# Fields and radiation of toroidal dipole moments moving uniformly in a medium

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We consider the fields and the Cherenkov radiation in the case of a toroidal dipole moment moving with constant velocity in a medium with permittivity $\varepsilon$ and permeability $\mu$. In vacuo the fields outside the moving toroidal dipole vanish as for a toroidal dipole at rest. However, in a medium fields occur outside the dipole which are proportional to $\varepsilon\mu - 1$ and are caused by the fact that in the framework of the macroscopic electrodynamics of a uniform medium this medium can also be thought to fill the dipole itself. Toroidal dipoles, including point dipoles, which are not filled by the medium do not radiate when they move uniformly. The problems analyzed here are mainly of methodological interest and they facilitate, in particular, the understanding of the peculiarities of the radiation by other sources moving in a medium, first of all by magnetic dipole moments.

It has only recently been fully recognized that there exists apart from electric and magnetic dipole moments a toroidal dipole moment independent of them (see Ref. 1 and the literature cited there and also Refs. 2–5). For instance, a "current toroid" possesses such a dipole moment and no other kind; this is a solenoid curved round into a torus with a winding guaranteeing the absence of a current component "along" the torus (the presence of such an azimuthal component would lead to the appearance in the torus of a magnetic moment also). Clearly, the magnetic field in the toroid is nonvanishing only inside it while the electric field is everywhere zero (we are now talking about an uncharged toroid at rest with a constant current).

The toroidal dipole moment is given by the expression[1]

$$\mathbf{T} = \frac{1}{10c}\int \{\mathbf{r}(\mathbf{jr}) - 2r^2\mathbf{j}\}\, d\mathbf{r}, \qquad (1)$$

where $\mathbf{j}(\mathbf{r})$ is the current density. In the case of a point toroid the toroidal moment density equals $\mathbf{T}\delta(\mathbf{r})$ and

$$\mathbf{j} = c \operatorname{rot}\operatorname{rot}\{\mathbf{T}\delta(\mathbf{r})\}. \qquad (2)$$

One can say that in that case the magnetization is equal to $\mathbf{M} = \operatorname{curl} \mathbf{T}\delta(\mathbf{r})$. For an extended distribution one can for qualitative considerations replace in (2) the $\delta$-function by some nonsingular form-factor. Without specifying it we shall in what follows associate a toroidal dipole with such a small current toroid.

The toroidal dipole moment $\mathbf{T}$ is a vector which transforms in the same way as $\mathbf{j}$, i.e., it is a polar vector, changing its sign under time-reversal. The toroidal moment $\mathbf{T}$ defined according to (1) has the same dimensionality as an electrical quadrupole moment $D_{\alpha\beta}$; in the case of point charges one can write

$$T_\alpha = \frac{1}{10c}\sum e\{r_\alpha(\mathbf{vr}) - 2r^2 v_\alpha\}, \qquad D_{\alpha\beta} = \sum e\{3r_\alpha r_\beta - r^2\delta_{\alpha\beta}\},$$

where $\mathbf{v}$ is the velocity of the charge, $\mathbf{r} = \{r_\alpha\}$ is its radius vector, and the summation is over all charges (we drop the corresponding index).

Of course, systems such as a current toroid have been considered for a long time but the vector $\mathbf{T}$ and Eqs. (1), (2) are usually not introduced or not connected with a completely explicit form of current toroid. Moreover, an explicit introduction of the toroidal dipole moment and its density greatly facilitates the understanding of the physics of the problem. This is clear from the example of toroidal magnetic structures in the case of solids.[5]

Because of what we have said a solution of various problems for systems possessing a toroidal moment is of interest. In particular, as far as we know nobody has as yet considered the fields and radiation of toroidal dipole moments moving in vacuo or in a medium (although this problem was briefly touched upon in Ref. 6). In what follows we dwell therefore on the calculation of the fields and radiation of a toroidal dipole moment (which is static in its own rest frame $K'$) moving uniformly in a fixed medium (laboratory frame $K$). In the case of a uniform medium when one need deal only with Cherenkov radiation, this is the simplest problem of the theory of radiation in a medium and moreover it is undoubtedly of methodological interest (see Refs. 6, 7 and the literature cited there). A characteristic feature of the fields of moving toroidal dipoles is that when there is no medium these fields are localized inside the sources, while if there is a medium present they extend outside the sources. Moreover, the latter conclusion is valid only for standard applications of the equations of macroscopic electrodynamics. For microscopic toroidal dipoles it is, in general, untrue. This specific feature encouraged us to present this work in considerable detail.

**1.** We shall assume that in the laboratory frame $K$ there is a nonabsorbing medium at rest which is isotropic and characterized by a dielectric permittivity $\varepsilon$ and a magnetic permeability $\mu$ which are independent of the field strengths. Except in the last sections of this paper in which we consider transition radiation and problems associated with the motion of a source in a channel, we assume the medium to be uniform. We neglect in our equations the frequency dispersion of $\varepsilon$ and $\mu$, since it is well known[7,8] how to include it in the final expressions for the radiation power. Thus we start from the usual equations

$$\operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{\varepsilon}{c}\frac{\partial\mathbf{E}}{\partial t}, \qquad \operatorname{div}\varepsilon\mathbf{E} = 4\pi\rho,$$

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$$\operatorname{rot}\mathbf{E}=-\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t},\quad \operatorname{div}\mu\mathbf{H}=0,$$
$$\mathbf{D}=\varepsilon\mathbf{E},\quad \mathbf{B}=\mu\mathbf{H}, \tag{3}$$

where $\mathbf{j}$ and $\rho$ are the "external" current and "external" charge densities.

