Quantum-mechanical properties of Čerenkov radiation in a crystal

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A new type of radiation is considered. Its spectral and angular characteristics are the same as those of the Čerenkov radiation and its intensity depends on the scattering of electrons by atomic planes (strings). Possible applications of this radiation to measurements of the energy of relativistic particles are discussed.

1. INTRODUCTION

When a relativistic electron is channeled in a crystal almost along an atomic plane, the classical relativistic equation of motion shows that the acceleration produced by the longitudinal force is greater by a factor of γ^2 than that due to the lateral force [$\gamma = (1-\beta_{\parallel}^2)^{-1/2}$ is the Lorentz factor]. It follows that, when the relativistic electron travels close to an atomic plane line), its interaction with the crystal can be described by the average potential over the plane (line) of the single crystal. The transverse motion of the channeling particle is nonrelativistic and is described by the Schroedinger equation with the relativistic mass. The longitudinal motion may be looked upon as free. Depending on the angle of incidence of the beam of electrons on the single crystal relative to a given crystallographic direction, the transverse motion can be either finite or infinite. Correspondingly, the transverse energy spectrum is discrete in the first case and continuous in the second. Spontaneous transitions between states associated with the transverse motion results in the emission of electromagnetic radiation by the channeling particle, which may then be looked upon as a relativistic "quasiatom." This is of considerable theoretical interest because it may be possible to use this process to investigate the wave properties of particles in macroscopic motion.

We have examined effects due to the quantization of the transverse motion of a channeling particle when it undergoes spontaneous transitions in the transverse continuum without transferring momentum or energy to the crystal. This process is possible when the radiating particle travels with a velocity greater than that of light in the medium.

2. DETERMINATION OF RADIATION PARAMETERS

Let us take the z axis along an atomic plane of the single crystal and the x axis perpendicular to it. Let us further suppose that the electron travels close to the z axis under the conditions of planar channeling. The interaction between the electron and the field due to the atomic plane will be taken into account exactly, and the emission of photons will be described in first-order perturbation theory:

$$W_{i\to f} = \frac{2\pi}{\hbar}\int |V_{i\to f}|^2 \delta(E_i - E_f - \hbar\omega)\, dN, \tag{1}$$

where $W_{i\to f}$ is the photon emission probability and $V_{i\to f}$ is the matrix element of the interaction, summed over the photon polarizations[1]:

$$|V_{i\to f}|^2 = \frac{2\pi n(\omega) e^2}{\omega c}\{|J_x(\varkappa_x)|^2 \sin^2\varphi + |J_x(\varkappa_x)\cos\theta\cos\varphi - J_z(\varkappa_x)\sin\theta|^2\}, \tag{2}$$

where

$$\begin{aligned} J_x(\varkappa_x) &= \frac{c}{E_{\parallel}}\int \exp(-i\varkappa_x x)\,\psi_f^*(x)\,\hat{P}_x\psi_i(x)\,dx, \\ J_z(\varkappa_x) &= \beta_{\parallel}\int \exp(-i\varkappa_x x)\,\psi_f^*(x)\,\psi_i(x)\,dx, \end{aligned} \tag{2.1}$$

$\beta_{\parallel} = v_{\parallel}/c$, $v_{\parallel}$ is the component of the electron velocity along the z axis, $\varkappa_x$ is the component of the wave vector $\varkappa$ of the photon along the x axis, θ is the angle between the $\varkappa$ and z axes, φ is the azimuthal angle of the vector $\varkappa$, $n(\omega)$ is the refractive index of the medium, $\psi(x)$ is the solution of the Schroedinger equation corresponding to the continuous spectrum of transverse motion, and dN is an element of phase space.

For motion without transfer of momentum or energy, the argument of the delta-function becomes

$$E_i - E_f - \hbar\omega = \hbar\omega[1 - n(\omega)\beta_{\parallel}\cos\theta]. \tag{3}$$

Since electron momentum is conserved in the problem formulated above, we may consider a single independent particle, i.e., $dN = \omega^2 d\omega d\Omega$. Expression (3) leads to the conditions for the emission of radiation and its directivity, namely, $n(\omega)\beta_{\parallel} = 1$ and $\theta = \arccos(1/n\beta_{\parallel})$. These are analogous to the corresponding conditions for the Vavilov emission. It is possible to obtain an exact analytic expression for the quantities in (2.1) in which $\psi_{1,2}(x)$ satisfies the Schroedinger equations

$$\psi_1''(x) + k_1^2\psi_1(x) = V(x)\psi_1(x), \tag{4}$$

$$\psi_2^{*\prime\prime}(x) + k_2^2\psi_2^*(x) = V(x)\psi_2^*(x). \tag{5}$$

Let us multiply (4) by $\psi_2^*(x)$ and (5) by $\psi_1(x)$, and then subtract one from the other:

$$\frac{d}{dx}[\psi_2^*(x)\psi_1'(x) - \psi_2^{*\prime}(x)\psi_1(x)] = (k_2^2 - k_1^2)\psi_2^*(x)\psi_1(x). \tag{6}$$

