Expansion of a relativistic proton cloud in a magnetized plasma

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The expansion of a cloud of fast protons in a cold plasma, accompanied by the excitation of Alfvén waves, is analyzed on the basis of quasilinear theory. Self-similar solutions are derived for the cases of nonrelativistic and ultrarelativistic particles. Various nonlinear mechanisms which may operate to reverse the direction of resonant particles and to trap them near a shock wave are examined.

1. The mechanism by which cosmic rays are accelerated by shock waves in the interplanetary medium, which has attracted considerable interest in recent years, is essentially a version of Fermi acceleration.[1] The acceleration by this mechanism occurs as the particles repeatedly cross a shock front; fast particles are trapped near the front as a result of their scattering by small-scale irregularities of the magnetic field. Krymskiĭ[2] and Bell[3] have shown that this acceleration gives rise to a universal power-law energy spectrum which agrees quite well with observations.

Bell[3] was the first to suggest that the role of the small-scale magnetic-field irregularities which scatter the particles might be played by Alfvén turbulence excited by fast protons as they disperse across the wavefront into cold plasma. The scattering of these protons by the Alfvén waves which they themselves excite causes the distribution function of the fast particles to approach isotropy and results in the trapping of these particles near the shock front.

In the present paper we use quasilinear theory to analyze the dispersal of a cloud of relativistic protons in a cold magnetized plasma, accompanied by the excitation of Alfvén waves.

2. The initial system of equations for this problem can be written as follows if we assume that the protons excite only Alfvén waves which propagate along the magnetic field:

$$\frac{\partial f}{\partial t}+v_z\frac{\partial f}{\partial z}=\frac{e^2}{4c^2}\frac{\partial}{\partial p_z}\left[v_\perp{}^2(w,p_z)\frac{|H_k|^2}{v_z}\frac{\partial f}{\partial p_z}\right], \qquad (1)$$

$$\frac{\partial |H_k|^2}{\partial t}+\frac{d\omega_k}{dk}\frac{\partial |H_k|^2}{\partial z}$$

$$=\frac{4\pi^3e^2}{\omega_k}\frac{\omega_{Hi}{}^2}{\omega_{pi}{}^2}|H_k|^2\int\limits_{w_{min}}^{\infty}v_\perp{}^2(w,p_z)\frac{\mathscr{E}^2}{c^4}\frac{\partial f}{\partial p_z}dw. \qquad (2)$$

This system of equations is written for waves which are polarized along the direction in which the electrons gyrate in the magnetic field. The condition for the resonance of these electrons with fast ions corresponds to the anomalous Doppler effect:

$$kv_z=\omega_k+\omega_{Hi}m_ic^2/\mathscr{E} \quad \text{or} \quad kp_z\approx m_i\omega_{Hi}, \qquad (3)$$

since the frequency of the waves which are excited satisfies $\omega_k = kv_A \ll \omega_{Hi} m_i c^2/\mathscr{E}$, where $v_A = H_0/(4\pi n_0 m_i)^{1/2} \ll c$ and $\omega_{Hi} = eH_0/m_i c$ are the Alfvén velocity and the ion cyclotron frequency, respectively, calculated from the properties of the cold plasma. The derivative of the distribution function $f(w, p_z)$ with respect to p_z on the right sides of (1) and (2) is evaluated along the lines of diffusion of resonant particles:

$$w=\mathscr{E}-\frac{\omega_k}{k}p_z(\mathscr{E},v_z)=\text{const}, \qquad (4)$$

where $\mathscr{E} = (m_i^2c^4 + (p_\perp^2 + p_z^2)c^2)^{1/2}$ is the energy of the particles. Figure 1 shows lines of diffusion in the $p_\perp, p_z$ plane, The quantity $w_{\min}(p_z) = (m_i^2c^4 + p_z^2c^2)^{1/2}$ is the minimum value of w at the given p_z.

Since the number density of relativistic protons is low, $n_r \ll n_0$, a small change in the density of the cold plasma is sufficient to restore quasineutrality, and the polarization electric field can be ignored. We can then write the initial distribution function along lines of diffusion, w = const, as follows:

$$f=\begin{cases}0, & v_z<z/t,\\ f_0(w,p_z), & v_z>z/t,\end{cases} \qquad (5)$$

where f_0 is the distribution function of the protons of the expanding cloud.

