Anomalies in the transverse impedance of sound in ^{3}He

K. D. Ivanova and A. É. Meĭerovich

Institute of Physical Problems, Academy of Sciences of the USSR

(Submitted 12 June 1984)
Zh. Eksp. Teor. Fiz. **87**, 1984–1996 (December 1984)

We calculate the transverse impedance of ^{3}He for a general form of the *f*-function with a large number of harmonics. We study the anomalies of the impedance which are connected with the pressure threshold for the propagation of transverse zero sound. We determine close to the threshold the anomalous temperature dependence and the magnetic-field dependence of the real and the imaginary parts of the impedance for a zero-sound propagation speed close to the Fermi speed. We discuss the possibility of explaining the observed low-temperature high-frequency anomaly of the impedance on the basis of an additional relaxation mechanism connected with the emission of zero-sound quanta when quasiparticles collide.

1. INTRODUCTION

One of the most striking quantum features of normal liquid ^{3}He is the possibility[1] of the propagation of weakly damped high-frequency oscillations of the transverse-sound wave-kind which do not exist in ordinary liquids. Experimental work in this field reduces basically[2–4] to a measurement of the transverse impedance of ^{3}He. Any theoretical description of the propagation of transverse zero sound must thus be accompanied by the calculation of the transverse impedance.

The transverse impedance contains contributions both from the transverse zero sound and from the usual quasiparticles in ^{3}He and a determination of the parameters of the zero-sound spectrum is then possible only through a quantitative comparison of the experimental data with theory. Unfortunately, the contributions from the transverse zero sound and from the free quasi-particles to the impedance Z are of the same order of magnitude and are as a rule characterized by the same temperature and frequency dependence. In fact, the dependence of Z on the frequency ω and the temperature T is determined for $T \ll T_F$ (T_F is the degeneracy temperature) solely by the value of the parameter $\omega\tau$ ($\tau \propto 1/T^2$ is a characteristic relaxation time). The interpretation of the experimental data on the transverse impedance on the basis of some approximate theoretical model is thus somewhat arbitrary and does not always lead to a reliable result. When we evaluate the transverse impedance of ^{3}He we usually restrict ourselves,[5–13] because of technical difficulties, to taking into account merely two harmonics of the Fermi-liquid function, equating all other harmonics to zero the collision integral in the τ-approximation; is used for the determination of the damping. We show in the present paper that in a certain range of pressures the transverse impedance has singularities which lead to quite specific dependences of Z on the temperature and on the external magnetic field H. This may facilitate the identification of the transverse zero sound. All calculations are performed in a general form without choosing a particular form of the Fermi-liquid function. However, the analysis of the obtained general expression for the impedance is relatively simple only near the singularity.

According to the experimental data the propagation of transverse zero sound is apparently impossible at low pressures (the wave propagation velocity c turns out to be less than the Fermi velocity v_F). The existence of zero sound turns out to be possible only starting from some threshold pressure of the order of 8 atm. For pressures close to the threshold pressure the wave velocity is close to v_F and there occurs in the theory a natural small parameter $(c - v_F)/v_F \ll 1$. Taking such a small parameter into account leads to the appearance of anomalies in $Z(T,H)$ and facilitates the analyzing of the general expression for the impedance with an arbitrary *f*-function. An important circumstance is then the fact that close to the threshold even at low temperatures $T \ll T_F$ the set of quasiparticles moving in phase with the zero-sound wave and given by the value

$$\exp\{-(T_F/T)(c-v_F)/v_F\},$$

may turn out not to be small and it is necessary to take into account the collisionless damping of the zero sound.[14]

In the next section of the paper we obtain for $\omega\tau \to \infty$ an expression for Z at arbitrary form of the *f*-function, while in the vicinity of the singularity we separate explicitly the anomalous temperature dependences of $\operatorname{Im} Z(T)$ and $\operatorname{Re} Z(T)$. These results can easily be generalized to the case of an external magnetic field as the results turn out to be sensitive to a weak field only close to the singularity. Correspondingly we determine in the third section the function $Z(H,T)$. In the fourth section we generalize the expression for Z for arbitrary *f*-functions to the case of finite $\omega\tau$. In the last section of the paper we discuss the possibility of explaining the experimentally observed low-temperature, high-frequency anomaly of the impedance at high pressures on the basis of taking into account the mechanism proposed by Landau[1] for the damping of zero sound, connected with the emission of zero-sound quanta in quasiparticle collisions.

2. TRANSVERSE IMPEDANCE AS $\omega\tau \to \infty$.

To determine the transverse impedance as $\omega\tau \to \infty$ we shall solve the collisionless kinetic equation by the Wiener-Hopf method using a calculation scheme close to that in Refs. 5 to 7. A principal differences lies in the fact that we

0038-5646/84/121143-08$04.00

shall give the discussion for a general form of the Fermi-liquid function and shall not restrict ourselves to merely the first harmonics. The main difficulty is then connected with the fact that as the result of the factorization of the nondegenerate kernel of the integral equation the kinetic equation reduces to an infinite set of equations for which there does not exist a consistent general way of solving it by the Wiener-Hopf method. However, one can in this case succeed to construct a recurrence procedure for reducing this set to a single equation and the analysis of its features can be done relatively easily and one can now use for its solution the Wiener-Hopf method.

