shows that the wave propagation is determined by coupled spin temperature and entropy oscillations, and its velocity differs from the propagation velocity of temperature waves in unpolarized solutions[5,9] by the replacement $TS^2/C \to TS^2/C + M_z^2/\chi$, i.e.,

$$u^2 = \left(\frac{TS^2}{C} + \frac{M_z{}^2}{\chi}\right)\frac{\rho_s}{\rho_3[\rho_3 - (M/m_3)\rho_s]}, \tag{19}$$

where $\rho_3 = m_3 N_3$.

Since magnetic measurements are, in many respects, simpler than thermal measurements, studies of spin temperature waves are more readily performed by exciting the oscillations by varying the longitudinal component of the external magnetic field. The exciting force $-M_z\delta H$ must then be added to (18).

However, the wave propagation velocity (19) is very small, and the damping of these waves is relatively high, so that difficulties arise in the measurements. Thus, near the transition point,

$$u^2 \sim \frac{M_z{}^2}{\chi}\frac{\rho_s}{\rho_3{}^2} \sim \frac{\beta H}{m_3}\frac{\beta H}{T_F}\frac{\rho_s}{\rho_3} \leqslant \frac{T_F}{m_3}\left(\frac{T_c}{T_F}\right)^2\frac{\rho_s}{\rho_3}, \tag{20}$$

where we have used the condition $\beta H \lesssim T_c$, since s-pairing is not at all possible[5] in fields such that $\beta H > 1.3T_c$. Apart from viscosity and thermal conduction, the attenuation of the wave is determined by spin diffusion and longitudinal relaxation. The corresponding processes are described by adding the terms $D\Delta M_z$ *and* $\delta M_z/\tau$ to (17). Under resonance excitation, the fact that attenutation is small means that

$$D\omega/u^2 \ll 1, \quad \omega\tau \gg 1. \tag{21}$$

If we use the following estimate[5] for the diffusion coefficient

$$D \sim (\hbar/M)x^{-2/3}(T_F/T)^2$$

(x is the ^{3}He concentration in the solution) together with (20), we find that the first of the inequalities in (21) is equivalent to

$$\omega \ll (T_c/\hbar)x^{2/3}(T_c/T_F)^3(\rho_s/\rho_3), \tag{22}$$

and, when T_c is low enough, only the quasistationary oscillations are weakly attenuated. The conditon for both inequalities in (21) to be satisfied is

$$u^2\tau/D \gg 1 \tag{23}$$

and whether or not this will be satisfied will depend on which particular type of longitudinal relaxation takes place. For dipole–dipole relaxation,

$$\tau^{-1} \sim (\beta^4 N_3{}^2/\hbar T_F)(T/T_F)^2$$
$$\sim 10^{-15}(T_c/\hbar)(T_c/T_F)x^{2/3},$$

and the inequality (23), which is equivalent to

$$(T_c/T_F)^2(\rho_s/\rho_3) \gg 10^{-15},$$

is readily satisfied. Condition (22) shows that resonant excitation of weakly damped waves is very difficult, whereas nonresonant excitation reduces the signal amplitude.

Attempts to observe the spin temperature waves encounter further difficulties connected with the increase in the mean free path of the excitations as the temperature is reduced. When $T \lesssim 0.1$ mK, the mean free path of the excitations is of the order of some centimeters, and is comparable with the characteristic dimensions L of the specimen.[3] Relaxation is then determined by collisions with the walls, with characteristic times

$$\tau \sim (1/w)(L/v_0) \sim (x^{-1/3}/w)(L/s_0),$$

where w is the accommodation coefficient for spin (for relaxation of magnetization) or for energy (thermalization processes) in collisions with the wall, and $s_0 \sim 10^5$ cm/s. Condition (23) may then not be satisfied, and the condition for hydrodynamic oscillations is violated because the wavelength turns out to be shorter than the quasiparticle mean free path.

The above difficulties in observing spin temperature waves arise not only in the magnetic method of wave generation, but also in the usual thermal method. The role of spin diffusion and longitudinal relaxation is then played by thermal conduction and scattering by walls, and the restriction (22) on the frequency of the oscillations remains practically the same even in the absence of magnetization ($M = H = 0$).

