Wave interaction in an active conservative medium

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Truncated equations describing the interaction of waves in an unstable medium are derived by a Hamiltonian formalism. It is shown in first-order perturbation theory that three pairs of waves with opposite wave vectors interact with each other; the instability correspondingly becomes explosive. The truncated equations are integrable by the method of the inverse problem.

Truncated equations describing the nonlinear dynamics of waves in a continuous medium are usually derived by the following approach (see Ref. 2, for example). The initial equations are of the form $D(\omega,\mathbf{k})f_\omega = j_\omega(f)$, where the zeros of $D(\omega,\mathbf{k})$ determine the linear spectrum, while $j_\omega(f)$ is the nonlinear current, expanded in powers of f. If j_ω and f_ω are a set of narrow lines, one of which lies near a zero $D(\omega_0,\mathbf{k})=0$, then we can expand $D(\omega,k)$ near ω_0. If the analysis is restricted to the first term of this expansion, the equations take the form $(\omega-\omega_0)D_0' f_\omega = j_\omega$. This procedure is also used in the case of waves which are damped or which grow in the linear approximation, under the obvious condition $\mathrm{Im}\omega \ll \mathrm{Re}\omega$. It is clear that an expansion of $D(\omega,\mathbf{k})$ is legitimate only if the line width $\Delta\omega$, which is inversely proportional to the scale time of the nonlinear interaction, satisfies the inequality $\Delta\omega \ll \Omega$, where the scale frequency Ω is determined by the linear dispersion relation. For both stable and unstable waves the assumption $\Omega \sim |\omega_0|$ is usually adopted.

There are, on the other hand, many media in which this latter condition does not hold. Let us assume that $D(\omega)$ is a polynomial in ω with real coefficients. If there exists a natural mode with $\mathrm{Im}\omega_0 > 0$, then there also exists a wave with $\mathrm{Im}\omega_0 < 0$, so that we would have $\Omega \sim \mathrm{Im}\omega_0$. The condition $\Delta\omega \ll \mathrm{Im}\omega_0$ essentially strips the corresponding truncated equations of their meaning: Over a time much shorter than $\Delta\omega^{-1}$ the wave amplitudes grow markedly, and an expansion in powers of the field variables cannot be used. This situation is typical of media for which canonical variables can be introduced.

In a stable conservative medium, the interaction of waves can also be described by a Hamiltonian formalism.[1] In this approach the decay conditions imposed on the frequencies and wave vectors of the interacting waves—conditions which are usually interpreted as momentum and energy conservation laws—appear as the "small denominators" which are familiar in mechanics.[3] In other words, under the decay conditions there is no canonical transformation of any sort which would make it possible to prune certain terms from the interaction Hamiltonian.

Our purpose in the present paper is to systematically derive equations describing the wave interaction in an unstable medium. We assume that the initial equations are Hamilton's equations and are written in terms of canonical variables.

We will see that in the case of an unstable medium the waves interact under the same decay conditions, but the frequencies which appear in these conditions are complex quantities. We derive truncated equations and analyze their solutions for the particular case of the interaction of three unstable wave modes and the generation of a stable second harmonic of an unstable mode. We conclude with a discussion of the applicability of this theory to processes in a multistream plasma.

1. CHOICE OF A QUADRATIC HAMILTONIAN

We assume that the medium is described by a pair of canonical variables $p(\mathbf{x},t)$ and $q(\mathbf{x},t)$ and by a Hamiltonian $H(p,q)$. We expand the Hamiltonian in powers of p and q: $H = H_0 + H_1 \ldots$. In a spatially homogeneous medium the term of this expansion which is quadratic in p and q is

$$H_0 = {}^1/_2 \int d\mathbf{k}[A_\mathbf{k}|p_\mathbf{k}|^2 + 2B_\mathbf{k} p_\mathbf{k} q_\mathbf{k}^* + C_\mathbf{k}|q_\mathbf{k}|^2], \tag{1}$$

where $p_\mathbf{k}(t)$ and $q_\mathbf{k}(t)$ are the Fourier transforms of $p(\mathbf{x},t)$ and $q(\mathbf{x},t)$, and $A_\mathbf{k}$, $B_\mathbf{k}$, and $C_\mathbf{k}$ are functions which are determined by the particular model adopted for the medium. It follows from the reality of $p(\mathbf{x}), q(\mathbf{x})$, and H_0 that

$$p_{-\mathbf{k}} = p_\mathbf{k}^*, \quad q_{-\mathbf{k}} = q_\mathbf{k}^*, \quad B_{-\mathbf{k}} = B_\mathbf{k}^*, \tag{2}$$

and that $A_\mathbf{k}$ and $C_\mathbf{k}$ are real function of $\mathbf{k}$ of even parity.