Another feature of the calculations performed here is that we take into account the possibility that the system may be close to the threshold for propagation of transverse zero sound:

$$|\ln|\alpha||\gg 1,\quad \alpha=(c-v_F)/v_F. \tag{1}$$

Far from the threshold (1) the temperature dependence of the impedance is as $\omega\tau\to\infty$, connected only with a small temperature spread of the quasi-particle equilibrium distribution function $n_0(\varepsilon)$ and reduces to a small correction of the order $(T/T_F)^2\ll 1$ which we shall neglect. Close to the threshold (1) there occurs an anomalous temperature dependence connected with the appearance of singular integrals of the kind[14]

$$\int d\varepsilon\int_{-1}^{1} dy\frac{\partial n_0}{\partial\varepsilon}\frac{f(x,y,\varepsilon)(1-y^2)}{(v_F-v)/v_F+xy}, \tag{2}$$

where ε and $\mathbf{v}$ are the quasi-particle energy and velocity while f is some smooth function. When condition (1) holds, when $1-x\sim\alpha$ the real part of the integrals (2) are of the order $\alpha|\ln|\alpha||\gg\alpha$ and the imaginary part is determined by $\exp(-\alpha T_F/T)$. This means that in all expression, apart from the integrals (2) close to the threshold (1), one can always assume that $\partial n_0/\partial\varepsilon=-\delta(\varepsilon-\varepsilon_F)$ and close to the threshold we must use a more exact expression for the equilibrium distribution function when evaluating the imaginary parts of (2). Moreover, even in the integrals (2) when condition (1) holds, if we neglect contributions of order α in comparison with $\alpha\ln\alpha$ we can, for any smooth function f in (2), replace ε by ε_F in the argument. Far from the threshold $f(\varepsilon)$ is also changed to $f(\varepsilon_F)$ as in that case always $\partial n_0/\partial\varepsilon=-\delta(\varepsilon-\varepsilon_F)$.

The kinetic equation for quasiparticles with a general form of f-function is an integrodifferential equation with a nondegenerate kernel. An expansion of the f-function in Legendre polynomials reduces this equation to an infinite set of integral equations with degenerate kernels. A Laplace transformation transforms this set to a set of linear functional equations. We reduce this set to a single functional equation with singularities determined by integrals of the type (2). The formal solution of this equation is easily obtained by the Wiener-Hopf method. Far from the threshold a direct use of this solution is very difficult since the higher harmonics of the f-function are not known at the present time and we shall show that the higher harmonics make an appreciable contribution to Z. However, if condition (1) holds the obtained solution can be used for a comparison with experiments thanks to its anomalous temperature dependence.

We shall evaluate the transverse impedance Z of the half-space $z>0$ of normal liquid ^{3}He for oscillations excited by transverse oscillations (in the x-direction) of the boundary plane $z=0$ with a frequency ω and a velocity $\mathbf{u}$ ($u_y=u_z=0$, $u_x\equiv u$). The linearized collisionless kinetic equation has the form

$$-i\omega\delta n+v\cos\theta\frac{\partial}{\partial z}\left\{\delta n-\frac{\partial n_0}{\partial\varepsilon}\int f(\mathbf{p},\mathbf{p}')\delta n'\,d\Gamma'\right\}=0, \tag{3}$$

where $\delta n(\mathbf{p},z)$ is the deviation of the quasi-particle distribution function from the equilibrium one $n_0(\varepsilon)$, $\delta n'=\delta n(\mathbf{p}',z)$, $d\Gamma=2d^3p/2\pi\hbar)^3$, θ is the angle between the momentum $\mathbf{p}$ and the z-axis, $f(\mathbf{p},\mathbf{p}')$ is the Fermi-liquid function averaged over spins, while the time-dependence is given by the factor $\exp(-i\omega t)$.

The boundary conditions for the distribution function are that there be no perturbations for incoming particles as $z\to\infty$:

$$n(\cos\theta<0,\ z\to\infty)=0 \tag{4}$$

and that the reflection of the quasi-particles from the moving boundary $z=0$ be diffuse:

$$n(\cos\theta>0,z=0)=n_0(\varepsilon+\delta\varepsilon-\mathbf{pu}),$$
$$\delta\varepsilon=\int f(\mathbf{p},\mathbf{p}')\delta n'\,d\Gamma'. \tag{5}$$

In fact, for a dissipative medium the boundary condition (4) can be replaced by

$$n(z\to\infty)=0.$$

The Fermi-liquid function f depends for an isotropic medium only on the quantities $|\mathbf{p}|$, $|\mathbf{p}'|$, and the angle x between the vectors $\mathbf{p}$ and $\mathbf{p}'$:

$$\cos\chi=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi'),$$

and the equations for oscillations with different azimuthal numbers separate. In that case the boundary condition (5) distinguishes only oscillations with azimuthal number $m=1$. According to what we said above we must put $|\mathbf{p}|=|\mathbf{p}'|=p_F$ in the argument of f and use the normal expansion in Legendre polynomials

$$f(\cos\chi)=\frac{d\Gamma}{d\varepsilon}\sum_l F_lP_l(\cos\chi),$$

where according to the folding theorem

$$P_l(\cos\chi)$$
$$=P_l(y)P_l(y')+2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(y)P_l^m(y')\cos(m[\varphi-\varphi']),$$

with $y=\cos\theta$, $y'=\cos\theta'$. After multiplication by φ and integration over φ, Eq. (3) becomes

$$-i\omega n_1+vy\frac{\partial n}{\partial z}\left\{n_1-\frac{1}{2\sin\theta}\frac{\partial n_0}{\partial\varepsilon}\times\sum_l\frac{F_l}{l(l+1)}P_l^1(y)\int d\varepsilon'\,dy' n_1'\sin\theta' P_l^1(y')\right\}=0,\tag{6}$$

where we used the substitution

$$\delta n=\sin\theta\cos\varphi n_1(\varepsilon,y,z).$$

It is convenient to introduce as new variables the functions

$$\nu(\varepsilon,y,z)=n_1-\frac{1}{2\sin\theta}\frac{\partial n_0}{\partial\varepsilon}\sum_l\frac{F_l}{l(l+1)}P_l^1(y)\times\int d\varepsilon'\,dy' n_1'\sin\theta' P_l^1(y'),$$

$$\nu_l(z)=\int d\varepsilon\,dy\nu\sin\theta P_l^1(y)$$

and make the substitution $z\to(iv_F/\omega)z$. Equation (6) then takes the form

$$\nu+\frac{1}{2\sin\theta}\frac{\partial n_0}{\partial\varepsilon}\sum_l\frac{F_l/l(l+1)}{1+F_l/(2l+1)}\nu_l P_l^1(y)+\frac{v}{v_F}y\frac{\partial}{\partial z}\nu=0,\tag{7}$$

and the boundary condition (5) becomes

$$\nu(y>0,\ z=0)=-p_F u\partial n_0/\partial\varepsilon.\tag{8}$$

As a result of Laplace transforming the functions ν and ν_l

$$\nu(x)=\int_0^\infty e^{-xz}\nu(z)\,dz$$

Eq. (7) reduces to

$$\left(1+xy\frac{v}{v_F}\right)\nu(x)+\frac{1}{2\sin\theta}\frac{\partial n_0}{\partial\varepsilon}\sum_l\frac{F_l/l(l+1)}{1+F_l/(2l+1)}P_l^1(y)\nu_l(x)=\frac{v}{v_F}y\nu\ (z=0).\tag{9}$$