4. VORTICES IN A SUPERFLUID SOLUTION

The dependence of thermodynamic functions of the superfluid solution on $\mathbf{Q}$ [Eq. (5)] has a considerable effect on the energy and structure of the vortices. The energy and size of the vortex core will be determined below for $T < T_c$.

The quantization condition for the circulation of velocity around a vortex line must be satisfied for each of the condensates separately, and automatically ensures nonzero $\mathbf{Q}$ in the neighborhood of a vortex. The velocity distribution around a homogeneous vortex line is

$$v_3 = (n_3/2m_3)(\hbar/r), \quad v_4 = (n_4/m_4)(\hbar/r),$$

and, when $\mathbf{v}_n = 0$, we have

$$Q = \frac{\hbar}{r}\left| n_3 + \frac{2m_3}{m_4}\left(\frac{M}{m_3} - 1\right)n_4 \right|, \tag{24}$$

where the integers n_3 and n_4 are the numbers of quanta for the circulations of $\mathbf{v}_3$ and $\mathbf{v}_4$. For fixed n_4, which determines the circulation of the superfluid velocity above the ^{3}He transition to the superfluid state in solution (the superfluid transition in ^{3}He cannot change the circulation of ^{4}He because, in solution, $\rho_3 \ll \rho_4$, the minimum energy corresponds to the vortex with circulation n_3 that minimizes Q in (24). The circulation n_3 of ^{3}He for vortices below the transition thus turns out to depend on the effective mass M and, hence, on pressure in the system. Thus, for $n_4 = 1$, minimum energy corresponds to the vortex with $n_3 = -2$ for $8/3 > M/m_3 > 2$, $n_3 = -3$ for $10/3 > M/m_3 > 8/3$, $n_3 = -4$ for $4 > M/m_3 > 10/3$, and so on. The vortex energy (due to the superfluidity of ^{3}He is then proportional to the square of Q as given by (24), and turns out to be a periodic function of M with minima at $M/m_3 = 7/3; 3; 11/3 \ldots$ and maxima at $M/m_3 = 2; 8/3; 10/3 \ldots$. The circulation n_3 for (stable) minimum-energy vortices changes abruptly at the maxima. This corresponds to transitions of the first order in pressure for the vortices. One such transition necessarily occurs because[5,8] $M/m_3 \approx 8/3$ for $P \sim 10$ atm. Whether there is a second transition ($M/m_3 = 10/3$) at maximum pressure is not as yet clear.

At the minimum energy points $M/m_3 = 7/3; 3; 11/3 \ldots$, the quantity Q given by (24) vanishes at the appropriate values of n_3. This means that the superfluid transition in ^{3}He at such points has no effect on the ^{3}He and ^{4}He fluxes, so that the transition in ^{3}He at these points occurs with $Q = 0$, and turns out to be insensitive to the presence of vortices in ^{4}He above the transition point (with the exception of a narrow region of atomic linear dimensions near the vortex core). It is useful to note here that it is precisely such a point that corresponds to the saturated vapor pressure ($M/m_3 = 7/3$ to within 1%)!

The structure of the Δ (r) vortex for ^{3}He is determined by the same relationships as for superconductors, except for the replacement $v_s \to Q/2m_3$. However, since Q as given by (24) depends in this case on M (pressure), the radius of the vortex core must also depend on $M(P)$. The size of the minimum-energy vortex is therefore proportional to $|n_3 + (2/m_4)(M - m_3)n_4|$, and is an oscillating function of M. At points corresponding to minimum Q, i.e., when $M/m_3 = 7/3; 3; 11/3 \ldots$ ($n_4 = 1$), the radius of the vortex line is of the atomic order of magnitude even in the Ginzburg-Landau region, and is determined by the structure of the vortex in ^{4}He.