It follows from Hamilton's equations

$$dq_\mathbf{k}/dt = \delta H/\delta p_{-\mathbf{k}}, \quad dp_\mathbf{k}/dt = -\delta H/\delta q_{-\mathbf{k}} \tag{3}$$

for the Hamiltonian H_0 that, in the linear approximation, we have

$$p_k(t),\ q_k(t) \propto \exp(\lambda t), \quad \lambda = i\{\mathrm{Im}\,B_\mathbf{k} \pm [A_\mathbf{k}C_\mathbf{k} - (\mathrm{Re}\,B_\mathbf{k})^2]^{1/2}\}.$$

The medium is stable with respect to the growth of small perturbations if $A_\mathbf{k}C_\mathbf{k} > (\mathrm{Re}B_\mathbf{k})^2$, and this case has been studied thoroughly by Zakharov.[1] In the present paper we focus on the situation in which the opposite inequality can hold for some values of $\mathbf{k}$; i.e., we assume that the medium is unstable with respect to the growth of certain modes.

We know quite well that in the stability region we can transform (with a purely imaginary determinant) from the variables $p_\mathbf{k}, q_\mathbf{k}$ to the complex amplitudes $a_\mathbf{k}$, with Hamiltonian (1) taking the form

$$H_0 = \int d\mathbf{k}\,\omega_\mathbf{k}^0 a_\mathbf{k} a_\mathbf{k}^*, \quad \omega_\mathbf{k}^0 = \mathrm{Im}\,B_\mathbf{k} + [A_\mathbf{k}C_\mathbf{k} - (\mathrm{Re}\,B_\mathbf{k})^2]^{1/2}. \tag{4}$$

Again in the unstable region it is possible to transform to complex amplitudes, but Hamiltonian (1) takes a different form:

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$$H_0=\int d\mathbf{k}[\omega_\mathbf{k}a_\mathbf{k}a_\mathbf{k}^* - {}^1/_2\gamma_\mathbf{k}(a_\mathbf{k}a_{-\mathbf{k}}+a_\mathbf{k}^*a_{-\mathbf{k}}^*)], \quad (5)$$

where now

$$\omega_\mathbf{k}=\mathrm{Im}\,B_\mathbf{k}, \quad \gamma_\mathbf{k}=[(\mathrm{Re}\,B_\mathbf{k})^2-A_\mathbf{k}C_\mathbf{k}]^{1/2}. \quad (6)$$

By virtue, of relations (2), the identities $\omega_\mathbf{k} = -\omega_{-\mathbf{k}}$, $\gamma_\mathbf{k} = \gamma_{-\mathbf{k}}$ hold for unstable excitations. Furthermore, we can assume $\gamma_\mathbf{k} \geqslant 0$ without any loss of generality.

In several situations, e.g., in superconductivity theory, a quadratic Hamiltonian in the form in (5) is quite natural, but for our purposes in this paper it is more convenient to choose a different representation. It is not difficult to see that in the instability region we can make the canonical transformation $p_\mathbf{k}, q_\mathbf{k} \rightarrow P_\mathbf{k}, Q_\mathbf{k}$, with Hamiltonian (1) becoming

$$H_0=i\int d\mathbf{k}\Omega_\mathbf{k}P_\mathbf{k}Q_{-\mathbf{k}}, \quad \Omega_\mathbf{k}=\omega_\mathbf{k}+i\gamma_\mathbf{k}=-\Omega_{-\mathbf{k}}^*. \quad (7)$$

The reality condition (2) again takes the form $P_{-\mathbf{k}} = P_\mathbf{k}^*, Q_{-\mathbf{k}} = Q_\mathbf{k}^*$. If we set

$$P_\mathbf{k}=a_\mathbf{k}, \quad Q_\mathbf{k}=-ia_{-\mathbf{k}}^*, \quad \omega_\mathbf{k}^0=\Omega_\mathbf{k},$$

then Hamiltonian (4) also takes the form in (7), but the condition on the reality of $p(x)$ and $q(x)$ changes: $Q^*_{-\mathbf{k}} = iP_\mathbf{k}, P^*_{-\mathbf{k}} = iQ_\mathbf{k}$.