We multiply Eq. (9) by $\sin\theta P_l^1(y)$ and integrate over $d\varepsilon dy$. As the coefficients in (9) do not contain the singular integrals (2), there appears the following set of equations for the functions ν_l:

$$\nu_l\left[1+\frac{F_l}{2l+1}\right]^{-1}+\frac{x}{2l+1}\{l\nu_{l+1}+(l+1)\nu_{l-1}\}=\int d\varepsilon\,dy\sin\theta yP_l^1(y)\nu(z=0).\tag{10}$$

The function $\nu_l(x)$ does not have singularities as $x\to0$. This means that the quantities on the right-hand side of (10) are equal to $\nu_l(0)[1+F_l/(2l+1)]^{-1}$ and the set (17) has the form

$$\left[1+\frac{F_l}{2l+1}\right]^{-1}\{\nu_l-\nu_l(0)\}+\frac{x}{2l+1}\{l\nu_{l+1}+(l+1)\nu_{l-1}\}=0.\tag{11}$$

Equations (11) allow us to express all functions $\nu_l(x)$ in terms of the function $\nu_1(x)$ and its derivatives at the point $x=0$:

$$\nu_l(x)=\sum_{k=0}^{l-1}A_{kl}\varphi_k,$$

$$\varphi_k(x)=\frac{1}{x^k}\left\{\nu_1(x)-\nu_1(0)-x\nu_1'(0)-\ldots-x^{k-1}\frac{\nu_1^{(k-1)}(0)}{(k-1)!}\right\},$$

$$\varphi_0(x)=\nu_1(x),\quad \varphi_k(0)=\nu_1^{(k)}(0)/k!,\quad x\varphi_{k+1}=\varphi_k-\varphi_k(0),\tag{12}$$

where the constant coefficients A_{kl} are determined by the recurrence relations

$$A_{k,l+1}=-\frac{2l+1}{l}\frac{1}{1+F_l/(2l+1)}A_{k-1,l}-\frac{l+1}{l}A_{k,l-1},$$
$$A_{01}=1,\quad A_{02}=0,\quad A_{kl}(k>l-1)=0.\tag{13}$$

In this way Eq. (9), which can easily be transformed to the form

$$\nu_1+\frac{1}{2}\int\frac{d\varepsilon\,dy}{v_F/v+xy}\frac{\partial n_0}{\partial\varepsilon}P_1^1(y)\sum_l\frac{F_l/l(l+1)}{1+F_l/(2l+1)}P_l^1(y)=-\int\frac{(1-y^2)y\nu(z=0)}{v_F/v+xy}d\varepsilon\,dy,\tag{14}$$

turns out to be an equation for only the function $\nu_1(x)$.

We can write the function $P_1^1P_l^1/(xy+v_F/v)$ on the left-hand side of Eq. (14) in the form

$$\frac{P_1^1P_l^1}{xy+v_F/v}\equiv\frac{1-y^2}{xy+v_F/v}\frac{dP_l}{dy}=\sum_{k=0}^{l}R_{kl}P_k(y)+\frac{Z_l(y^2-1)}{xy+v_F/v},\tag{15}$$

where the functions $R_{0l}(x)$ and $Z_l(x)$ are polynomials of degree $l-1$ in $1/x$. As a result the left-hand side of (14) takes the form

$$\nu_1+\sum_l\frac{F_l/l(l+1)}{1+F_l/(2l+1)}\nu_l\left\{-R_{0l}+\frac{2}{3}Z_lU(x)\right\},$$

$$U(x)=\frac{3}{4}\int\frac{\partial n_0}{\partial\varepsilon}\frac{(y^2-1)\,d\varepsilon\,dy}{v_F/v+xy}.\tag{16}$$

Using the boundary condition (8) we can write the right-hand side of (14) in the form

$$-\int\frac{(1-y^2)y\nu(z=0)}{v_F/v+xy}d\varepsilon\,dy=\int\frac{\partial n_0}{\partial\varepsilon}\frac{p_Fu(1-y^2)y}{v_F/v+xy}d\varepsilon\,dy-\int d\varepsilon\int dy\frac{(1-y^2)y}{v_F/v+xy}\left[p_Fu\frac{\partial n_0}{\partial\varepsilon}+\nu(z=0)\right]=-\frac{4}{3}\frac{p_Fu}{x}[1-U(x)]+Q(x),$$

$$Q(x)=-\int d\varepsilon\int_{-1}^{0}dy\frac{(1-y^2)y}{v_F/v+xy}\left[\nu(z=0)+p_Fu\frac{\partial n_0}{\partial\varepsilon}\right],\tag{17}$$

and Eq. (14) becomes

$$\nu_1-\sum_l\frac{F_l/l(l+1)}{1+F_l/(2l+1)}\nu_l\left\{R_{0l}-\frac{2}{3}Z_lU(x)\right\}=-\frac{4}{3}\frac{p_Fu}{x}[1-U(x)]+Q(x).\tag{18}$$