The proton distribution function (5) is unstable with respect to the excitation of Alfvén waves through the anomalous Doppler effect, since it is obvious that under the condition $v_z \approx z/t$ the condition $\partial f/\partial p_z > 0$ will hold. The quasilinear diffusion of resonant protons in the field of these waves causes the distribution function of these protons to approach isotropy and thereby determines the dynamics of the expansion of the proton cloud. The results of the solution of the corresponding problem are given below.

The analogous problem for hot electrons, whose relaxation in a cold plasma involves the excitation of plasma waves by these electrons, was solved previously by Ryutov and Sagdeev.[5] We will draw extensively from their results. As in Ref. 5, we assume that on the time scale of the dispersal of the

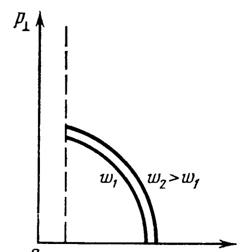

FIG. 1. Lines of diffusion of resonant particles on the $p_z, p_\perp$ plane.

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cloud the instability is fast, and a solution can be sought through an expansion in the parameter $\varepsilon = (\gamma t)^{-1} \ll 1$, where γ is a typical growth rate of the instability, found from (2). This solution method is an analog of the standard gasdynamic approach in which the solution of the kinetic equation is found through expansion in the reciprocal of the collision rate (Ref. 4, for example). To lowest order in ε, the distribution function f is equal to that stable function f^∞ for which the right sides of (1) and (2) vanish. Let us determine this function more specifically. For $w \leqslant m_i c^2/(1 - z^2/t^2c^2)^{1/2}$, there are no particles at all, since on such lines of diffusion the condition $v_z < z/t$ holds. We thus have

$$f=0, \quad w \leqslant m_i c^2/(1-z^2/c^2t^2)^{1/2}. \tag{6}$$

We assume $m_i c^2/(1 - z^2/c^2t^2)^{1/2} < w \leqslant (m_i^2c^4 + \pi^2c^2)^{1/2}$, where π is the maximum longitudinal momentum of a resonant particle, found from resonance condition (3): $\pi \approx m_i\omega_{Hi}/k_{\min}$. The interaction with the waves then gives rise to a plateau along the line $w = \text{const}$ for all values of p_z on this line; i.e.,

$$f=h(t,\ z,\ w), \quad p_{min}<p_z<p_{max}=(w^2/c^2-m_i^2c^2)^{1/2}. \tag{7}$$

The height of this plateau is found from the conservation of the number of particles on each line of diffusion. Integrating (1) over p_z from $p_{\min}$ to $p_{\max}(w)$, and using $|H_k|^2 = 0$ at the lower limit and $v_\perp = 0$ at the upper limit, we find

$$\frac{\partial h}{\partial t}+\frac{c^2}{2w}\left(p_{max}(w)+p_{min}\right)\frac{\partial h}{\partial z}=0. \tag{8}$$

The solution of this equation under the condition $p_{\min} \ll p_{\max}$, which satisfies the obvious initial and boundary conditions $f = 0$ at $t = 0$ and $f = f_0(w, p_z = 0)$ at $z = 0$, is

$$h=\begin{cases} 0, & z>{}^1/_2c(1-m_i^2c^4/w^2)^{1/2}t, \\ f_0(w,p_z=0), & z<{}^1/_2c(1-m_i^2c^4/w^2)^{1/2}t; \end{cases} \tag{9}$$

i.e., the relaxation front on the line $w = \text{const}$ moves at a velocity $\frac{1}{2}c(1 - m_i^2c^4/w^2)^{1/2}$. An expression for the spectral energy density of the Alfvén waves excited during the expansion of a proton cloud in a cold plasma is found from the energy integral of quasilinear equations (1) and (2). Ignoring in (2) the term which describes the drift of Alfvén waves (the group velocity of these waves is low in comparison with the scale velocity of the relativistic protons), we find, by the standard approach, the following expression for the spectral density of magnetic-field fluctuations:

$$|H_k|^2=\frac{16\pi^3}{\omega_k}\frac{\omega_{Hi}^2}{\omega_{pi}^2}p_z^2\left[\int_{w_{min}}^{\infty}hw\,dw+\frac{c^2p_z}{2}\int_0^t dt'\int_{w_{min}}^{\infty}\frac{\partial h}{\partial z}dw\right],$$

$$k=\frac{m_i\omega_{Hi}}{p_z}, \tag{10}$$

$$w_{min}\approx(c^2p_z^2+m_i^2c^4)^{1/2}.$$

3. We first consider the case of a power-law energy distribution of the relativistic particles—a simple case but the one most important for applications. The energy distribution observed in the cosmic rays is $\mathscr{E}^{-2.5}$ and corresponds to a momentum distribution

$$f_0(\mathscr{E})=K/\mathscr{E}^{3,5}(\mathscr{E}^2-m_i^2c^4)^{1/2}.$$

Determining the constant K from the normalization condition $\int f_0 dp = n_R$, and using (9), we find

$$h(w)=\frac{3n_R}{8\pi}\frac{m_i^{3/2}c^6}{w^{3,5}}\frac{1}{(w^2-m_i^2c^4)^{1/2}},$$

$$\xi=\frac{z}{t}<\frac{c}{2}\left(1-\frac{m_i^2c^4}{w^2}\right)^{1/2}.$$

In this case expression (10) leads to the following expression for the spectral energy density of the waves in the long-wave region ($kc < \omega_{Hi}$):

$$|H_k|^2=\frac{12\pi^2}{5}\frac{\omega_{Hi}^2}{\omega_k\omega_{pi}^2}n_Rm_ic\frac{\omega_{Hi}^2}{k^2}\left[1-\left(\frac{2\xi}{c}\right)^2\right]^{5/4} \times\left[1-\frac{5}{7}\frac{\omega_{Hi}}{kc}\left[1-\left(\frac{2\xi}{c}\right)^2\right]^{1/2}\right]. \tag{11}$$

As time passes ($\xi < \frac{1}{2}c(\omega_{Hi}^2/(k^2c^2 + \omega_{Hi}^2))^{1/2}$, the steady-state spectrum shown in Fig. 2 is established:

$$|H_k^\infty|^2=\frac{24\pi^2}{35}\frac{\omega_{Hi}^2}{\omega_{pi}^2}\frac{n_Rm_ic^3}{\omega_k}\left(\frac{kc}{\omega_{Hi}}\right)^{1/2}. \tag{12}$$

The spectral density has an integrable singularity ($|H_k|^2 \propto k^{-1/2}$) as $k \to 0$, i.e., in the limit of high energies of the resonant particles. Nevertheless, the relaxation length for these particles—the distance over which their distribution function becomes isotropic—turns out to be extremely large, and it increases rapidly with increasing energy of the particles. From diffusion equation (1) and (12) we find the following estimate of the relaxation length:

$$l_{REL}\approx\frac{70}{3\pi}\frac{n_0}{n_R}\frac{v_A}{\omega_{Hi}}\left(\frac{\mathscr{E}}{m_ic^2}\right)^{3/2}. \tag{13}$$

This solution, which corresponds to the approximation $\varepsilon \ll 1$, i.e., $z \gg l_{\rm REL}$, is evidently inapplicable at high energies of the resonant particles. The question of whether effective "trapping" near the shock wave of the highest-energy component of the cosmic rays can occur apparently cannot be solved in the quasilinear theory. The induced scattering of Alfvén waves by thermal plasma particles may prove important. As usual, this scattering causes a spectral pumping of energy to long wavelengths, $k \to 0$, which is responsible for the scattering of high-energy particles.