For a toroid at rest (rest frame $K'$ the same as $K$) we assume that the current $\mathbf{j}$ has the form (2) and that $\rho = 0$ or, choosing the $z$ — and $z'$-axes along $\mathbf{T} = \mathbf{T}'$,

$$j_z=-c\Delta_\perp\{T\delta(\mathbf{r})\},\quad \mathbf{j}_\perp=c\frac{\partial}{\partial z}\frac{\partial}{\partial \mathbf{r}_\perp}\{T\delta(\mathbf{r})\},\quad \rho=0, \tag{4}$$

where $T = T_z$, $T_x = T_y = 0$ and $r_{\perp,y} = y$, $r_{\perp,x} = x$. The solution of Eqs. (3) for such a $\mathbf{j}$ has the form $\mathbf{H} = 4\pi\operatorname{curl}\{\mathbf{T}\delta(\mathbf{r})\}$ or

$$H_z=0,\quad H_x=4\pi\frac{\partial}{\partial y}T\delta(\mathbf{r}),\quad H_y=-4\pi\frac{\partial}{\partial x}T\delta(\mathbf{r}),$$
$$E_z=E_x=E_y=0, \tag{5}$$

i.e., the field is concentrated inside the toroid; if we have in mind an extended toroid, the field $\mathbf{H}$ inside it can be assumed to be constant in magnitude and directed along the azimuth.

Now let the toroid be moving with a velocity $\mathbf{v}$ along the $z$-axis along which the vector $\mathbf{T}$ lies. Equations (4), (5) then refer to the rest frame $K'$, and using a Lorentz transformation for $j^i = \{c\rho, \mathbf{j}\}$ we find in the laboratory frame $K$

$$j_z=-c\Delta_\perp T\delta(\mathbf{r}_\perp)\delta(z-vt),$$
$$\mathbf{j}_\perp=c\left(1-\frac{v^2}{c^2}\right)\frac{\partial}{\partial z}\frac{\partial}{\partial \mathbf{r}_\perp}T\delta(z-vt)\delta(\mathbf{r}_\perp),$$
$$\rho=-\frac{v}{c}\Delta_\perp T\delta(z-vt)\delta(\mathbf{r}_\perp). \tag{6}$$

Here we have used the fact that

$$\delta\left(\xi\Big/\left(1-\frac{v^2}{c^2}\right)^{1/2}\right)=\left(1-\frac{v^2}{c^2}\right)^{1/2}\delta(\xi).$$

**2.** At first sight it may appear that the fields in the system $K$ can also be obtained from (5) using a Lorentz transformation. When a medium is present, however, this is no longer the case, since we can obtain only the fields in the frame $K''$ in which the medium moves with a velocity $\mathbf{v}$ (i.e., with the velocity with which the source moves in the medium). However, in the vacuum case the fields of a moving toroidal moment are, of course, obtained from (5) as the result of a Lorentz transformation. Then

$$H_z=0,\quad H_x=\frac{4\pi}{(1-v^2/c^2)^{1/2}}\frac{\partial}{\partial y}T\delta(\mathbf{r}_\perp)\delta\left(\frac{z-vt}{(1-v^2/c^2)^{1/2}}\right)$$
$$=4\pi\frac{\partial}{\partial y}T\delta(z-vt)\delta(\mathbf{r}_\perp),\quad H_y=-4\pi\frac{\partial}{\partial x}T\delta(z-vt)\delta(\mathbf{r}_\perp),$$
$$\mathbf{E}=-[\mathbf{v}\mathbf{H}]/c. \tag{7}$$

Naturally, the fields $\mathbf{H}$ and $\mathbf{E}$ are nonvanishing only inside the toroid. Writing the velocity $\mathbf{v} = (0, 0, v)$ we can put the current (6) in the form

$$\mathbf{j}=\rho\mathbf{v}+\mathbf{j}_T,\quad \operatorname{div}\mathbf{j}_T=0;\quad j_{T,z}=-c\left(1-\frac{v^2}{c^2}\right)\Delta_\perp T\delta(z-vt)\delta(\mathbf{r}_\perp),$$
$$\mathbf{j}_{T,\perp}=c\left(1-\frac{v^2}{c^2}\right)\frac{\partial}{\partial z}\frac{\partial}{\partial \mathbf{r}_\perp}T\delta(z-vt)\delta(\mathbf{r}_\perp). \tag{8}$$

We consider now the case when in the laboratory frame there is only a toroidal moment for which

$$\mathbf{j}=\mathbf{j}_T{}^v=c\operatorname{rot}\operatorname{rot}\{\mathbf{T}_v\delta(z-vt)\delta(\mathbf{r}_\perp)\};\quad \rho=0,\quad T_v=(1-v^2/c^2)T, \tag{9}$$

which for a velocity $\mathbf{v}$ along the $z$-axis corresponds to Eq. (8) with $\rho = 0$.