5. ^{3}He–^{4}He SUPERFLUID LIQUID CRYSTALS

For magnetic fields $1.33 > \beta H/T_c > 1.06$, the transition temperature T_{QH} [Eq. (7)] has a minimum for $Q_0(H) \neq 0$ [plots of T_{Q_0H} and $Q_0(H)$ can be found, for example, in Refs. 5 and 16]. This means that, even in equilibrium, pairing occurs with nonzero total momentum and the resulting spatially inhomogeneous system[11,12] is a kind of superfluid liquid crystal.[5,7] The order parameter in this system is

$$\Delta(\mathbf{r}) = \exp(i\Phi_3) \sum \Delta_m \exp(i\mathbf{Q}_m\mathbf{r}/\hbar), \tag{25}$$

where $|\mathbf{Q}_m| = Q_0(H)$ near the transition point for all vectors $\mathbf{Q}_m$ in (25). As we depart from the transition point, the higher-order harmonics $\mathbf{Q}_m$ become significant in the expansion for Δ (r). The specific form of the expansion (25) that corresponds to minimum free energy is at present unknown. One usually investigates the so-called Fulde-Ferrell phase

$$\Delta(\mathbf{r}) = \exp(i\Phi_3)\,\Delta_0 \exp(i\mathbf{Q}_0\mathbf{r}/\hbar), \tag{26}$$

the layered structure

$$\Delta(r) = \exp(i\Phi_3)\,(\Delta_0/2)\cos(\mathbf{Q}_0\mathbf{r}/\hbar) \tag{27}$$

and the cubic structure

$$\Delta(\mathbf{r}) = \exp(i\Phi_3)\,(\Delta_0/2)\,\{\cos(Q_0x/\hbar) + \cos(Q_0y/\hbar) + \cos(Q_0z/\hbar)\}. \tag{28}$$

We shall be interested in the orienting property of the flux and in spin temperature waves in inhomogeneous phases in solution.

The formal solution of Gor'kov equations (3) for the inhomogeneous phases (26) has the form

$$\begin{aligned} \tilde{F}(\mathbf{p},\mathbf{p}') &= \tilde{G}_\mathbf{p} G_{\mathbf{p}'} \sum_m \Delta_m^* \psi_m(\mathbf{p}+\mathbf{Q}_m)\,\delta(\mathbf{p}'-\mathbf{p}-\mathbf{Q}_m), \\ G^\pm(\mathbf{p},\mathbf{p}') &= G_\mathbf{p}\delta(\mathbf{p}-\mathbf{p}') - G_\mathbf{p} \sum_m \Delta_m \tilde{F}(\mathbf{p}+\mathbf{Q}_m, \mathbf{p}'), \end{aligned} \tag{29}$$

where $\tilde{F}(\mathbf{p},\mathbf{p}')$ and $G^\pm(\mathbf{p},\mathbf{p}')$ are the Fourier components of the Green functions $\tilde{F}(\mathbf{r},\mathbf{r}')$ and $G^\pm(\mathbf{r},\mathbf{r}')$ and

$$\begin{gathered} G_\mathbf{p} = [i\zeta_s - (^1/_2M)(\mathbf{p}+\mathbf{Q}/2)^2 + \mu_3 \pm \beta H]^{-1}, \\ \tilde{G}_\mathbf{p} = G_\mathbf{p}^*(-H, -\mathbf{Q}), \quad \psi_m(\mathbf{p}) = \sum_n (1-\hat{A})_{mn}^{-1}, \\ A_{mn}(\mathbf{p}) = -\frac{\Delta_n^*}{\Delta_m^*} {\sum_{i,k}}' \Delta_i^* \Delta_k G_{\mathbf{p}+\mathbf{Q}_i-\mathbf{Q}_m} \tilde{G}_{\mathbf{p}-\mathbf{Q}_n}, \end{gathered} \tag{30}$$

where the sum in the last expression is evaluated over vectors $\mathbf{Q}_i$, $\mathbf{Q}_k$ for which $\mathbf{Q}_i - \mathbf{Q}_k = \mathbf{Q}_m - \mathbf{Q}_n$.