Another important question which essentially cannot be solved in the quasilinear theory involves the diffusion of resonant protons to the region of small values of the longitu-

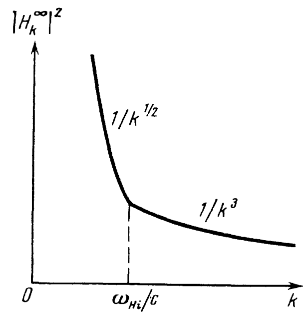

FIG. 2. Spectral density of the waves in the steady state.

dinal momentum, $-p_{\min} < p_z < p_{\min}$. If the interaction with the Alfvén turbulence is to in fact result in a trapping of the resonant protons near the shock front, the direction in which these particles are moving must be reversed. In the short-wave part of the spectrum, however, which is the part responsible for the diffusion at small values of p_z, the cyclotron absorption of Alfvén waves by the cold plasma becomes important, and the excitation of these waves becomes impossible. Achterberg[6] has analyzed various nonlinear mechanisms for a diffusion of resonant particles at small values of v_z. One of these mechanisms involves nonlinear broadening of a wave-particle resonance (Dupree broadening), which is described in our case of Alfvén waves by

$$\Delta v_z^{NL} \approx \left(\frac{D_k}{k} \frac{c^4}{\mathscr{E}^2} \right)^{1/3} = \left(\frac{|H_k|^2}{4H_0^2} v_\perp^2 \omega_{Hi} \frac{m_i^2 c^4}{\mathscr{E}^2} \right)^{1/3}, \qquad (14)$$

where D_k is the coefficient for the diffusion of the resonant particles among the waves, determined from (1). The cyclotron absorption in the cold plasma has the consequence that the wave numbers of the Alfvén waves are restricted to $k \ll m_i \omega_{Hi} / p_{\min}$, where $p_{\min} = a m_i v_T$, v_{Ti} is the thermal velocity of the ions of the cold plasma, and the numerical factor a is $a \approx 2$–3. Accordingly, if the direction of the resonant particles on a line $w = \text{const}$ is to be reversed, the nonlinear mechanism must cause a particle to pass through a region of width $\Delta v_z \approx p_{\min} c^2 / w$. From a comparison with (14) we find the following condition under which the nonlinear broadening is effective:

$$\left. \frac{k |H_k|^2}{4H_0^2} \right|_{k \approx k_{max}} \geqslant \frac{p_{min}^2}{m_i^2 v_\perp^2} \frac{m_i c^2}{\mathscr{E}}. \qquad (15)$$

The energy spectrum of the short ($kc > \omega_{Hi}$) Alfvén waves can also be found without difficulty from Eq. (10). The result is

$$|H_k|^2 = \frac{6\pi^2 \omega_{Hi}^2}{\omega_k \omega_{pi}^2} n_R m_i c^3 \frac{\omega_{Hi}^2}{k^2 c^2} \int_{\psi(\xi)}^{\infty} \frac{dt}{\mathrm{ch}^{5/2} t}. \qquad (16)$$

Here $\psi(\xi) = (1 - (2\xi/c)^2)^{-1/2} - 1$. As time elapses, and we have $\xi \ll c/2$, $\psi(\xi) \to 0$, we find the following result for the integral in (16):

$$\int_0^\infty dt/\mathrm{ch}^{5/2} t = \frac{4^{1/4} \Gamma(5/2)}{\Gamma^2(5/4)} \approx 0.87.$$

A steady-state wave distribution is established; this distribution has a rather rapid decay ($\sim 1/k^3$) on the side of large wave numbers. This distribution is also shown in Fig. 2. Substituting $|H_k|^2$ from (16) along with $v_\perp = c$ into condition (15), we find that the direction can reverse is possible only for relativistic particles of sufficiently high energy

$$\frac{\mathscr{E}}{m_i c^2} \geqslant \frac{4}{5\pi} \frac{n_0}{n_R} \frac{c}{v_A}. \qquad (17)$$