A moving toroidal moment described by (9)

$$j'_{T,z'}=-c\Delta_\perp{}'T\delta(\mathbf{r}');\quad \mathbf{j}'_{T,\perp'}=c\frac{\partial}{\partial z'}\frac{\partial}{\partial \mathbf{r}_\perp{}'}T\delta(\mathbf{r}'),$$
$$\rho_T{}'=\frac{v}{c}\Delta_\perp{}'T\delta(\mathbf{r}') \tag{10}$$

in its rest frame. For such a toroidal moment the electric field no longer vanishes in the frame $K'$; it is given by the equation

$$\operatorname{div}\mathbf{E}'=-\Delta\varphi'=4\pi\rho_T{}'=\frac{4\pi v}{c}\Delta_\perp{}'T\delta(\mathbf{r}'). \tag{11}$$

Hence $\varphi' = (v/c)\Delta'_\perp T(r')^{-1}$ and when $r' \neq 0$, by virtue of $\Delta'(r')^{-1} = 0$,

$$\varphi'=\frac{D_{\alpha\beta}{}'}{6}\frac{\partial^2}{\partial r_\alpha{}'\partial r_\beta{}'}\frac{1}{r'},\quad D_{\alpha\beta}{}'=\frac{2v}{c}T\begin{vmatrix}1&0&0\\0&1&0\\0&0&-2\end{vmatrix}, \tag{12}$$

i.e., the field $\mathbf{E}' = -\nabla\varphi'$ is the field of the electrical quadrupole moment $D'_{\alpha\beta}$. On the other hand, the magnetic field $\mathbf{H}'$ retains its form (5). In other words, the toroidal moment (9) at rest is a combination of the toroidal dipole (4), (5) and an electric quadrupole. One can check this also for the model of an extended current toroid—in that case the quadrupole forms electric dipoles fanned out in the $y'$, $x'$-plane (i.e., at right angles to the moment $\mathbf{T}$).

The field of the toroidal moment (9) in the frame $K$ in contrast to the field of the moving toroidal dipole (6)–(8) is no longer localized in the region of the moment itself. This is clear from general considerations. It follows from the more general solution given below which is valid when there is a medium present.

**3.** We turn to finding the fields of toroidal moments moving in a medium by means of Eqs. (3). They give rise to the equation

$$\operatorname{rot}\operatorname{rot}\mathbf{E}+\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=-\frac{4\pi\mu}{c^2}\frac{\partial \mathbf{j}}{\partial t}. \tag{13}$$

In (13) we shall substitute for $\mathbf{j}$ expressions (6) or (9). The limits of applicability of such an approach and, especially, of Eq. (13) itself, will be touched upon in what follows. Expressions (6) and (9) are such that $\partial/\partial t = -v\partial/\partial z$ and we can put

$$E_z=-\frac{\partial\psi}{\partial z},\quad E_x=-\frac{\partial\varphi}{\partial x},\quad E_y=-\frac{\partial\varphi}{\partial y}. \tag{14}$$

Therefore Eq. (13) gives for the case (6)

$$\left(1-\frac{v^2\varepsilon\mu}{c^2}\right)\varphi-\psi=\frac{4\pi\mu v}{c^2}\left(1-\frac{v^2}{c^2}\right)T\delta(z-vt)\delta(\mathbf{r}_\perp),\quad(15)$$

$$\Delta_\perp\psi+\left(1-\frac{v^2}{c^2}\varepsilon\mu\right)\frac{\partial^2\psi}{\partial z^2}=-\frac{4\pi v(\varepsilon\mu-1)}{c^2\varepsilon}\Delta_\perp T\delta(z-vt)\delta(\mathbf{r}_\perp).\quad(16)$$

More precisely, from (13) one gets Eq. (15) operated on by $(\partial/\partial y)(\partial^2/\partial z^2)$ or $(\partial/\partial x)(\partial^2/\partial z^2)$ and Eq. (16) operated on by $\partial/\partial z$. However, in the problems of interest to us we can use Eqs. (15) and (16). For a current with the density (9) one must replace the right-hand side of Eq. (16) by

$$-\frac{4\pi}{c}\mu v\left(1-\frac{v^2}{c^2}\right)\Delta_\perp T\delta(z-vt)\delta(\mathbf{r}_\perp).\quad(17)$$

In vacuo when $\varepsilon=\mu=1$ the right-hand side of Eq. (16) vanishes and one can put $\psi=0$, i.e., $E_z=0$, and we have from (15)

$$\varphi=(4\pi v/c)T\delta(z-vt)\delta(\mathbf{r}_\perp).\quad(18)$$

Hence [see (14)] the field $\mathbf{E}$ turns out to be equal to the field (7) as it should.

When a medium is present, if $\varepsilon\mu\neq1$, the fields of a moving toroidal dipole are no longer localized in the region it occupies. The same is true for the toroidal moment (9), when the right-hand side of Eq. (16) is non-vanishing, already in vacuo by virtue of the substitution (17). Of course, such a toroidal moment now has the indicated electric field in the rest frame (see (11), (12)).

**4.** We turn to the solution of Eqs. (15), (16). We shall do this in the Fourier transformation

$$\psi=\int\psi_{\varkappa,k}\exp\{i\boldsymbol{\varkappa}\mathbf{r}_\perp+ik(z-vt)\}d\boldsymbol{\varkappa}\,dk,$$
$$\varphi=\int\varphi_{\varkappa,k}\exp\{i\boldsymbol{\varkappa}\mathbf{r}_\perp+ik(z-vt)\}d\boldsymbol{\varkappa}\,dk.\quad(19)$$