According to (12) the function

$$\Phi(x)=\frac{2}{3}\sum_l \frac{F_l/l(l+1)}{1+F_l/(2l+1)} Z_l\nu_l(x),$$
$$\Psi(x)=\nu_1(x)-\sum_l \frac{F_l/l(l+1)}{1+F_l/(2l+1)} R_{0l}(x)\nu_l(x) \tag{19}$$

can be written in the form

$$\Phi(x)=\Phi_1(x)\nu_1(x)+\Phi_2(x),\quad \Psi(x)=\Psi_1(x)\nu_1(x)+\Psi_2(x),$$
$$\Phi_1(x)=\frac{2}{3}\sum_{l>0}\sum_{k=0}^{l-1}\frac{F_l/l(l+1)}{1+F_l/(2l+1)}\frac{A_{kl}}{x^k}Z$$
$$\Phi_2(x)=\frac{2}{3}C\,\nu^s(0)$$
$$C_s=\sum_{l>s-1}\sum_{k=s+1}^{l-1}\frac{F_l/l(l+1)}{1+F_l/(2l+1)}Z_l(x)A_{kl}\frac{x^{s-k}}{s!}, \tag{20}$$

and similarly for $\Psi_{1,2}$. In this notation Eq. (14) has the form

$$\Delta\left\{x\left[\nu_1+\frac{\Phi_2}{\Phi_1}\right]-\frac{4}{3}\frac{p_F u}{\Phi}\right\}=-\frac{4}{3}p_F u\left(1+\frac{\Psi_1}{\Phi}\right)$$
$$+x\left(-\Psi_2+\frac{\Phi_2\Psi_1}{\Phi_1}\right)+xQ, \tag{21}$$
$$\Delta(x)=\Psi_1(x)+\Phi_1(x)U(x).$$

We shall solve Eq. (21) by factorization using the Wiener-Hopf method. For a degenerate system the function $\partial n_0/\partial\varepsilon$ is nonvanishing only in a narrow region around ε_F. We shall assume that for some $0<s<1-\alpha$ the function $\partial n_0/\partial\varepsilon=0$ for $v/v_F<s$. The function $U(x)$ in (16) then has no singularities in the range $|\mathrm{Re}(x)|<s$. In that case $\Delta(x)$ in (21) also has no singularities in that range as the divergence of the polynomials $\Psi_{1,2}$ and $\Phi_{1,2}$ as $x\to 0$ is unimportant, inasmuch as for $x\to 0$ we have

$$(\Psi_1+\Phi_1 U)\to 1/(1+F_1/3),\qquad (\Psi_2+\Phi_2 U)\to 0.$$

Moreover, $\Delta(x\to\infty)=1$ and the equation $\Delta(x)=0$ has, by assumption, a single solution $x=x_0$ determining the velocity of zero-sound propagation $\mathrm{Re}\,x_0=1-\alpha$. Accordingly the function

$$G(x)=\Delta(x)(x^2-s^2)/(x^2-x_0^2) \tag{22}$$

does not have any singularities in the range $|\mathrm{Re}(x)|<s$ or zeroes; $G(\infty)=1$. The function G factorizes in the usual way:

$$G=\frac{g_+}{g_-},\qquad \ln g_\pm=\frac{1}{2\pi i}\int_{\pm s-i\infty}^{\pm s+i\infty}\frac{\ln G(t)}{t-x}dt, \tag{23}$$

where the functions $g_\pm$ are analytical in the half-planes $\pm\,\mathrm{Re}\,x<s$. As a result of standard calculations (cf., e.g., Ref. 7) we get for $g_\pm$ the following expressions:

$$g_+(-x)=1/g_-(x),\quad g_-(x)=G^{-1/2}(x)\exp(-x\zeta),$$
$$\zeta(x)=-\frac{1}{2\pi i}\int_s^\infty\frac{\ln\Delta_+(t)-\ln\Delta_-(t)}{t^2-x^2}dt,$$
$$\Delta_\pm=\Psi_1+\Phi_1U_\pm, \tag{24}$$

where the $U_\pm$ are the values of the function $U(t)$ in (16) above and below the cut along the real axis $s<t<\infty$.

Using (22) and (23) we can transform Eq. (21) to the form

$$\frac{x+x_0}{x+s}\frac{1}{g_-}\left\{x\left(\nu_1+\frac{\Phi_2}{\Phi_1}\right)-\frac{4}{3}\frac{p_F u}{\Phi_1}\right\}$$
$$=\frac{x-s}{x-x_0}\frac{1}{g_+}\left\{xQ+x\left(\frac{\Phi_2\Psi_1}{\Phi_1}-\Psi_2\right)\right.$$
$$\left.-\frac{4}{3}p_F u\left(\frac{\Psi_1}{\Phi_1}+1\right)\right\}. \tag{25}$$

The coefficients in (25) have singularities connected with the zeroes of $\Phi_1(x)$. A simple analysis of Eqs. (10), (12), (15), and (20) shows that $\Phi_{1,2}$ and $\Psi_{1,2}$ are even functions (polynomials) of $1/x$ and Φ_1 can be written in the form

$$\Phi_1(x)=\Lambda\prod_i\left(\frac{1}{x_i^2}-\frac{1}{x^2}\right)\qquad \mathrm{Re}\,x_i>0.$$

One can remove the singularities in (25) connected with the points $x=x_i$ by adding to the right- and left-hand sides of (25) pole terms

$$-\sum_i\frac{1}{x-x_i}\frac{x_i+x_0}{x_i+s}\frac{1}{2x_i\Phi_{1i}g_{-i}}\left\{x\Phi_{2i}-\frac{4}{3}p_F u\right\}, \tag{26}$$

and the singularities in the points $x=-x_i$ are removed by adding

$$\sum_i\frac{1}{x+x_i}\frac{x_i+s}{x_i+x_0}\frac{\Psi_{1i}}{2x_i\Phi_{1i}g_{+i}}\left\{x\Phi_{2i}-\frac{4}{3}p_F u\right\}, \tag{27}$$

where

$$g_{-i}=g_-(x_i),\quad g_{+i}=g_+(-x_i),$$
$$\Psi_{1i}=\Psi_1(x_i),\quad \Phi_{2i}=\Phi_2(x_i),$$
$$\Phi_{1i}=\{\Phi_1/(x^2-x_i^2)\}_{x=x_i}.$$