Let us examine in more detail the structure of the linear unstable waves. From Hamiltonian (7) we find the equations of motion

$$dP_\mathbf{k}/dt=-i\Omega_\mathbf{k}P_\mathbf{k}, \quad dQ_\mathbf{k}/dt=-i\Omega_\mathbf{k}^*Q_\mathbf{k}. \quad (8)$$

A solution of these equations corresponding to a plane wave with a wave vector $\mathbf{q}$ is

$$\begin{aligned} P_\mathbf{k}&=\pi_\mathbf{q}\exp(-i\Omega_\mathbf{q}t)\delta(\mathbf{k}-\mathbf{q})+\pi_\mathbf{q}^*\exp(i\Omega_\mathbf{q}^*t)\delta(\mathbf{k}+\mathbf{q}), \\ Q_\mathbf{k}&=\xi_\mathbf{q}\exp(-i\Omega_\mathbf{q}^*t)\delta(\mathbf{k}-\mathbf{q})+\xi_\mathbf{q}^*\exp(i\Omega_\mathbf{q}t)\delta(\mathbf{k}+\mathbf{q}). \end{aligned} \quad (9)$$

where $\pi_\mathbf{q}$ and $\xi_\mathbf{q}$ are arbitrary constants. The wave energy is equal to the value of Hamiltonian (7) with solutions (9); i.e., $\varepsilon \propto \mathrm{Im}(\pi_\mathbf{q}^*\xi_\mathbf{q}\Omega_\mathbf{q}^*)$. We wish to stress that although the moduli of the canonical variables vary over time (in this particular representation, $|P|$ increases exponentially, while $|Q|$ decreases), the energy ε does no depend on the time, as it should not in a conservative medium.

From these rather trivial transformations falls an important circumstance: In speaking of the interaction of entities of some sort we usually mean that in the absence of the interaction the energy (and other integrals) of these entities remains constant. In the instability region, the pairs of waves in (9), characterized by the two complex amplitudes $\pi_\mathbf{q}$ and $\xi_\mathbf{q}$, play the role of these initial entities, while in the stability region the role is played by a single wave characterized by the amplitude $a_\mathbf{q}$. In the discussion below we will call the pairs of plane waves (or wave packets) of the type in (9) "unstable modes."

2. INTERACTION OF UNSTABLE MODES

If the canonical variables are initially quite small, we can expand the Hamiltonian in powers of P and Q. This expansion will of course be legitimate only up to a certain time, $t<T$. Nevertheless, we could have a situation in which the mode interaction time is shorter than T. In this case the interaction of the unstable modes can be treated by perturbation theory.

In this section of the paper we restrict the discussion to the cubic term in the expansion of the Hamiltonian in powers of the canonical variables. As we have already mentioned, the quadratic term in the expansion can be written in the form in (7), regardless of the stability of the medium. The next term in the expansion is of the form

$$\begin{aligned} H_1=\int d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \\ \times[P_{\mathbf{k}_1}P_{\mathbf{k}_2}P_{\mathbf{k}_3}V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}+Q_{\mathbf{k}_1}Q_{\mathbf{k}_2}Q_{\mathbf{k}_3}U_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \\ +P_{\mathbf{k}_1}Q_{\mathbf{k}_2}Q_{\mathbf{k}_3}S_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}+Q_{\mathbf{k}_1}P_{\mathbf{k}_2}P_{\mathbf{k}_3}T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}]. \end{aligned} \quad (10)$$

The condition $\mathrm{Im}H_1=0$ imposes certain restrictions on the matrix elements V, U, S, and T. The well-known procedure for putting a Hamiltonian in its simplest possible form is "classical perturbation theory," which can be outlined as follows: We take the canonical transformation

$$Q_k=Q_k'+f_1(Q',P'), \quad P_k=P_k'+f_2(P',Q'),$$

and we attempt to choose functionals $f_{1,2}$ such that the Hamiltonian H_1 vanishes in terms of the new variables. As a result of calculations similar to those in Ref. 1, we find that, formally, the terms P^3 and Q^3 can be discarded everywhere except for those vectors $\mathbf{k}_i$ for which the relations

$$\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0, \quad \Omega_{\mathbf{k}_1}+\Omega_{\mathbf{k}_2}+\Omega_{\mathbf{k}_3}=0 \quad (11)$$

hold, and the terms PQ^2 and QP^2 remain nonvanishing on the surface

$$\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0, \quad \Omega_{\mathbf{k}_1}+\Omega_{\mathbf{k}_2}^*+\Omega_{\mathbf{k}_3}^*=0. \quad (12)$$