Substitution of (19) into (15), (16) leads to the result

$$\psi_{\varkappa,k}=-4\pi v(\varepsilon\mu-1)\varkappa^2T[(2\pi)^3c\varepsilon\tilde{k}^2]^{-1}=\psi^L_{\varkappa,k}+\psi^{NL}_{\varkappa,k},$$
$$\psi^{NL}_{\varkappa,k}=4\pi(\varepsilon\mu-1)k^2(1-\varepsilon\mu v^2/c^2)T[(2\pi)^3c\varepsilon\tilde{k}^2]^{-1},\quad(20)$$
$$\psi^L_{\varkappa,k}=-4\pi v(\varepsilon\mu-1)T[c\varepsilon(2\pi)^3]^{-1};$$

$$\varphi_{\varkappa,k}=4\pi vk^2T[\varkappa^2+k^2\varepsilon\mu(1-v^2/c^2)][(2\pi)^3c\varepsilon\tilde{k}^2]^{-1}=\varphi^L_{\varkappa,k}+\varphi^{NL}_{\varkappa,k},$$
$$\varphi^{NL}_{\varkappa,k}=4\pi vk^2(\varepsilon\mu-1)T[(2\pi)^3c\varepsilon\tilde{k}^2]^{-1},\quad(21)$$
$$\varphi^L_{\varkappa,k}=4\pi vT[(2\pi)^3c\varepsilon]^{-1},$$

where

$$\tilde{k}^2=\varkappa^2+k^2(1-\varepsilon\mu v^2/c^2).$$

The terms $\psi^L_{\varkappa,k}$ and $\varphi^L_{\varkappa,k}$ in (20) and (21) in the coordinate representation are proportional to $\delta(z-vt)\delta(\mathbf{r}_\perp)$, i.e., they correspond to a local potential of the toroidal dipole moving in the medium. In that case the expressions for $\psi^L_{\varkappa;k}$ and $\varphi^{NL}_{\varkappa;k}$ are proportional to $\varepsilon\mu-1$ and in vacuo vanish in agreement with what has been said earlier ($\psi_{\varkappa,k}$ vanishes also in the rather peculiar medium[9] with $\varepsilon\mu=1$). We can write the expression for $\varphi^L_{\varkappa,k}$ also in the form

$$\varphi^L_{\varkappa,k}=\frac{4\pi vT}{c}-\frac{4\pi(\varepsilon-1)vT}{c\varepsilon},\quad(22)$$

where the second term, like $\psi_{\varkappa,k}$, is connected with the currents in the medium.

The fields corresponding to $\psi^{NL}_{\varkappa,k}$ and $\varphi^{NL}_{\varkappa,k}$ are nonlocal, i.e., in the coordinate representation they extend outside the source,

$$\psi^{NL}_{\varkappa,k}=\left(1-\frac{v^2}{c^2}\varepsilon\mu\right)\varphi^{NL}_{\varkappa,k},$$
$$\varphi^{NL}(\mathbf{r},t)=\frac{4\pi vT}{(2\pi)^3c}\int\frac{\varepsilon\mu-1}{\varepsilon}\exp\{i\boldsymbol{\varkappa}\mathbf{r}_\perp+ik(z-vt)\}\frac{d\boldsymbol{\varkappa}\,dk}{\tilde{k}^2}.\quad(23)$$

Let the condition for the occurrence of the Cherenkov radiation,

$$(v/c)\sqrt{\varepsilon\mu}>1\quad(24)$$

not be satisfied. Even in that case the potentials $\psi^{NL}$ and $\varphi^{NL}$ and the corresponding electrical field decrease slowly with distance from the source. Indeed, integrating (23) over the angle between $\varkappa$ and $\mathbf{r}_\perp$ we have

$$\varphi^{NL}(\mathbf{r},t)=-\frac{vT}{\pi c}\frac{\partial^2}{\partial z^2}\int\frac{\varepsilon\mu-1}{\varepsilon}J_0(\varkappa r_\perp)\exp\{ik(z-vt)\}\frac{\varkappa d\varkappa dk}{\tilde{k}^2}$$
$$=-\frac{vT}{\pi c}\frac{\partial^2}{\partial z^2}\int\frac{\varepsilon\mu-1}{\varepsilon}K_0\left(kr_\perp\left(1-\frac{v^2}{c^2}\varepsilon\mu\right)^{1/2}\right)\cos k(z-vt)\,dk$$
$$=-\frac{vT}{c}\frac{\partial^2}{\partial z^2}\frac{\varepsilon\mu-1}{\varepsilon}\left[(z-vt)^2+r_\perp^2\left(1-\frac{v^2}{c^2}\varepsilon\mu\right)\right]^{-1/2}$$
$$=-\frac{vT}{c}\frac{\varepsilon\mu-1}{\varepsilon}\left[2(z-vt)^2-r_\perp^2\left(1-\frac{v^2}{c^2}\varepsilon\mu\right)\right]$$
$$\times\left[(z-vt)^2+r_\perp^2\left(1-\frac{v^2}{c^2}\varepsilon\mu\right)\right]^{-5/2}.\quad(25)$$

We put here $\varepsilon=\text{const}$ and $\mu=\text{const}$, where $J_0$ and $K_0$ are the familiar Bessel functions. If there is frequency dispersion, $\varepsilon=\varepsilon(kv)$, and one can formally assume that also $\mu=\mu(kv)$. Of course, one can then not integrate over $k$ in a general manner. According to (25) the potential decreases with distance in the same way as an electric quadrupole moment [thus, for $t=0$ and $\mathbf{r}_\perp=0$ the potential $\varphi^{NL}=-2v(\varepsilon\mu-1)/c\varepsilon z^3$].

A toroidal dipole is thus one example of a source for which in a vacuum the fields do not extend beyond the source but in a medium they are then nonvanishing also outside the source. Another example is an infinite solenoid with current moving in a medium, or a constant magnet (with axes at an angle to the velocity). In a familiar sense one can also relate to this a current magnetic moment for which in the rest state $\mathbf{j}=c\,\text{curl}\,\{\mathbf{m}\delta(\mathbf{r})\}$ and

$$\mathbf{H}=\frac{\mathbf{B}}{\mu}=\frac{3\mathbf{r}(\mathbf{mr})-\mathbf{m}r^2}{r^5}+4\pi\mathbf{m}\delta(\mathbf{r}).\quad(26)$$

The second term corresponds here to the field which is restricted within the source. However, when the latter moves in a medium the fields connected with the $\delta$-term are now nonvanishing also outside the dipole. At this point one can see the reason for the different radiation from current and "true" magnetic dipoles which move in a medium (see Ref. 6).