Multiplying (6) by $\exp(-i\varkappa_x x)$ and integrating by parts, we obtain

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$$(k_2^2-k_1^2)J_{\parallel}(\varkappa_x)=i\varkappa_x J_{\perp}(\varkappa_x)+F(\varkappa_x)-i\varkappa_x I(\varkappa_x), \quad (7)$$

where

$$J_{\parallel}(\varkappa_x)=\int_{-\infty}^{\infty}\exp(-i\varkappa_x x)\psi_2^*(x)\psi_1(x)\,dx, \quad (7.1)$$

$$J_{\perp}(\varkappa_x)=\int_{-\infty}^{\infty}\exp(-i\varkappa_x x)\psi_2^*(x)\psi_1'(x)\,dx,$$

$$I(\varkappa_x)=\int_{-\infty}^{\infty}\exp(-i\varkappa_x x)\times\psi_2^{*\prime}(x)\psi_1(x)\,dx,$$

$$F(\varkappa_x)=\{\exp(-i\varkappa_x x)[\psi_2^*(x)\psi_1'(x)-\psi_2^{*\prime}(x)\psi_1(x)]\}_{-\infty}^{+\infty}. \quad (7.2)$$

Using (7) and the definition of the primitive

$$f(x)=\int f'(x)\,dx=\exp(-i\varkappa_x x)\psi_2^*(x)\psi_1(x)$$

we finally have

$$2i\varkappa_x J_{\perp}(\varkappa_x)=[k_2^2-k_1^2-\varkappa_x^2]J_{\parallel}(\varkappa_x)-F(\varkappa_x)+\Phi(\varkappa_x) \quad (8)$$

where

$$\Phi(\varkappa_x)=\{i\varkappa_x\exp(-i\varkappa_x x)\psi_2^*(x)\psi_1(x)\}_{-\infty}^{+\infty}.$$

Since $\varkappa_x=\varkappa\sin\theta\cos\varphi$, there are angles θ and φ lying in the ranges $0\leqslant\theta\leqslant\pi$, $0\leqslant\varphi\leqslant 2\pi$ for which $\varkappa_x=0$. We recall[2] that, when $f(x)$ and $\varphi(x)$ are related to $xf(x)=\varphi(x)$ and can be zero, we have, in general,

$$f(x)=P.[\varphi(x)/x]+A\delta(x), \quad (9)$$

where $P_\bullet$ denotes the principal value of the integral in the Cauchy sense. From (8) and (9), it then follows that

$$J_{\perp}=\pi[\psi_2^*(x)\psi_1(x)]'_{x=0}+A\delta(\varkappa_x), \quad (10)$$

since

$$P.[\Phi(\varkappa_x)/\varkappa_x]=0, \quad P.[\varkappa_x J_{\parallel}(\varkappa_x)]=-2\pi i[\psi_2^*(x)\psi_1'(x)]'_{x=0},$$

$$P.[F(\varkappa_x)/\varkappa_x]-(k_2^2-k_1^2)P.[J_{\parallel}(\varkappa_x)/\varkappa_x]=0.$$

It is readily seen that the second term in (10) appears in (2) in the form $x\delta(x)=0$ and need not, therefore, be taken into account. Since $k_2^2-k_1^2-\varkappa_x^2\neq 0$, we find, using (8), that

$$J_{\parallel}(\varkappa_x)=[2i\varkappa_x J_{\perp}(\varkappa_x)+F(\varkappa_x)-\Phi(\varkappa_x)]/(k_2^2-k_1^2-\varkappa_x^2). \quad (11)$$

It follows from (3) that the radiation that we are considering lies in the optical range [it appears when $n(\omega)>1$] and we can neglect the effect of the emission of the photon on the scattering of the particle by the plane. We then have $F(\varkappa_x)=\Phi(\varkappa_x)=0$ and, according to (6), the Wronskian of the solutions $\psi_1(x)$ and $\psi_2(x)$ is independent of x, which allows us to obtain from (10) an explicit expression for $J_\perp(\varkappa_x)$. Moreover it is clear from (10) that initial and final state functions have different parities, Using the asymptotic form of the even $\psi_+(x)$ and $\psi_-(x)$ solutions of (5) and (4).

$$\lim_{x\to\infty}\psi_+(x)=2e^{i\varphi_+}\cos(k_\perp|x|+\varphi_+),$$
$$\lim_{x\to\infty}\psi_-(x)=2e^{i\varphi_-}\,\mathrm{sign}(x)\cos(k_\perp|x|+\varphi_-), \quad (12)$$
$$k_\perp=(2m_0E_\perp)^{1/2}/\hbar,$$

where $E_\perp$ is the transverse energy, we obtain the following expression for the Wronskian:

$$W=\psi_2^*(x)\psi_1'(x)-\psi_2^{*\prime}(x)\psi_1(x)$$
$$=-4\pi k_\perp|t|\exp\{i(\varphi_--\varphi_+)\}=\psi_+^*(0)\psi_-'(0). \quad (13)$$

Consequently,

$$J_\perp(\varkappa_x)=-4\pi k_\perp\exp\{i(\varphi_--\varphi_+)\}\sin(\varphi_--\varphi_+), \quad (14)$$

where $\sin(\varphi_--\varphi_+)=|t|$ is the modulus of the amplitude for the propagation of the particle above the potential well produced by the atomic plane.