The vector $\mathbf{Q}$ in (30) is defined by (5), and the superfluid velocity $\mathbf{v}_3$ is defined as $\mathbf{v}_3 = (\hbar/2m_3)\nabla\Phi_3$ for the inhomogeneous phases in (25). This definition of $\mathbf{v}_3$ is not entirely unambiguous for (26) because of possible rotation of the vector $\mathbf{Q}_0$ (see below). The consistency equation for the Green functions (29) reduces to the set of coupled equations

$$1 = g \sum_s \int \frac{d^3p}{(2\pi\hbar)^3} \tilde{G}_{\mathbf{p}-\mathbf{Q}_m} G_\mathbf{p} \psi_m(\mathbf{p}). \tag{31}$$

Expressions (29)–(31) have a simple form only for the phase (26), when there is only one function ψ_m, namely,

$$\psi = \{1 + |\Delta_0|^2 G_\mathbf{p} \tilde{G}_{\mathbf{p}-\mathbf{Q}_0}\}^{-1} \tag{32}$$

and one equation (31). The velocity dependence of all the thermodynamic quantities then has the same form as for the homogeneous phase, except for the replacement $\mathbf{Q} \to \mathbf{Q} + \mathbf{Q}_0$. Expressions (29)–(31) become simpler for the remaining phases only in the Ginzburg-Landau region where $(1 - \hat{A})^{-1} = 1 + \hat{A}$. However, even in this case, the consistency equations (31) have been solved only for the one-dimensional phase (27) (Refs. 5 and 15). Nevertheless, the following general expression for the flux for $Q \ll Q_0$ can be obtained in the Ginzburg-Landau region [compare this with (14)]:

$$\begin{aligned} \mathbf{j}_3 &= \frac{1}{2m_3} \sum_m \rho_s^{(m)} (\mathbf{Q}+\mathbf{Q}_m) + m_3 N_3 \mathbf{v}_n, \\ \rho_s^{(m)} &= \frac{3}{2\pi^2} \frac{m_3^2}{M} N_3 \frac{|\Delta_m|^2}{T^2_{Q_0H}} \varphi_3^{(m)}, \end{aligned} \tag{33}$$

where the function $\varphi_3^{(m)}$ is the same as φ_3 given by (15), except for the replacement $\mathbf{Q} \rightarrow \mathbf{Q} + \mathbf{Q}_m$. The function φ_3 in (15) will, however, vanish for $Q = Q_0(H)$ which means that there is no superfluid flow in the phases in (25) in equilibrium. The equation $\varphi_3(Q_0) = 0$ is actually identical with the equation[5,6] for $Q_0(H)$. Hence, the expression for the flux (33), linearized in $\mathbf{Q}$, has the form

$$\mathbf{j}_3=\frac{1}{2m_3}\sum\rho_s^{(m)}\frac{\mathbf{Q}\cdot\mathbf{Q}_m}{Q_0^2}\mathbf{Q}_m+m_3N_3\mathbf{v}_n,\quad \varphi_3^{(m)}=2\varphi_2(Q_0) \tag{34}$$

[a plot of the function $\varphi_2(Q_0)$ is contained in Ref. 7]. The relative change in the transition temperature (9) is given by

$$\delta T_{\mathbf{Q}_mH}/T_{Q_0H}=-2[\mathbf{Q}\mathbf{Q}_m/Q_0^2]^2q_0^2\varphi_2(Q_0). \tag{35}$$

Since the superfluid flux (34) and the transition temperature in (35) are very sensitive to a change in the angle between the velocity $\mathbf{Q}$ and the vectors $\mathbf{Q}_m$, the critical fluxes and the energy of the structure are also determined by the orientation of the flux relative to the set of vectors $\mathbf{Q}_m$. In the absence of fluxes, the spatial orientation of the structure (25) is set[7] by the walls of the vessel alone or, when the volume is very large, by the dipole-dipole interaction that is exceedingly weak in the solution. The appearance of a flux may lead to a change in the orientation in the volume. For example, for the one-dimensional phases (26) and (27), the maximum superfluid flux is reached for $\mathbf{Q}_0$ parallel to the direction of the flux. The critical flux for the phase (26) is then

$$j_c=\frac{1}{\pi}\left[\frac{3}{2}\varphi_2(Q_0)\right]^{1/2}\frac{m_3}{M}N_3\frac{|\Delta_0|^2}{v_0T_{Q_0H}}\tau^{1/2},\qquad \tau=\frac{T_{Q_0H}-T}{T_{Q_0H}}, \tag{36}$$

whereas for (27) it is greater by a factor of two. For the more complex structures (25), the current orientation depends on the ratio of diagonal and off-diagonal elements in the consistency equations (31). For example, for the cubic phase defined by (28), the equations in (31) reduce to the following set in the Ginzburg-Landau region when (35) is taken into account:

$$\tau-2(Q/Q_0)^2q_0^2\varphi_2(Q_0)\alpha_i^2=B_{ij}|\Delta_j|^2,$$

$$B_{i=j}=B_1,\quad B_{i\neq j}=B_2,$$

where i and j run through the values x, y, z, whereas α_i are the cosines of the angles between the vector $\mathbf{Q}$ and the axes of the crystal (28); $\Sigma\alpha_i^2 = 1$, and the elements of the matrix B_{ij} are proportional to linear combinations of the integrals (with respect to the momenta) of the products of the four Green functions in (31) and the matrix $\hat{A}$ in (30). The second-order transition then corresponds to the condition $B_1 + 2B_2 > 0$ and if, at the same time, $|B_1| > |B_2|$, a maximum critical flux differing from (36) by a factor of 6 is obtained when the flux is directed along the diagonal of the cube $\alpha_i^2 = 1/3$ and superfluidity is destroyed to a greater degree in the direction corresponding to maximum α_i. In the opposite case, superfluidity in the transverse direction is destroyed, and maximum superfluid flux corresponds to the direction of the current along one of the vectors $\mathbf{Q}_i$.

The above results are rigorous only for a second-order transition (or a transition close to second order). In reality, even in the mean-field approximation, the superfluid transition turns out to be of the first order[5,12,15] for some of the phases in (25), and inclusion of fluctuations always leads to first-order transition.[19]

The propagation of low-frequency hydrodynamic oscillations in the phases (25) was determined in Ref. 7, but without taking spin oscillations into account. Since superfluid motion and inhomogeneous deformations do not produce the transport of magnetic moment (an additonal hydrodynamic variable appears in the inhomogeneous phases, namely, the displacement vector $\mathbf{u}$ corresponding to the noninvariance of the order parameter $\Delta(\mathbf{r})$ [Eq. (25)] under the displacement transformation $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{u}$, and for symmetric phases $\Delta(\mathbf{r}) = \Delta(-\mathbf{r})$ there are no particle fluxes proportional to $\partial u_i/\partial x_k$), the hydrodynamic equations[7] must be supplemented by (17) and (18), as for the homogeneous phase. This leads to the replacement $TS^2/C \rightarrow TS^2/C + M_z^2/\chi$ in the dispersion relation and in the propagation velocity of temperature waves. Temperature oscillations are then accompnied by magnetization oscillations, as in the homogeneous phase, whereas the propagation velocity of transverse elastic oscillations with $\operatorname{div} \mathbf{v}_n = 0$ remains unaltered.

The fact that the results obtained in Ref. 7 for inhomogeneous phases (25) are almost identical to those obtained in Ref. 20 for the phase (26) is not surprising. This agreement is due to the fact that the flux vector (34) can be factorized in the Ginzburg-Landau region and consists of the sum of components along the vectors $\mathbf{Q}_m$. Hence, the velocity of the spin temperature waves also turns out to be proportional to

$$\sum\Delta_m^2\cos^2\varphi_m,$$

where φ_m is the angle between the direction of propagation of the oscillations and the vector $\mathbf{Q}_m$. Correspondingly, the results for the one-dimensional phases (26) and (27) are formally similar [the phase (26) is distinguished only by the fact that its inhomogeneous deformation is totally equivalent to superfluid motion; this and the absence of inversion $\Delta(\mathbf{r}) \neq \Delta(-\mathbf{r})$ lead to the coupling between spin temperature and elastic waves]. This similarity should vanish as the temperature is reduced.

The dispersion relation given in Ref. 7 is valid, when (17) and (18) are taken into account, even well away from the phase transition, because all we need to derive the hydrodynamic equations are the relations (obtained as a consequence of Galilean invariance and of the linearity of mass fluxes and momentum in velocities and deformations) between the superfluid densities and the elastic moduli. All the coefficients in the dispersion relation are then expressed in terms of the single tensor $\rho_{ik}^{(s)}$ whose form is, of course, different from (34). Spin and temperature oscillations are then still coupled, as before, and the orienting effect of the flux, depends on the symmetry properties of the $\rho_{ik}^{(s)}$.

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