In that case the singularities in the half-plane $\mathrm{Re}\,x<s$ vanish on the right-hand side of (25), and those for $\mathrm{Re}\,x>-s$ on the left-hand side; the vanishing of the singularities in the range $|\mathrm{Re}\,x|<s$ is connected with the fact that in that range

$$g_+/g_-=[(x^2-s^2)/(x^2-x_0^2)](\Psi_1+\Phi_1U)$$

and for $|\mathrm{Re}\,x_i|<s$

$$\frac{\Psi_{1i}}{g_+(x_i)}=\frac{x_i^2-x_0^2}{x_i^2-s^2}\frac{1}{g_-(x_i)}.$$

The function ν_1 is analytical for $\mathrm{Re}\,x>0$ and thereby the whole of the left-hand side of (25), taking the addition of (26) and (27) into account, turns out to be analytical in the half-plane. The right-hand side is then analytical in the half-plane $\mathrm{Re}\,x<s$, as the function Q of (17) is analytical in the half-plane. As a result the analytical function which coincides the right-hand side of the equation for $\mathrm{Re}\,x<s$ and with left-hand side for $\mathrm{Re}\,x>s$ turns out to be a constant. We determine the value of the constant in the point $x=0$ and substitute it into Eqs. (25) to (27):

$$v_1(x)=\frac{4}{3}p_F u$$
$$\times\left\{\frac{1}{x\Phi_1}+\frac{x+s}{x+x_0}g_- -\sum_i\frac{1}{2x_i\Phi_{1i}}\frac{x_i+x_0}{x_i+s}\left[\frac{1}{g_{-i}(x-x_i)}\right.\right.$$
$$\left.\left.-\frac{\Psi_{1i}}{(x+x_i)g_{+i}}\left(\frac{x_i+s}{x_i+x_0}\right)^2\right]\right\}-\frac{x+s}{x+x_0}g_-\sum_i\Phi_{2i}$$
$$\times\frac{1}{2x_i\Phi_{1i}}\frac{x_i+x_0}{x_i+s}\left\{\frac{1}{g_{-i}(x-x_i)}+\frac{\Psi_{1i}}{(x+x_i)g_{+i}}\left(\frac{x_i+s}{x_i+x_0}\right)^2\right\}-\frac{\Phi_2}{\Phi_1}.\tag{28}$$

Expression (28) is merely the purely formal solution of the kinetic equation, as the function $\Phi_2(x)$ of (20) contains in it as parameters a linear combination of derivatives $\nu_1^{(n)}(x)$ at the point $x=0$. We determine the values of $\nu^{(n)}(0)$ by differentiating (28) and solving the resultant set of linear equations. As the structure of the functions φ_k of (12) is such that

$$\frac{d^k}{dx^k}\left(v_1+\frac{\Phi_2}{\Phi_1}\right)_{x\to 0}=0,$$

the corresponding set of equations for the $\nu^{(n)}(0)$ has the form ($k=0; 1; 2\ldots$)

$$\sum_i\frac{1}{2x_i\Phi_{1i}}\frac{x_i+x_0}{x_i+s}\frac{d^k}{dx^k}\left\{\frac{4}{3}p_F u\left[\frac{1}{(x-x_i)g_{-i}}\right.\right.$$
$$\left.-\frac{\Psi_{1i}}{(x+x_i)g_{+i}}\left(\frac{x_i+s}{x_i+x_0}\right)^2\right]$$
$$\left.-\Phi_{2i}\left[\frac{1}{(x-x_i)g_{-i}}+\frac{\Psi_{1i}}{(x+x_i)g_{+i}}\left(\frac{x_i+s}{x_i+x_0}\right)^2\right]\right\}_{x\to 0}=0.\tag{29}$$

The parameters can tend to zero and the set (29) changes to

$$B_{ki}\Phi_{2i}=\frac{4}{3}p_F uR_k,\qquad B_{ik}=\frac{1}{x_i^{k+1}}[A_i+(-1)^kB_i],$$
$$R_k=\sum_i\frac{1}{x_i^{k+1}}[A_i-(-1)^kB_i],$$
$$A_i=\frac{1+x_0/x_i}{\Phi_{1i}g_{-i}},\qquad B_i=\frac{\Psi_{1i}}{(1+x_0/x_i)\Phi_{1i}g_{+i}},\tag{30}$$

and the solution of this set is equal to

$$\mathbf{v}(0)=-\frac{3}{2}\hat{C}^{-1}\mathbf{\Phi}_2=-2p_F u\hat{C}^{-1}\hat{B}^{-1}\mathbf{R},$$
$$\mathbf{v}(0)\equiv\{v_1^{(s)}(0)\},\quad \mathbf{\Phi}_2\equiv\{\Phi_{2i}\},\quad C_{is}=C_s(x_i),\tag{31}$$

where the functions $C_s(x_i)$ are given by Eq. (20).

We are interested in the transverse impedance Z which is connected by definition with the momentum flux tensor $\Pi_{ik}(z)$ through the relation

$$Z=\frac{1}{u}\Pi_{xz}(z=0).\tag{32}$$

The momentum flux tensor is equal to

$$\Pi_{xz}(z=0)\equiv\int d\Gamma\frac{p_xp_z}{M}\left\{\delta n(z=0)-\frac{\partial n_0}{\partial\varepsilon}\delta\varepsilon(z=0)\right\}$$
$$=\frac{p_F^3}{4\pi^2\hbar^3}\int d\varepsilon\, dy\,(1-y^2)\,y\nu(z=0),\tag{33}$$

where M is the effective quasiparticle mass of ^{3}He. We can determine the last integral in (33) from the first of Eqs. (10) with $x\to 0$ and it turns out to be equal to $-\nu_1(0)/(1+F_1/3)$. As a result we get from (31) to (33)

$$Z=\frac{3}{2}p_F NC_{0i}^{-1}B_{ik}^{-1}R_k,$$
$$N=p_F^3/3\pi^2\hbar^3,\tag{34}$$

where N is the density of the ^{3}He particles.