These terms in Hamiltonian (10) are important not only on surfaces (11) and (12) but also in neighborhoods of these surfaces of size $\Delta\mathbf{k} \lesssim |PS||\partial\Omega_k/\partial k|^{-1}$, i.e., in neighborhoods in which the linear growth rates are small in comparison with the reciprocal of the scale time for the nonlinear interaction. Accordingly, for the unstable modes to interact with each other it is sufficient that the matching conditions be satisfied for only the real parts of the frequencies and of the wave vectors. In the present paper, however, we restrict the discussion to the interaction of modes with wave vectors which lie precisely on resonant surface (12).

If the $\mathbf{k}_i$ are d-dimensional vectors, Eqs. (11) and (12) describe $2(d-1)$-dimensional surfaces; i.e., the vectors $\mathbf{k}_i$ are generally not collinear. Since in this representation of the quadratic Hamiltonian (7) we have $\gamma_\mathbf{k} \geqslant 0$, the relation holds only if all the $\gamma_{\mathbf{k}_i}$ are zero, so that in the case of interest here the terms of the type P^3 and Q^3 in the interaction Hamiltonian are inconsequential. As for (12), we note that if all the $\gamma_{\mathbf{k}_i}$ are greater than zero then the corresponding terms in the Hamiltonian H_1 describe the interaction of three unstable modes; if, on the other hand, we have $\gamma_{\mathbf{k}_2}=0$ or $\gamma_{\mathbf{k}_3}=0$, then we are dealing with the interaction of two unstable modes and one stable one. These two cases correspond to different equations.

3. TRUNCATED EQUATIONS

We begin with the interaction of three unstable modes. For the Hamiltonian H_0+H_1, where H_0 is given by (7), and we have $U=0$ and $V=0$ in H_1 in (10), while S and T are nonzero near surface (12), the equations of motion are

$$dQ_{\mathbf{k}}/dt=-i\Omega_{\mathbf{k}}^{*}Q_{\mathbf{k}}+\int d\mathbf{k}_1\,d\mathbf{k}_2\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\times[Q_{\mathbf{k}_1}Q_{\mathbf{k}_2}S_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2}+2P_{\mathbf{k}_1}Q_{\mathbf{k}_2}T_{\mathbf{k}_2\mathbf{k}_1-\mathbf{k}}],$$
$$dP_{\mathbf{k}}/dt=-i\Omega_{\mathbf{k}}P_{\mathbf{k}}+\int d\mathbf{k}_1\,d\mathbf{k}_2\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\times[-2Q_{\mathbf{k}_1}P_{\mathbf{k}_2}S_{\mathbf{k}_2\mathbf{k}_1-\mathbf{k}}-P_{\mathbf{k}_1}P_{\mathbf{k}_2}T_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2}]. \quad (13)$$

The condition $\mathrm{Im}H_1=0$ in this case yields

$$S_{-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3}=S^{*}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3},\quad T_{-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3}=T^{*}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}.$$

From (13) we can easily find equations describing the interaction of packets of unstable modes. We assume $\mathbf{q}_1=\mathbf{q}_2+\mathbf{q}_3$, $\Omega_1=\Omega_2+\Omega_3(\Omega_i=\Omega_{\mathbf{q}_i})$. We seek a solution of Eqs. (13) in the form

$$P_{\mathbf{k}}=\sum_{i=1}^{3}[\pi_i(\mathbf{k}-\mathbf{q}_i,t)\exp(-i\Omega_i t)+\pi_i^{*}(-\mathbf{k}-\mathbf{q}_i,t)\exp(i\Omega_i^{*}t)],$$
$$Q_{\mathbf{k}}=\sum_{i=1}^{3}[\xi_i(\mathbf{k}-\mathbf{q}_i,t)\exp(-i\Omega_i^{*}t)+\xi_i^{*}(-\mathbf{k}-\mathbf{q}_i,t)\exp(i\Omega_i t)], \quad (14)$$