Substituting (14) in (1), and using (3) and (11), we obtain the following expression for the energy emitted by the particle per unit path length:

$$\frac{dI}{dl}=|t|^2\frac{dI_c}{dl}, \quad (15)$$

where

$$\frac{dI_c}{dl}=4\pi^2\frac{e^2}{hc^2}\int h\nu\left(1-\frac{1}{n^2\beta_\parallel^2}\right)d\nu,$$

is the Čerenkov energy. It is clear that the influence of the scattering process on the Cerenkov emission intensity is a purely quantum-mechanical effect. It manifests itself strongly when the particle moves near the top of the well. With increasing distance from the top of the well, the particle transmission coefficient above the well $|t|^2$ tends to unity and we obtain the usual Čerenkov radiation. The periodicity of the potential acting in the transverse direction can be readily taken into account. The point is that, in the one-dimensional case, the solid-state problem can be solved exactly in terms of the t matrix formalism for scattering by an individual periodicity element.[3] The dispersion relation is then

$$\cos(k_\perp d+\delta)/|t|=\cos Kd, \quad (16)$$

where $\hbar K$ is the quasimomentum, $k_\perp$ is the transverse wave number of the electron, $\delta=\arg t$, t is the amplitude for the propagation of the electron above the well, and d is the separation between atomic planes. For a symmetric potential, the Bloch wave function has the form[4] $(0<Kd<\pi)$

$$\psi(x)=\left[1+\frac{\psi_-(d/2)}{\psi_+(d/2)}\mathrm{ctg}^2\frac{Kd}{2}\right]^{-1/2}\times\left[\psi_-(x)-i\frac{\psi_-(d/2)}{\psi_+(d/2)}\mathrm{ctg}^2\frac{Kd}{2}\psi_+(x)\right], \quad (17)$$

so that, according to (16) and (17),

$$\frac{dI}{dl}=\frac{1}{4}|t|^2\sin^2 Kd\frac{dI_c}{dl}. \quad (18)$$

At the edges of energy bands, $Kd=\pi n$ and the function $\psi(x)$

has a definite parity. Hence, the radiation emitted as a result of spontaneous transitions between the bottom and the top of the energy band is described by (15), so that the intensity of the radiation can be controlled by tilting the crystal, first, because of the change in $|t|^2$ and, secondly, because of the periodicity of the potential acting in the transverse direction. In the special case of the Pöschl-Teller potential $V(x) = -V_0/\cosh^2\alpha x$, the transmission coefficient above the well[5] is

$$|t|^2 = \mathrm{sh}^2\pi\varepsilon(\sinh^2\pi\varepsilon + \sin^2\pi s)^{-1}, \tag{19}$$

where

$$\varepsilon = \left[\frac{2E_\parallel E_\perp}{(\cosh\alpha)^2}\right]^{1/2}, \quad s = \frac{1}{2}\left\{-1 + \left[1 + \frac{8m_0c^2V_0\gamma}{(\cosh\alpha)^2}\right]^{1/2}\right\},$$

$E_\perp$ is the transverse energy of the electron, $E_\parallel$ is the longitudinal energy of the electron, V_0 is the well depth, m_0 is the electron rest mass, and α^{-1} is a measure of the well width. Since $|t|^2$ oscillates as the γ-factor is varied, the radiation intensity will behave in an analogous manner. This feature can be exploited in measurements of the transmission coefficient above the well. For values of γ for which s is an integer, (16) shows that the energy spectrum above the well spreads out into a continuum (4). The radiation intensity is then described by (15) across the entire energy spectrum. The above results show that we have a qualitative explanation of the experimentally established[6] orientational dependence of the optical radiation emitted when an electron is channeled in a single crystal.

We note in conclusion that the fact that the Pöschl-Teller potential provides a good approximation to the potential between the planes is not fortuitous. It was shown in Ref. 4 on the basis of an anlysis of experimental data on the channeling of electrons in a single crystal that the potential in the space between atomic planes is of the soliton type. This is an indirect confirmation of the suggestion made in Ref. 7 that the nonlinear interaction between a channeling particle and a crystal takes a portion of the latter that is close to the particle trajectory to an excited state of the one-dimensional or two-dimensional lattice, so that the model of continuous atomic planes (lines) used in the theory of channeling corresponds to physical reality and is not merely a convenient approximation.

3. CONCLUSION

1. It is clear from (18) and (19) that the radiation described above depends not only on the velocity, but also on the rest mass of the particle (the dependence on mass is included in the transmission coefficient). Hence, by measuring the transmission coefficient and the position of the radiation cone, it is possible to determine the velocity and rest mass of the particle (which cannot be done with the usual Čerenkov counters).

2. An analogous type of radiation can also be emitted in the case of glancing incidence of relativistic electrons on the surface of single crystals. If the condition for the emission of the radiation is then satisfied only in the second medium, the entire radiation is emitted into this medium (crystal).

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