Equation (34) enables us to determine the impedance $Z(\omega\tau\to\infty)$ taking any number of harmonics into account. The problem in that case in fact reduces to determining the roots x_i of the equation $\Phi_1(x_i)=0$ for $\Phi_1(x)$ of (20) and the root x_0 of the equation $\Delta(x_0)=0$ for $\Delta(x)$ of (21). However, it is difficult to use (34) directly, as we cannot from a comparison with experimental data determine right away all parameters $F_{l>1}$ which occur in (34).

Far from threshold the impedance (34) is purely real as in that case

$$U_\pm(x)=\frac{3}{4}\left\{\frac{2}{x^2}+\frac{x^2-1}{x^3}\left[\ln\left|\frac{1+x}{1-x}\right|\pm i\pi\theta(|x|-1)\right]\right\},\tag{35}$$

x_0 is real and all the roots x_i come in complex conjugate pairs. Because of (35) there is then also no temperature dependence of Re Z. Close to the threshold $|\alpha|\ll 1$ the situation changes. Instead of (35) we have ($|1-x|\ll 1$)

$$\frac{4}{3}U_\pm(x)=\frac{2}{x^2}+\frac{1}{x^3}\left\{(x^2-1)\ln\left|\frac{1+x}{1-x}\right|\pm i\pi\Omega(x)\right\},$$
$$\Omega(x)=\frac{T}{T_F}\ln\left\{1+\exp\left(-\frac{2T_F}{T}\frac{1-x}{x}\right)\right\},\tag{36}$$

while x_0, at our accuracy $|\ln|\alpha||\gg 1$, is equal to (cf. Ref. 14).

$$x_0=1+ix_0'',\quad x_0''=\frac{\pi T}{2T_F\ln|\alpha|}\ln\left\{1+\exp\left(-\frac{2T_F}{T}\alpha\right)\right\}\tag{37}$$

The appearance of an imaginary part for x_0 causes an appreciable imaginary part for Z and the temperature dependence $\Omega(T)$ leads to a temperature dependence of the phases ζ of (24). As the temperature correction to Ω in (36) fast tends to zero far from the point $x=1$ the phases ζ of (24) are given by the relation

$$\zeta(x_i)=\zeta_0(x_i)+\delta\zeta(T)/(x_i^2-1),$$

where

$$\delta\zeta(T)=-2\int_0^\infty dt\frac{f(t)\,\partial\Omega/\partial t+(\ln|1-t|+1)\Omega}{4f^2(t)+\pi^2\Omega^2}$$
$$+\int_1^\infty dt\frac{(\ln|1-t|+1)(t-1)/t+f(t)/t^2}{f^2(t)+\pi^2(t-1)^2/t^2}$$
$$f(t)=\alpha\ln\alpha+(1-t)\ln|1-t|.$$

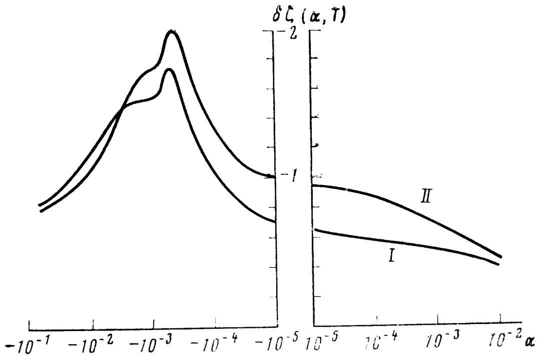

FIG. 1. Plot of the function $\delta\zeta(\alpha)$ of (38) for two values of the temperature: I) $T/T_0 = 0.01$; II) $T/T_0 = 0.001$.

A plot of the function $\delta\zeta(T)$ is shown in Fig. 1, and ζ_0 is the value of the phases as $T\to 0$. The impedance (34) then takes the form

$$Z=\frac{3}{2}p_F N C_{0i}^{-1}B_{ik}^{-1}\{R_k+[\delta\zeta(T)+ix_0''](r_k-B_{kl}^{-1}b_{lm}R_m)\},\tag{38}$$

where we have introduced the notation

$$b_{ik}=\frac{x_i^k}{x_i^2-1}[A_i-(-1)^k B_i],$$
$$r_k=\sum_i \frac{x_i^k}{x_i^2-1}[A_i+(-1)^k B_i],$$

and the components of the vector $\mathbf{R}$ and of the matrices $\hat{B}$ and $\hat{C}$ must be evaluated at $T=0$. Equations (36) to (38) also determine the anomalous temperature dependence of the real and imaginary parts of the impedance near the threshold.

3. MAGNETIC ANOMALIES OF THE IMPEDANCE

Far from the singular point (threshold), the effect of a magnetic field H on the transverse impedance Z is, like that on most of the other Fermi-liquid characteristics of normal ^{3}He, small because the parameter

$$h=\beta H/T_F(1+F_0^a),\tag{39}$$

is small; here $\beta\sim 0.08$ mK/kOe is the magnetic moment of the ^{3}He nucleus, F_0^a is the zeroth harmonic of the anti-symmetric part of the Fermi-liquid function. All corrections to the impedance are then of the order of h^2 and are hardly observable in realistically reachable fields. However, close to the singularity the magnetic-field dependence becomes important as even a small shift in the Fermi velocities

$$v_\pm=v_F(1\pm h/2)$$

for particles with different spin orientations lead (in the α scale) to an appreciable shift of the poles of the singular integrals (2). There then appear in the problem instead of α two small parameters:

$$\alpha_\pm=(c-v_\pm)/v_\pm=\alpha\mp h/2.\tag{40}$$

The accuracy of the further calculations near the singularity will, as above, imply that $|\ln|\alpha_\pm||\gg 1$ and we shall neglect terms of order h as compared to $h\ln h$. This means[14,15] that we can neglect all regular corrections in the magnetic field and restrict ourselves to taking the field into account only in the integrals (2), i.e., in fact, only in $U(x)$ of (16).