From an intuitive point of view the extension of the field beyond the limits of a source which moves in a medium can be explained very simply. In macroscopic electrodynamics [and particularly when one uses Eqs. (3)] the medium is assumed to fill all of space, including the region inside the sources. Therefore for a source which moves in a fixed medium this medium continually passes through the source (or, using the graphic terminology of M. A. Miller, "is blown out" of the source. This blowing out is still more pictorial when we consider the process in the rest frame of the source, so that the medium moves with a velocity $-\mathbf{v}$). However, the medium in the source is polarized even by the field localized in it, as a result of which the polarization together with the corresponding part of the field now appears outside the source.

**5**. In the light of what we have said it is clear that when condition (24) is satisfied the toroidal dipole moment (6) will be a source of Cherenkov radiation.

The energy $Q$ emitted per unit time can be evaluated by different methods (see, e.g., Ref. 7). One of them consists in evaluating the work done by the field on the source (current):

$$Q=-\int \mathbf{jE}\,d\mathbf{r}=-(2\pi)^3\int \mathbf{j}_{-\varkappa,-k}\mathbf{E}_{+\varkappa,k}\,d\varkappa\,dk. \tag{27}$$

Substituting here the Fourier components corresponding to expressions (6), (14), (15) we have [see also (19) to (21)]

$$\begin{aligned} Q_T&=i(2\pi)^3\int \{j_{\perp,-\varkappa,-k}\varkappa\varphi_{\varkappa,k}+j_{z,-\varkappa,-k}k\psi_{\varkappa,k}\}\,d\varkappa\,dk \\ &=\int_{-\infty}^{\infty}dk\int_0^{\infty}\frac{\varkappa^2\,d\varkappa^2}{2ck}(\varepsilon\mu-1)(vk)^3T^2\delta(\bar{k}^2) \\ &=\frac{T^2}{c^4v}\int_0^{\infty}\mu(\varepsilon\mu-1)^2\left(1-\frac{c^2}{v^2\varepsilon\mu}\right)\omega^5\,d\omega, \end{aligned} \tag{28}$$

where we have used the fact that $\omega=|k|v$ and when integrating over $\varkappa^2$ have used the relation

$$\operatorname{Im}\frac{1}{\bar{k}^2}=i\pi\frac{k}{|k|}\delta(\bar{k}^2).$$

If one is dealing with the emission from a "pure moving" toroidal moment (9) we get similarly

$$Q_{T_v}=\frac{T_v^2}{c^4v}\int_0^{\infty}\mu(\varepsilon\mu)^2\left(1-\frac{c^2}{v^2\varepsilon\mu}\right)\omega^5\,d\omega. \tag{29}$$

Of course, $\sqrt{(\varepsilon\mu)}=n$ is the refractive index of the transparent medium considered; the integration over the frequency is carried out over the region where $v\geqslant c/n(\omega)$. Expression (29) is the same as the one given in Ref. 6 where it was specified through $T_v=T$.

Another method of evaluating the emitted power consists in calculating the change in the energy of the field per unit time. The calculation was given in detail in Chapter 7 of Ref. 7 for a source of charge $e$, electric dipole $\mathbf{p}$, and magnetic moment $\mathbf{m}$ and a current density of the form

$$\mathbf{j}=e\mathbf{v}\delta(\mathbf{r}-\mathbf{v}t)+c\,\mathrm{rot}\{\mathbf{m}\delta(\mathbf{r}-\mathbf{v}t)\}+\frac{\partial}{\partial t}\{\mathbf{p}\delta(\mathbf{r}-\mathbf{v}t)\}. \tag{30}$$

In this case we put $\mu=1$ in Ref. 7, but the generalization to the case $\mu\neq 1$ can be made very simply (see Ref. 6). If the source, however, also has a toroidal moment one must clearly add to (3) expressions (6), (8), or (9). We restrict ourselves here to the last case. We then get instead of Eq. (7.44) of Ref. 7

$$\begin{aligned}\ddot{q}_{\lambda j}+\omega_\lambda^2 q_{\lambda j}=(4\pi/\varepsilon)^{1/2}\{&e(\mathbf{a}_{\lambda j}\mathbf{v})-c(\mathbf{a}_{\lambda j}\mathbf{T}_v)k_\lambda^2+i(c\mathbf{m}[\mathbf{a}_{\lambda j}\mathbf{k}_\lambda] \\ &-(\mathbf{p}\mathbf{a}_{\lambda j})(\mathbf{k}_\lambda\mathbf{v}))\}\exp\{-i(\mathbf{k}_\lambda\mathbf{v})t\},\end{aligned} \tag{31}$$

where $\omega_\lambda^2=(c^2/\varepsilon\mu)k_\lambda^2$ and the $\mathbf{a}_{\lambda j}$ are polarization vectors for waves of wavevector $\mathbf{k}_\lambda$ where for an isotropic medium $\mathbf{a}_{\lambda i}\,\mathbf{a}_{\lambda j}=\delta_{ij}$, $\mathbf{k}_\lambda\,\mathbf{a}_{\lambda j}=0$, $i,j=1,2$. It is clear from (31) that the emission from a toroidal moment interferes with the emission of a charge but not with the emission of an electric or magnetic dipole. Without performing any new calculations one can easily obtain a general expression for $Q$, generalizing Eqs. (6.61) and (7.45) from Ref. 7. We restrict ourselves here to the results for the case when only the moment $T_v$ is present, i.e., $e=0$, $\mathbf{m}=0$, and $\mathbf{p}=0$. In that case

$$Q_{T_v}=\frac{1}{2\pi c^4v}\sum_{j=1,2}\int_0^{\infty}d\omega\int_0^{2\pi}\mu n^4\omega^5(\mathbf{a}_{\lambda j}\mathbf{T}_v)^2\,d\varphi. \tag{32}$$