A second mechanism which may be involved in reversing the direction of the resonant particles is a magnetic-mirror mechanism. This mechanism, also proposed by Achterberg,[6] is based on the circumstance that the long-wave ($kc < \omega_{Hi}$) part of the spectrum of Alfvén waves gives rise to a rather slow modulation (on the scale of the ion Larmor radius) of the magnetic field and to the formation of magnetic mirrors for particles with a low longitudinal velocity. The increase in the transverse momentum of a relativistic particle for this type of modulation of the magnetic field is

$$\Delta p_\perp^2 = p_\perp^2 - p_{\perp 0}^2 \approx \frac{\mathscr{E}^2}{c^2} \frac{\delta H}{H_0},$$

$$\delta H = \left(H_0^2 + \sum_{kc < \omega_{Hi}} |H_k|^2 \right)^{1/2} - H_0,$$

and it follows from the condition $\mathscr{E} = \text{const}$ that the magnetic mirror is capable of stopping particles with a velocity $v_z = p_{\min} c^2 / w$ if

$$\sum_{kc < \omega_{Hi}} \frac{|H_k|^2}{2H_0^2} \geqslant \frac{p_{min}^2}{m_i^2 c^2} \left(\frac{m_i c^2}{\mathscr{E}} \right)^2. \qquad (18)$$

Comparison of (18) and (15) shows that the mirror mechanism is more important, primarily because in the spectrum described by (12) and (16) most of the energy is in the long-wave region; another contributing factor is the greater effectiveness of the mirror mechanism at high energies of the relativistic particles.

Substituting $|H_k|^2$ from (12) into condition (18) and integrating over k in the long-wave part of the spectrum, we find that the mirror mechanism causes trapping of particles with energies

$$\frac{\mathscr{E}}{m_i c^2} \geqslant \left(\frac{35 n_0}{6 n_R} \frac{p_{min}^2}{m_i^2 c^2} \frac{v_A}{c} \right)^{1/3}. \qquad (19)$$

If the direction in which the particles are moving is reversed, an instability of the Alfvén waves will also occur in the region of negative p_z, but in this case the instability involves the normal Doppler effect. The interaction with these waves gives rise to a plateau along the lines of diffusion, $w = \text{cost}$, at $-p_{\max}^- < p_z < p_{\max}^+$, where

$$p_{max}^{\pm}(w) = \frac{1}{c} \left\{ \left(w \pm \frac{v_A}{c} (w^2 - m_i^2 c^4)^{1/2} \right)^2 - m_i^2 c^4 \right\}^{1/2}$$

$$\approx \frac{(w^2 - m_i^2 c^4)^{1/2}}{c} \left(1 \pm \frac{v_A}{c} \frac{1}{(1 - m_i^2 c^4 / w^2)^{1/2}} \right)^{1/2}.$$

For the case under consideration, with

$$h(w) \sim \frac{1}{w^{3,5}} (w^2 - m_i^2 c^4)^{-1/2},$$

the average proton velocity is

$$\langle v_z \rangle = \left\{ \int dp_z p_\perp dp_\perp h \right\}^{-1} \int dp_z p_\perp dp_\perp \frac{p_z c^2}{\mathscr{E}} h \approx v_A. \qquad (20)$$

The relaxation front in Eq. (8) moves at the same velocity, $u = (c^2/2w)(p_{\max}^+ - p_{\max}^-) \approx v_A$. Under the condition that the region $v_z \approx 0$ is traversed, the proton trapping near the shock front is thus quite effective.

4. A distinctive feature of this solution is that there is no lower boundary $k = k_{\min}$ on the spectrum, at which $|H_k|^2 = 0$, in the long-wave part of the spectrum. Accordingly, the plateau is established on the distribution function at all possible values of p_z for each line of diffusion $w = \text{const}$. A solution of another type occurs when a cloud of relativistic protons with a Maxwellian distribution function

$$f_m(\mathscr{E})=A\exp\left(-\frac{\mathscr{E}}{T_R}\right), \tag{21}$$

where

$$A=\frac{n_R}{4\pi(m_ic)^3}\frac{m_ic^2}{T_RK_2(m_ic^2/T_R)},$$

expands in a cold plasma. In this case, a will be shown below, the spectral density vanishes ($|H_k|^2=0$) at $k\leqslant k_{\min}$. Accordingly, a plateau is established at all values of p_z on the lines of diffusion for which $w\leqslant(m_i^2c^4+\pi^2c^2)^{1/2}$, where $\pi=m_i\omega_{Hi}/k_{\min}$. The distribution function at these values of w is found from Eq. (9).