As a result the kinetic equation, the boundary condition, and the calculation scheme are the same as in the preceding section and the only difference is that in the expression for $U(x)$ of (16) we must make the change

$$\frac{\partial n_0}{\partial\varepsilon}=\frac{1}{2}\left(\frac{\partial n_+}{\partial\varepsilon_+}+\frac{\partial n_-}{\partial\varepsilon_-}\right),$$

where $n_\pm$ and $\varepsilon_\pm$ are the equilibrium distribution function and the energy of quasi-particles with different spin orientations. This leads to a change in the functions Ω of (36):

$$\Omega(T,H)=\frac{1}{2}(\Omega_++\Omega_-),$$
$$\Omega_\pm=\frac{T}{T_F}\ln\left\{1+\exp\left(-\frac{2T_F}{T}\frac{1-x_\pm}{x_\pm}\right)\right\},\tag{41}$$

where $x_\pm=xv_\pm/v_F$ and correspondingly to a change in x_0'' of (37):

$$x_0''(H,T)=\frac{\pi}{4}\left\{\frac{\Omega_+(1-x_+=\alpha_+)}{\ln|\alpha_+|}+\frac{\Omega_-(1-x_-=\alpha_-)}{\ln|\alpha_-|}\right\}.\tag{42}$$

The temperature dependence and the magnetic-field dependence of the impedance are then determined by Eq. (38) with the substitution of (41) and (42) for (36) and (37). We note the vanishing of the coefficient α_+ of (40) corresponds to the effect of the suppression of transverse zero sound by a weak magnetic field.[14,15]

4. TRANSVERSE IMPEDANCE FOR FINITE $\omega\tau$.

The general expression (34) for the transverse impedance can easily be generalized also to the case of finite $\omega\tau<\infty$. If we take collisions into account the results become considerably more complicated although when we take a large number of harmonics into account the calculation scheme remains as before.

On the right-hand side of the kinetic Eq. (3) we must add the collision integral $I(n)$. One shows easily (cf. Refs. 6 and 11) that if we take into account the conservation laws we can in the τ-representation write $I(n)$ in the form of the following expansion in harmonics:

$$I(\delta n)=-\frac{1}{\tau}\delta\tilde{n}-\sum_{l=2}^{\infty}(2l+1)\left(\frac{1}{\tau_l}-\frac{1}{\tau}\right)\int d\Gamma' P_l(\cos\chi)\delta\tilde{n}',$$

$$\delta\tilde{n}=\delta n-\frac{dn_0}{\partial\varepsilon}\int d\Gamma'\delta n'\sum_{l=2}^{\infty}F_lP_l(\cos\chi)$$
$$+3\frac{\partial n_0}{\partial\varepsilon}\int d\Gamma'\,\delta n' P_1(\cos\chi),\tag{43}$$

where $\tau_l = \lambda_l/\tau$ are the relaxation times for the various harmonics, $\tau \propto 1/T^2$, while the constants λ_l are independent of the temperature. After integration over $d\varphi$ and the introduction of the new variable n_1 from (6), the kinetic Eq. (3) with the right-hand side (43) becomes the equation (cf. (6))

$$\left(-i\omega+\frac{1}{\tau}\right)n_1+vy\frac{\partial}{\partial z}n_1 + \frac{\partial n_0/\partial\varepsilon}{2\sin\theta}\sum_{l>0}\frac{F_l}{l(l+1)}P_l^1(y)\left\{-vy\frac{\partial}{\partial z} + \frac{2l+1}{\tau}\left[\frac{1}{F_l}-\lambda_l\left(\frac{1}{F_l}+\frac{1}{2l+1}\right)\right]\right\} \times\int d\varepsilon'\,dy'\,P_l^1(y')\sin\theta' n_1'=0, \quad (44)$$

where by definition $\lambda_1 \equiv 0$. Moreover, as in the second section of this paper, we introduce the variable ν and perform a Laplace transformation. The kinetic Eq. (44) is then easily brought to the form (9) accurate to a simple transformation:

$$F_l\to\tilde{F}_l=F_l+\frac{2l+1}{1-i\omega\tau}\left\{1-\lambda_l\left[1+\frac{F_l}{2l+1}\right]\right\}\equiv F_l+\delta F_l. \quad (45)$$

As a result the transverse impedance Z is given as before by Eq. (34) also for finite $\omega\tau$, if we take into account the notation in (45). However, in this case the analysis of (34) is considerably more complex even when $\omega\tau \gg 1$, when

$$\delta F_l = \frac{i}{\omega\tau}\left[1-\frac{i}{\omega\tau}\right](2l+1)\left[1-\lambda_l\left(1+\frac{F_l}{2l+1}\right)\right]. \quad (46)$$

The imaginary part of Z arises both from the direct effect of complex corrections to F_l (the matrix $\hat{C}$ of (20), (31)) and as the result of the fact that the coefficients of the polynomial Φ_1 of (20) become complex and its roots x_i are no longer complex conjugates when $\omega\tau \gg 1$

$$\delta x_i=\sum_l\frac{\partial x_i}{\partial F_l}\delta F_l, \qquad \delta Z=\frac{1}{2}\sum_{i,l}\left\{\frac{\partial Z}{\partial x_i}+\frac{\partial Z}{\partial x_i^*}\right\}\frac{\partial x_i}{\partial F_l}\delta F_l.$$

Moreover, also important is the appearance in the phases ζ [Eq. (24)] of an imaginary part connected with the presence of imaginary parts of the coefficients of the polynomials Ψ_1 and Φ_1 in (20) and the expressions for $\Delta_\pm$ of (24):

$$\delta\zeta(x)=\frac{1}{\pi}\int^{\infty}\frac{(\delta\Psi_1\Phi_1+\delta\Phi_1\Psi_1)\,\mathrm{Im}\,U_+}{(x^2-x_i^2)\Delta_+\Delta_-}.$$

It is clear that when $\omega\tau \gg 1$, according to (46), the correction is of the order of $1/\omega^2\tau^2$ to the real part of the impedance and of the order $1/\omega\tau$ to the imaginary part.