If the moment $\mathbf{T}_v$ is directed along $\mathbf{v}$ we get from this the result (29) since one of the vectors $\mathbf{a}_{\lambda j}$ may be assumed to be orthogonal to $\mathbf{T}_v$ and for the other polarization

$$(\mathbf{a}_\lambda\mathbf{T}_v)=(T_v)^2\sin^2\theta,\qquad \cos\theta=c/nv.$$

The third method for finding $Q$ consists in evaluating the flux of the Poynting vector through a surface surrounding the charge trajectory. This way, which was used in the first paper on the theory of the Cherenkov effect,[10] is, in particular, convenient for evaluating the emission when there is a circular channel present in a dielectric.[11,12] We touch upon the problem of the emission by a toroidal dipole in a channel in what follows.

**6**. For reasons expounded in the next section of the paper the consideration of Cherenkov radiation and transition radiation for toroidal moments using macroscopic electrodynamics has only a very restricted value. Nonetheless it is expedient to dwell here briefly upon the transition radiation of a toroidal dipole at the boundary of two media. We restrict ourselves to the simplest case (at least, the one simplest for calculations), in which the toroidal moment $\mathbf{T}$ is directed along the velocity $\mathbf{v}$, the boundary dividing the media perpendicular to $\mathbf{v}$ and, moreover,

$$v\to c\quad (\text{i.e., } E/Mc^2=(1-v^2/c^2)^{-1/2}\gg 1).$$

A qualitative idea about the nature of the radiation can be obtained also without detailed calculations. As for the radiation of a quadrupole, the radiation by a toroidal moment must contain (compared to the radiation from a charge) an extra factor proportional to $\omega^4$. The spectral density of the radiation of an ultra-relativistic charge $W_\omega$ in the forward direction (along the direction of motion of the charge) is well known[8] to be approximately constant up to frequencies of the order $\omega_{pe}(E/Mc^2)$, where $\omega_{pe}=(4\pi e^2N/m)^{1/2}$ is the plasma frequency of the medium, and after that decreases as $1/\omega^4$. Hence, the spectral density of the radiation from a toroidal moment will be proportional to $\omega^4$ when $\omega \ll \omega_{pe}(E/Mc^2)$ and after that will be constant when $\omega \gg \omega_{pe}(E/Mc^2)$ (in the spectral region considered $\mu=1$ and $\varepsilon=1-\omega_{pe}^2/\omega^2$).

Such a flat spectrum can, in principle, stretch up to very high frequencies (we indicate the bounds below). We find the constant determining the spectral density of the radiation on the flat part of the spectrum. By a standard method[8] for a sharp boundary, for instance, when a toroidal dipole passes from the medium into vacuum, we get for the forward emission at a small angle $\theta^2 \ll 1$ to the velocity and for $E/Mc^2 \ll 1$ the expression

$$W_\omega=\frac{T^2}{\pi c^5}\int_0^{\theta_{max}}\omega_{pe}^4\theta^2\,d\theta^2\left\{\theta^2+\left(\frac{Mc^2}{E}\right)^2+\frac{\omega_{pe}^2}{\omega^2}\right\}^{-2}. \quad (33)$$

The upper limit is here determined solely by the fact that we have made the approximation $\theta^2 \ll 1$. When $\omega^2/\omega_{pe}^2 \gg (E/Mc^2)^2$ we can neglect the frequency dependence in the integrand of (33) and when $\ln(E/Mc^2) \gg 1$ we can neglect terms of order 1 in comparison with $\ln(E/Mc^2)$. As a result

$$W_\omega=2\frac{T^2\omega_{pe}^4}{\pi c^5}\ln\frac{E}{Mc^2}. \quad (34)$$

At lower energies, when the condition $\ln(E/Mc^2) \gg 1$ is not valid, emission at an angle $\theta \sim 1$ becomes important although it only determines a numerical factor in the argument of the logarithm. We can then systematically neglect terms $\omega_{pe}^2/\omega^2$ compared to $1-v/c$ and 1 and rigorously integrate over angles. For the forward emission ($0<\theta<\pi/2$) we get when $E/Mc^2 \gg 1$

$$W_\omega=\frac{T^2\omega_{pe}^4}{\pi c^5}\left(\ln 2\frac{E^2}{Mc^2}-\frac{13}{6}\right). \quad (35)$$

There is thus in (34) an additional additive constant $\ln\sqrt{2}-\frac{13}{12}=-0.737$ which must be taken into account when $\ln(E/Mc^2)$ is not too large. One can also write Eq. (35) in the form

$$W_\omega=\frac{2T^2\omega_{pe}^4}{\pi c^5}\ln\frac{E}{2.09Mc^2}. \quad (36)$$

It is interesting that in this case the spread in the boundary can determine the maximum emitted frequency. Let $\Delta z$ be a characteristic size determining the change of $\varepsilon$ at the boundary from its value $\varepsilon=1-\omega_{pe}^2/\omega^2$ to $\varepsilon=1$. The zone where the radiation is formed is given by the expression[8]

$$L_f\sim\frac{2\pi c}{\omega}\left[\theta^2+\left(\frac{Mc^2}{E}\right)^2+\frac{\omega_{pe}^2}{\omega^2}\right]^{-1}. \quad (37)$$