On lines of diffusion with $w>(m_i^2c^4+\pi^2c^2)^{1/2}$ a plateau is established only at $p_x\leqslant\pi$; at $\pi<p_z<p_{\max}(w)$ the proton distribution function is the same as the initial function, $f_m(w,p_z)$, since there is no resonance with waves at these values of p_z. We thus have

$$f=\begin{cases} h(t,z,w), & p_{min}<p_z\leqslant\pi,\\ f_m(w,p_z), & \pi<p_z<p_{max},\end{cases} \qquad w>(m_i^2c^4+\pi^2c^2)^{1/2}. \tag{22}$$

At these values of w, a discontinuity occurs at $p_z=\pi$ on the lines of diffusion. Integrating Eq. (1) over p_z, we find the following equation for this case, instead of (9):

$$\frac{\partial}{\partial t}(h\pi)+\frac{\partial}{\partial z}\left(h\frac{c^2\pi^2}{2w}\right)-f_m(w,\pi)\left(\frac{\partial\pi}{\partial t}+\frac{c^2\pi}{w}\frac{\partial\pi}{\partial z}\right)=0. \tag{23}$$

For the values of w under consideration here, the height of the plateau, $h(t,z,w)$, is determined along with its boundary $\pi(t,z)$. A second equation is found from energy integral (10) of the initial system of quasilinear equations. Setting $p_z=\pi$ in this equation, we find the following equation from the condition $|H_k|^2|_{k=k_{\min}}=0$:

$$\int_{w_{min}}^{\infty} hw\,dw+\frac{\pi c^2}{2}\int_{w_{min}}^{\infty}dw\frac{\partial}{\partial z}\int_0^t h\,dt'=0. \tag{24}$$

System of equations (23), (24) can be used to determine $h(t,z,w)$ and $\pi(t,z)$.

As in Ref. 5, we seek a self similar solution of these equations:

$$h=h(\xi,w),\quad \pi=\pi(\xi), \tag{25}$$

where $\xi=z/t$ is the self-similar variable. In this solution, as in the solution discussed in the preceding section, the number of resonant particles in the region $z>0$ increases linearly with time, in accordance with a constant flux density of particles across the boundary. An important distringuishing feature of quasihydrodynamic equations (23), (24)—as well as of Eq. (18)—is that these equations were derived to lowest order in the parameter $\varepsilon=1/\gamma t$ and therefore do not contain a relaxation length, just as the equations of ordinary gasdynamics do not include a collision rate. As has been pointed out elsewhere,[5] it is this distinguishing feature of the quasihydrodynamic equations of the quasilinear theory which allows us to seek a self-similar solution. The self-similar formulation of the problem puts Eqs. (23) and (24) in the form

$$\pi\frac{\partial h}{\partial\xi}\left(\frac{c^2\pi}{2w}-\xi\right)+(h-f_m(w,\pi))\left(\frac{c^2\pi}{w}-\xi\right)\frac{\partial\pi}{\partial\xi}=0, \tag{26}$$

$$\int_{w_{min}}^{\infty} hw\,dw+\frac{c^2\pi}{2}\int_{w_{min}}^{\infty}\frac{\partial}{\partial\xi}\left[\xi\int_{\xi}^{\xi_{max}}h\frac{d\xi'}{\xi'^2}\right]dw=0. \tag{27}$$

In the last of these equations,

$$\xi_{max}=\frac{c}{2}\left(1-\frac{m_i^2c^4}{w^2}\right)^{1/2}$$

is the maximum value of ξ, found from (9). We note that $h(\xi_{\max})\neq 0$.

We have examined the solution of system (26), (27) in the two limiting cases of nonrelativistic and ultrarelativistic protons. In the nonrelativistic case, we can use the approximation $w=m_ic^2$ in Eq. (26), and this distribution function h can be written in the form

$$h(\xi,w)=Ag(\xi)\exp(-w/T_R). \tag{28}$$

For $g(\xi)$ and for the upper boundary of the plateau along the velocity scale, $u(\xi)=\pi(\xi)/m_i$ we find the system of equations

$$u\frac{dg}{d\xi}\left(\frac{u}{2}-\xi\right)+(g-\tilde f_0(u))(u-\xi)\frac{du}{d\xi}=0,$$
$$u\frac{dg}{d\xi}\left(\xi-\frac{u}{2}\right)-g\xi\frac{du}{d\xi}=0, \tag{29}$$

where

$$\tilde f_0(u)=\exp[-m_iv_Au/T_R].$$

A similar system of equations was solved in Ref. 5. The solution is

$$\xi=\frac{\tilde f_0-g}{\tilde f_0-2g}u,\qquad g=\frac{u^2}{2}\left[\int_0^u du'u'/\tilde f_0(u')\right]^{-1}. \tag{30}$$