Under the condition $\alpha > 0$ the contribution to Im Z of the collisionless damping (37), (38) exceeds the collisional corrections $1/\omega\tau$ at low temperatures $[\ln(T_F/T) \sim |\ln\alpha| \gg 1]$ and at high frequencies: $\hbar\omega \sim T$. When $\alpha < 0$ (but $|\ln|\alpha|| \gg 1$) the restrictions on frequency and temperature are no longer so rigid. The experimental separation of the collisionless contribution to Im Z is apparently simplest due to the anomalous magnetic field dependence (42) in weak fields: in such fields the collisional correction to Z is not at all sensitive to the field.

5. HIGH-FREQUENCY ANOMALY OF THE THE IMPEDANCE.

Measurements of the transverse impedance[4] revealed a hitherto unexplained low-temperature (high-frequency) anomaly. As such an anomaly was observed at high pressures when the transverse sound speed certainly considerably exceeds the Fermi velocity, this anomaly has no direct bearing the impedance threshold features discussed above. However, the fact that this anomaly is not present at pressures below 8 atm may be yet another confirmation of the fact that the singularity $\alpha = 0$ indeed occurs at pressures of the order of 8 atm.

According to (45) the temperature dependence of the impedance, apart from small corrections of the order $(T/T_F)^2 \ll 1$, is determined far from the threshold solely by the factor $\omega\tau$, i.e., by the temperature dependence of τ. The data from Ref. 4 therefore indicate that the temperature dependence of τ cannot be reduced to a simple factor $1/T^2$ but at low temperatures turns out to be a more complicated function of the temperature. One must check whether the observed anomaly of Z is the manifestation of the additional low-temperature collisional absorption predicted by Landau[1] according to which the characteristic damping time of zero sound is connected by the relation

$$\frac{1}{\tau}=\frac{1}{\tau_0}\left[1+\left(\frac{\hbar\omega}{2\pi T}\right)^2\right] \quad (47)$$

with the usual collisional relaxation time $\tau_0 \propto 1/T^2$. The absence of such an anomaly at low pressures would then be understandable, for in that case zero sound cannot propagate at all and the dispersion (47) of τ is then absent.

In the experiments of Ref. 4 at the lowest temperatures 3.6 mK the parameter $(\hbar\omega/2\pi T)^2$ was 5×10^{-3} for frequencies of 30 MHz and 5×10^{-4} at 10 MHz, so that the dispersion of (47) of τ at high frequencies is observable. However, it is so far impossible to state unequivocally that the observed anomaly is a reflection of the dispersion (47). According to (46) for large $\omega\tau$

$$\mathrm{Re}\,Z=Z_0\left\{1-\frac{\gamma}{\omega^2\tau^2}\right\}\approx Z_0\left\{1-\frac{\gamma}{\omega^2\tau^2}\left[1+2\left(\frac{\hbar\omega}{2\pi T}\right)^2\right]\right\},$$
$$\mathrm{Im}\,Z=\frac{\varkappa}{\omega\tau}Z_0\sim Z_0\frac{\varkappa}{\omega\tau_0}\left[1+\left(\frac{\hbar\omega}{2\pi T}\right)^2\right], \quad (48)$$

where $Z_0 = Z(\omega\tau\to\infty)$ while γ and $\varkappa$ are some dimensionless coefficients which are independent of $\omega\tau$. The tendency of Z to decrease with increasing ω at constant $\omega\tau$ is, indeed, observed experimentally. However, substitution of the values used in the experiments for the parameters in the expansion (48) of Re Z in $1/\omega\tau$ (in this case $\omega\tau \sim 5$) leads to the value $\gamma \sim 100$ for the coefficient γ in (48). The cause of so large value of γ is so far unclear. For a final answer to the question whether the observed anomaly can be explained on the basis of the dispersion (47) of τ, additional measurements at large $\omega\tau$ are necessary.

[1]L. D. Landau, Zh. Eksp. Teor. Fiz. **32**, 59 (1957) [Sov. Phys. JETP **5**, 101 (1957)].
[2]P. R. Roach and J. B. Ketterson, Phys. Rev. Lett. **36**, 736 (1976); in Quantum Fluids and Solids (Eds. S. B. Trickey and E. D. Adams) Plenum, N. Y. 1977.
[3]M. J. Lea, K. J. Butcher, and E. R. Dobbs, Commun. Phys. **2**, 59 (1977).
[4]F. P. Milliken, R. W. Richardson, and S. J. Williamson, Physica **108BC**, 1201 (1982).
[5]I. L. Bekarevich and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **39**, 1699 (1960) [Sov. Phys. JETP **12**, 1187 (1961)].
[6]I. A. Fomin, Zh. Eksp. Teor. Fiz. **54**, 1881 (1968); Pis'ma Zh. Eksp. Teor. Fiz. **24**, 90 (1976) [Sov. Phys. JETP **27**, 1010 (1968); JETP Lett. **24**, 77 (1976)].
[7]E. G. Flowers and R. W. Richardson, Phys. Rev. **B17**, 1238 (1978).
[8]G. G. Brooker, Proc. Phys. Soc. **90**, 397 (1967).
[9]L. R. Corruccinin, J. S. Clarke, N. D. Mermin, and J. W. Wilkins, Phys. Rev. **180**, 225 (1969).
[10]E. G. Flowers, R. W. Richardson, and S. J. Williamson, Phys. Rev. Lett. **37**, 309 (1976).
[11]R. W. Richardson, Phys. Rev. **B18**, 6122 (1978).
[12]R. E. Nettleton, J. Low Temp. Phys. **26**, 277 (1977); J. Phys. **C11**, L725 (1978).
[13]D. Einzel, H. Høfgaard Jensen, H. Smith, and P. Wölfle, J. Low Temp. Phys. **53**, 695 (1983).
[14]E. P. Bashkin and A. É. Meĭerovich, Zh. Eksp. Teor. Fiz. **77**, 383 (1979) [Sov. Phys. JETP **50**, 196 (1979)].
[15]E. P. Bashkin and A. É. Meĭerovich, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 517 (1978) [JETP Lett. **27**, 485 (1978)].

Translated by D. ter Haar