In the case of emission by a charge the spread in the boundary can be important only if $\Delta z$ (the characteristic size of the spread) is larger than the zone $L_f$ where the "maximum" frequency $\omega_{pe}(E/Mc^2)$ is formed. As for the latter frequency $L_f \sim (4\pi c/\omega_{pe})(E/Mc^2)$ the condition for a sharp boundary, $\Delta z \ll L_f$ or $E/Mc^2 \gg \omega_{pe}\Delta z/4\pi c$ is usually well satisfied when $E/Mc^2 \gg 1$, if $\Delta z < 4\pi c/\omega_{pe}$. For normal solids this inequality means $\Delta z < 10^{-4}$ cm and is usually easily satisfied.

For a toroidal dipole the situation is radically changed. The main interest lies in the frequency range $\omega^2/\omega_{pe}^2 \gg (E/Mc^2)^2$ where the spectral density $W_\omega$ is constant [see (36)] and

$$L_f\sim\frac{4\pi c}{\omega}\left[\theta^2+\left(\frac{Mc^2}{E}\right)^2\right]^{-1}. \quad (38)$$

For fixed frequency $\omega$ the formation zone is largest when $\theta=0$ (strictly forward). The intensity is cut off if

$$L_f \leqslant \Delta z. \quad (39)$$

In the radiation the first thing that happens is that those waves will cut off which propagate at an appreciable angle to the velocity–the radiation becomes more directed.

When $\theta=0$

$$L_{f,max}\sim\frac{4\pi c}{\omega}\left(\frac{E}{Mc^2}\right)^2, \quad (40)$$

and we get from (39)

$$\omega_{max}\sim\frac{4\pi c}{\Delta z}\left(\frac{E}{Mc^2}\right)^2. \quad (41)$$

For a plateau $W_\omega=$ const to exist the value $\omega_{max}$ must lie on that plateau, i.e., $\omega_{max} \gg \omega_{pe}(E/Mc^2)$.

Hence it follows that under the condition

$$E/Mc^2 \gg (\omega_{pe}/c)\Delta z \quad (42)$$

the maximum emitted frequency is, indeed, given by (41).

When inequality (42) is satisfied we get an estimate for the total emitted energy:

$$W=\int W_\omega\,d\omega\sim\frac{T^2\omega_{pe}^4}{\Delta z c^4}\left(\frac{E}{Mc^2}\right)^2\ln\frac{E}{Mc^2}. \quad (43)$$

This expression is just an estimate because, as we noted, with increasing $\theta$ the angular distribution is limited (waves are mainly emitted along $\mathbf{v}$). Calculations show that for a boundary described by the relation

$$\varepsilon(z)-1=\frac{\omega_{pe}^2}{2\omega^2}\left(\frac{2}{\pi}\operatorname{arctg}\frac{z}{\Delta z}-1\right),$$

$$\varepsilon(-\infty)=1-\frac{\omega_{pe}^2}{\omega^2},\qquad \varepsilon(+\infty)=1,$$

there is no logarithmic factor in the integral intensity and

$$W=\frac{T^2\omega_{pe}^4}{2\Delta z\pi c^4}\left(\frac{E}{Mc^2}\right)^2. \quad (44)$$

For an extended toroidal dipole this result, which shows that the total emitted energy is proportional to $E^2$, is valid only when

$$c/\omega_{max} \gg l(Mc^2/E), \quad (45)$$

where $l$ is the characteristic size of the toroid in the direction of its motion in its proper frame (the factor $Mc^2/E$ arises in (45) due to Lorentz contraction). According to (41) condition (45) takes the form

$$\frac{E}{Mc^2} \gg \frac{l\omega_{max}}{c}=\frac{4\pi l}{\Delta z}\left(\frac{E}{Mc^2}\right)^2$$

or

$$E/Mc^2 \ll \Delta z/4\pi l. \quad (46)$$

Together with (42) this means that Eq. (44) under the conditions discussed is applicable as long as

$$l \ll c/\omega_{pe}. \tag{47}$$

It should be noted that we do know of no realistic problems in which transition radiation by relativistic toroidal dipole moments would be meaningful.

7. In the foregoing we used macroscopic electrodynamics and assumed that the medium is uniform everywhere, including the inside of the toroidal dipole (the point dipole assumption was used essentially only to simplify the calculations). It is clear that such an approximation (model) can be of real value only in exceptional cases. For instance, in a plasma there may be currents present corresponding to a current toroid and the dielectric permittivity of the plasma $\varepsilon$ may be almost the same inside and outside the toroid. However, even in that case it is in general necessary to take into account the effect of a magnetic field on $\varepsilon$ or, more precisely, to consider the tensor $\varepsilon_{\alpha\beta}(\omega, \mathbf{H})$. It is true that it is difficult to produce a fast-moving toroid even in a plasma but, in general, the analysis of toroidal dipoles moving in a medium may turn out to be useful for several plasma applications (see Ref. 2). (One should then bear in mind that there are, in general, slow waves in a plasma and the Cherenkov condition may thus be satisfied also for low velocities.) For toroidal dipoles of atomic or even smaller size[1,13] application of macroscopic electrodynamics without taking into account spatial dispersion is already, in general, inadmissible.