where the functions $\pi_i(\mathbf{k})$ and $\xi_i(\mathbf{k})$ are assumed to be nonzero in a sufficiently small neighborhood of $\mathbf{k}=0$. If we now substitute (14) into (13), expand $\Omega_{\mathbf{k}}$ around $\mathbf{k}=\mathbf{q}_i$, ignoring the dependence of S and T on $\mathbf{k}$, and take the inverse Fourier transforms, we find truncated equations for packets of unstable modes:

$$\frac{\partial\pi_1}{\partial t}+\mathbf{v}_1\frac{\partial\pi_1}{\partial\mathbf{x}}=-T^{*}\pi_2\pi_3,\quad \frac{\partial\xi_1}{\partial t}+\mathbf{v}_1^{*}\frac{\partial\xi_1}{\partial\mathbf{x}}=S^{*}\xi_2\xi_3,$$
$$\frac{\partial\pi_2}{\partial t}+\mathbf{v}_2\frac{\partial\pi_2}{\partial\mathbf{x}}=-S\pi_1\xi_3^{*},\quad \frac{\partial\xi_2}{\partial t}+\mathbf{v}_2^{*}\frac{\partial\xi_2}{\partial\mathbf{x}}=T\xi_1\pi_3^{*}, \quad (15)$$
$$\frac{\partial\pi_3}{\partial t}+\mathbf{v}_3\frac{\partial\pi_3}{\partial\mathbf{x}}=-S\pi_1\xi_2^{*},\quad \frac{\partial\xi_3}{\partial t}+\mathbf{v}_3^{*}\frac{\partial\xi_3}{\partial\mathbf{x}}=T\xi_1\pi_2^{*},$$

where $\mathbf{v}_i=\partial\Omega_i/\partial q_i$ are complex group velocities,

$$T=2(2\pi)^{d/2}T_{\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3},\quad S=2(2\pi)^{d/2}S_{\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3}.$$

We wish to stress that in deriving (15) we have made no assumptions of any sort regarding the time dependence of π_i and ξ_i. We also note that, without any loss of generality, we can set $S=T=1$; this can always be done by a scale transformation.

In this paper we write solutions of Eqs. (15) which do not depend on the spatial coordinates. It is not difficult to see that system (15) with $\partial/\partial x=0$ is a system of Hamilton's equations with the complex Hamiltonian

$$h=\pi_1\xi_2^{*}\xi_3^{*}+\xi_1^{*}\pi_2\pi_3 \quad (16)$$

and the Poisson brackets

$$\{f,g\}=\sum_{i=1}^{3}\left[\frac{\partial f}{\partial\xi_i^{*}}\frac{\partial g}{\partial\pi_i}-\frac{\partial f}{\partial\pi_i}\frac{\partial g}{\partial\xi_i^{*}}\right]. \quad (17)$$

This system has two additional integrals of motion:

$$I_\alpha=\pi_1\xi_1^{*}+\pi_\alpha\xi_\alpha^{*}, \quad (18)$$

where $\alpha=2,3$. The conditions $I_\alpha=\mathrm{const}$ are evidently analogs of the Manley-Rowe relations.[2] A direct check shows that the integrals I_α and h are in involution; i.e., their paired Poisson brackets vanish. Consequently, the system of Hamilton's equations (16), (17) is completely integrable.[3]

We transform to the new canonical momenta I_2, I_3, $p=\pi_1\xi_2^{*}\xi_3^{*}$ and coordinates $\Phi_\alpha=\ln\xi_\alpha^{*}$, $q=\xi_1^{*}/\xi_2^{*}\xi_3^{*}$. In terms of these variables, Hamiltonian (16) becomes

$$h=p+q(I_2-pq)(I_3-pq). \quad (19)$$

The trajectories of the system lie on the manifold Γ: $h=h_0$, where h_0 is determined by the initial conditions. It follows from the form of Hamiltonian (19) that the manifold Γ is a hyperelliptic Riemann surface of the first kind; i.e., the surface Γ is homeomorphic to a torus.[4] We known that in this case there exist doubly periodic functions $p(\zeta)$ and $q(\zeta)$ of a complex variable which perform a one-to-one mapping of the parallelogram of periods onto the Γ surface and thus satisfy the relation $h(p(\zeta),q(\zeta))=h_0$. Using formulas from the theory of elliptic functions,[5] we can easily find this mapping explicitly:

$$q(\zeta)=\frac{1}{2}\frac{\wp'(\zeta)-\wp'(\zeta_0)}{\wp(\zeta)-\wp(\zeta_0)}-\frac{h_0}{(I_2-I_3)^2},$$
$$p(\zeta)=\frac{1}{2q(\zeta)^3}\{q(\zeta)^2(I_2+I_3)-1+(I_2-I_3)[\wp(\zeta)-\wp(\zeta+\zeta_0)]\}, \quad (20)$$

where $\wp(\zeta)$ is the Weierstrass function with the invariants

$$g_2=\frac{4}{3}\frac{I_2^2-I_2I_3+I_3^2}{(I_2-I_3)^4},$$
$$g_3=-\frac{h_0^2}{(I_2-I_3)^6}+\frac{4}{27}\frac{(I_2+I_3)(2I_2-I_3)(2I_3-I_2)}{(I_2-I_3)^6},$$

and ζ_0 is determined by the equation

$$\wp(\zeta_0)=\frac{h_0^2}{(I_2-I_3)^4}-\frac{1}{3}\frac{I_2+I_3}{(I_2-I_3)^2}.$$

From the equations of motion for p and q we now find an equation for $\zeta(t)$:

$$d\zeta(t)/dt=I_2-I_3=\mathrm{const}. \quad (21)$$

Equations (20) and (21) give us the solutions of system (15).

Since the variable ζ is defined in a parallelogram with identical opposite sides, the trajectories of the system with Hamiltonian (19) wind around the torus Γ. Depending on the relation between the integrals I_α and h_0, these trajectories may be either closed or unclosed; if they are closed, there is a periodic exchange of energy between the interacting unstable modes.

As for the unclosed trajectories—this is the situation for most values of I_α and h_0—we note that they fill the entire torus Γ; i.e., a trajectory passes near any point of the surface. On Γ there exists a discrete set of points L_i near which these solutions are inapplicable; these points correspond to a situation in which one of the quantities π_i or ξ_i becomes infinite. A trajectory can pass through one of the points L_i only if the initial conditions are chosen in some special way. However, any unclosed trajectory sooner or later reaches the vicinity of any of the points L_i, and in this sense the interaction of the unstable modes is of an explosive nature for nearly all values of I_α and h_0. The dependence of the explosion time on the initial conditions is extremely odd.

Finally, we emphasize that all these processes unfold against the background of exponential changes in the amplitudes $P_{\mathbf{k}}$ and $Q_{\mathbf{k}}$; i.e., the scale time for the explosion must be shorter than the reciprocal of the linear growth rate.

4. SECOND-HARMONIC GENERATION

We now consider the interaction of two unstable modes and one stable mode. For definiteness we assume $\gamma_{\mathbf{k}_2} = 0$ we then find $\gamma_{\mathbf{k}_1} = \gamma_{\mathbf{k}_3}$ from (12). For simplicity we also set $\mathbf{k}_1 = \mathbf{k}_3 = \mathbf{q}$; there would be no difficulty in studying a more general case. We are thus studying the generation of a stable second harmonic of an unstable mode with a wave vector $\mathbf{q}$. The corresponding truncated equations are found by a method completely analogous to that used to derive Eqs. (15);

$$\frac{\partial a}{\partial t}+\mathbf{v}_2\frac{\partial a}{\partial \mathbf{x}}=-i\xi_1^*\pi_1^*,\quad \frac{\partial \pi_1}{\partial t}+\mathbf{v}_1\frac{\partial \pi_1}{\partial x}=-a^*\pi_1^*, \qquad (22)$$
$$\frac{\partial \xi_1}{\partial t}+\mathbf{v}_1^*\frac{\partial \xi_1}{\partial \mathbf{x}}=a^*\xi_1^*.$$

The fundamental unstable mode here is described by the pair of amplitudes π_1, ξ_1 with the wave vector $\mathbf{q}$, with the frequency $\Omega_{\mathbf{q}} = \omega_{\mathbf{q}} + i\gamma_{\mathbf{q}}$, and with the complex group velocity $\mathbf{v}_1$; the stable second harmonic is described by the single amplitude a with the wave vector $-2\mathbf{q}$, the frequency $-2\omega_{\mathbf{q}}$, and the group velocity $\mathbf{v}_2$. The Manley-Rowe relation expressing the conservation of energy and momentum takes the form $\pi_1\xi_1^* + i|a|^2 = iI =$ const in this case. In addition to the quadratic part of the energy, the interaction energy is of course also conserved: Re $(a\pi\xi) =$ const.