In this respect the fields of toroidal dipoles (and similar sources with a field localized inside them) are particularly sensitive to the state of the medium near the sources. Indeed, it is well known that a charge at rest with a spherically symmetric density $\rho(r)$ in a continuous medium with permittivity $\varepsilon$ outside the charge gives rise to a field which is independent of whether the medium penetrates the charge itself or the permittivity $\varepsilon_0 \neq \varepsilon$ in the region where $\rho \neq 0$ (charge in an empty spherical cavity, and so on). Even for an electric dipole this is not the case and, for instance, for a dipole in the center of a spherical cavity with a permittivity $\varepsilon_0$ the electric field outside changes by a factor $3\varepsilon/(\varepsilon_0 + 2\varepsilon)$ in comparison with the case when $\varepsilon = \varepsilon_0$. For a magnetic moment the situation is similar (the role of the "filling" of the magnet by a medium with permeability $\mu_0$ placed in a medium with permeability $\mu$ was analyzed in Ref. 14, § 74). The effect of the vicinity of the dipole on the field produced by it in the medium is closely connected (as is already clear, for instance, from the reciprocity theorem) with the problem of the field acting upon a dipole in a medium (we are dealing here with the difference between the acting or effective field $\mathbf{E}_{\text{eff}}$ and the average macroscopic field $\mathbf{E}$; for magnetics one must, of course, also distinguish $\mathbf{H}_{\text{eff}}$ and $\mathbf{H}$). It is well known that $\mathbf{E}_{\text{eff}}$ in its general form cannot be expressed in terms of $\varepsilon$ and $\mathbf{E}$. The result depends on the structure of the medium and the problem continues to be debated.[15,16]

For a toroidal dipole at rest [with a given value of $\mathbf{T}$, i.e., with a given current density (2)] the field $\mathbf{H}$ inside the toroid does not depend on $\mu$ outside and inside the toroid, while the induction $\mathbf{B} = \mu\mathbf{H}$ outside the toroid always vanishes. For a toroidal dipole (2), (4), (6) moving in a uniform medium with constant velocity the picture was explained above and the Cherenkov emission is proportional to $(\varepsilon\mu - 1)^2$. If the medium does not penetrate into the dipole there will be no Cherenkov radiation (see, however, footnote*). In particular, this statement is valid under conditions such that the problem can be rigorously solved in the framework of macroscopic electrodynamics. In fact, let us consider a toroidal dipole (with current density (6)) moving along the axis of a circular channel with radius $a$ in a medium with permittivity $\varepsilon$ and permeability $\mu$ while in the channel the permittivity is $\varepsilon_0$ and the permeability $\mu_0$. One of us (V.N.T.) has completely solved this problem (under the assumption that $\varepsilon\mu v^2/c^2 > 1$ and $\varepsilon_0\mu_0 v^2/c^2 < 1$). For the emitted power in the case of a narrow channel one then gets Eq. (28) with $\varepsilon\mu - 1$ replaced by $\varepsilon_0\mu_0 - 1$. Hence, for an empty channel ($\varepsilon_0 = \mu_0 = 1$) the emission vanishes. As stated this result is a completely general and explicit result and it will be valid for any empty channel, gap, etc. If one has to deal with problems of the fields and emission of microscopic toroidal dipoles, one can state that the fields outside the dipole may arise only if the medium surrounding it to some extent penetrates into the dipole (at a microscopic level one may be dealing with, for example, the entrance of separate particles from the surrounding medium into the dipole, or something like that) or, at any rate, it interacts with them.[1)] What we have said refers, of course, only to a uniformly moving toroidal dipole with a time-independent toroidal moment $T = T'$ in the rest frame. If the moment $T$ changes with time and (or) the dipole undergoes acceleration, radiation can, of course, take place.

As the problem of the effect of a medium on the fields of toroidal dipoles (and similar sources) has mainly a methodological character, especially in application to micro-dipoles, it is particularly relevant to emphasize the usefulness of understanding this problem for other sources. Specifically we have in mind in the first instance current and "true" (made up from magnetic monopoles) magnetic dipoles. The difference in the power of the Cherenkov radiation and of the transition radiation for these two kinds of dipoles in the past would appear to be real also in the application to point (microscopic) dipoles (see Refs. 7, 8 and the literature cited there). However, now it is clear that there are no differences whatever in those cases (both kinds of point dipoles radiate in the same way and have exactly the same fields far from the dipoles). For macroscopic magnetic dipoles the whole problem is connected with the role of the medium, which may fill different dipoles differently, while for a current magnetic dipole the presence of the term $4\pi\mathbf{m}\delta(\mathbf{r})$ in Eq. (29) is important. This problem is elucidated in detail in Ref. 6.

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[1)]Any interaction between the source and the medium for $v > c/n(\omega)$ gives some Cherenkov emission. In particular, the impenetrability of the source to the external medium means that particles of that medium somehow are repelled from its surface and therefore produce some perturbation. The intensity of the resulting emission can, however, not be calculated in the framework of macroscopic electrodynamics neglecting spatial dispersion. It seems to us rather obvious from physical considerations that the intensity of the Cherenkov radiation for microscopic toroidal dipoles will be very small, even without taking into account the fact that the values of $T$ themselves are small. If one is dealing with a dipole which is impenetrable to particles of the surrounding medium, this smallness is connected with the fact that the corresponding scattering cross-section is small. If, however, the particles of the surrounding medium (say, a plasma) can enter the dipole, the value of the wavefunction of the plasma electrons inside the toroidal dipole will be significant, but even in that case one may expect the effect to be small. The corresponding class of problems clearly deserves a more detailed analysis.


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