It is not difficult to derive an explicit solution of Eqs. (22). In the most interesting case we have $a = 0$ at the initial time; i.e., the interaction energy is zero. Restricting the discussion to this case, we find, with $\partial/\partial x = 0$,

$$|\pi_1|=C|\xi_1|,\quad |\pi_1\xi_1|=\frac{2|I|\,\mathrm{dn}(2|I|^{1/2}t)}{1+\mathrm{cn}(2|I|^{1/2}t)},$$
$$|a|=|I|^{1/2}\frac{1-\mathrm{cn}(2|I|^{1/2}t)}{\mathrm{sn}(2|I|^{1/2}t)}, \qquad (23)$$
$$\arg(a)=\mathrm{const},\quad \arg(a)+\arg(\pi_1)+\arg(\xi_1)=-\pi/2,$$
$$\sin\psi=\sin\psi_0\frac{1+\mathrm{cn}(2|I|^{1/2}t)}{2\,\mathrm{dn}(2|I|^{1/2}t)},$$

where dn, cn, and sn are the Jacobi elliptic functions of modulus $k = \cos(\psi_0/2)$; $\psi_0 = -\arg I$; and $\psi = \arg a + 2\arg\xi_1$.

Solutions (23) are explosive: All the amplitudes become infinite at the time $t_0 = |I|^{-1/2}K(k)$. If $|I|^{1/2} > \gamma_q$, then $P_k(t)$ decreases at $t \ll t_0$. In particular, if $\psi_0 = 0$ then $t_0 = \infty$, and the instability is stopped completely by the pumping of energy into the stable harmonic. The energy of the unstable mode in this notation is

$$\varepsilon\propto\omega_{\mathbf{q}}(|\pi_1\xi_1|\cos\psi-|a|^2)+\gamma_{\mathbf{q}}|\pi_1\xi_1|\sin\psi,$$

i.e., if $\psi_0 = 0$ the energy is determined exclusively by the real part of the frequency. Interestingly, damping of the second harmonic is not required for a dynamic stopping of the instability in this case.

5. CONCLUSION

There is yet another way in which Eqs. (15) appear. In a stable medium we know that truncated equations describing three-wave processes can be written in the form[7]

$$[J, Q_t]-[I, Q_x]+i[[J, Q], [I, Q]]=0, \qquad (24)$$

where the anti-Hermitian matrix Q is constructed from the amplitudes of the interacting waves, and the elements of the real diagonal matrices I and J are related by algebraic relations with the group velocities. If we discard the assumption that I and J are real—i.e., if we allow an amplification or absorption of the waves—then the reduction $Q^+ = -Q$ is no longer compatible with Eq. (24). In this case we must abandon the condition that Q is anti-Hermitian, and we find that Eqs. (15) and (24) become identical, as can be shown easily by direct calculation. System (15) thus turns out to be integrable by the method of the inverse scattering problem. It can be shown that the same is true of system (22).

In conclusion we consider a physical example to which these equations seem to apply: the two-stream instability of a plasma. If electrons are drifting (at a velocity greater than the thermal velocity) with respect to the ions in a plasma, then we know that an aperiodic Buneman instability can occur. This instability can be described in dissipationless two-fluid hydrodynamics, in which we can introduce canonical variables.[6] The scale times for the onset of this instability are easy to estimate. If the deviation of the electron density from a uniform distribution is initially characterized by a parameter $\delta n_e/n_0$, then the nonlinear growth rate for the most unstable resonant mode has the behavior $|I|^{1/2} \sim \omega_L(\delta n_e/n_0)$. The linear growth rate, in contrast, is given in order of magnitude by $\gamma_q \sim \omega_L(m/\mu)^{1/3}$($\omega_L$ is the electron plasma frequency, and m/μ is the ratio of the electron and ion masses). Accordingly, within the range of applicability of perturbation theory ($\delta n_e/n_0 \ll 1$), these equations can be used to analyze the Buneman instability.

We will also mention an example to which this theory does not apply. If a low-density electron beam penetrates a cold plasma, the medium will be unstable with respect to the excitation of plasma waves. In this case we would have $|I|^{1/2} \sim (\delta n_b/n_0)\gamma_q \ll \gamma_q$, and before the unstable modes which are excited have time to exchange energy the instability will be in the highly nonlinear regime, in which beam electrons are trapped in the wave field.

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