Under the condition $m_iv_Au\ll T_R$, we find, in particular, from these equations an expression describing the shift of the upper boundary of the plateau: $u^2\approx 3T_R\xi/m_iv_A$. In the opposite limiting case, $m_iv_Au\gg T_R$, this expression becomes $u\approx 2\xi$.

In the ultrarelativistic case we assume $\pi c\gg T_R\gg m_ic^2$. We can then set $w\approx w_{\min}\approx\pi c$ in Eq. (26). As before, we seek the distribution function of the resonant particles in the form (28). Integrating over w in (27), we find the following system of equations from (26) and (27):

$$\pi\frac{dg}{d\xi}\left(\frac{c}{2}-\xi\right)+\frac{d\pi}{d\xi}\left(g-\exp\left(-\frac{\pi v_A}{T_R}\right)\right)(c-\xi)=0,$$
$$\pi\frac{dg}{d\xi}\left[\pi\left(\xi-\frac{c}{2}\right)+\frac{T_R}{c}\xi\right]-\frac{T_R}{c}g\xi\frac{d\pi}{d\xi}=0. \tag{31}$$

Again, there is no difficulty in solving system (31). Under the condition $\pi v_A\ll T_R$, the solution is

$$g=1+\frac{T_R\tau}{2\pi c(1-\tau/2)}-\frac{\pi v_A}{T_R},$$
$$\pi=\frac{T_R}{c}\left(\frac{c}{v_A}\right)^{1/2}z(\tau), \tag{32}$$

where the variable $\tau=2\xi/c$ has the range $0\leqslant\tau<1$, and the

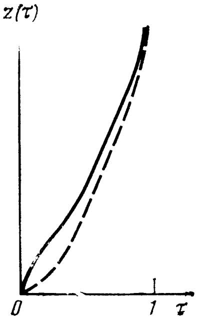

FIG. 3. The solution of Eq. (33) (the solid curve) and the discriminant curve for this equation (the dashed curve).

function $z(\tau)$ satisfies the differential equation

$$\frac{dz}{d\tau}\left[z(1-\tau)-\frac{\tau/2}{1-\tau/2}\frac{1}{z}\right]=\frac{1-\tau}{2(1-\tau/2)^2}. \quad (33)$$

Figure 3 shows the solution of Eq. (33). At $\tau \ll 1$ we have $z \approx (3/2\tau)^{1/2}$. Also shown in this figure is the discriminant curve of Eq. (33):

$$z_D(\tau)=(\tau/2(1-\tau)(1-\tau/2))^{1/2},$$

which has the property that on this curve we have $dz/d\tau \rightarrow \infty$ for any solution of Eq. (33). A solution which runs above the discriminant curve at small values of τ obviously cannot intersect this curve, and for such a solution we have $z \rightarrow \infty$ as $\tau \rightarrow 1$. The asymptotic law by which $z(\tau)$ increases at $\tau \approx 1 (\xi \approx c/2)$ can easily be found from Eq. (33):

$$z(\tau)=\left[\frac{\tau/2}{(1-\tau)(1-\tau/2)}+4(1-\tau)\right]^{1/2}. \quad (34)$$

Consequently, in the limit $\xi \rightarrow c/2$, which [as follows from (9)] corresponds to particles with an infinitely high energy, $w \rightarrow \infty$, the upper boundary along the longitudinal-momentum scale of the relaxation region also increases without bound, $\pi \rightarrow \infty$. In other words, as in the case in the preceding section, the distribution function at these values of ξ becomes isotropic for essentially all values of p_z. The question of the reversal of the direction of the resonant particles is also resolved by analogy with the approach in the preceding section for the case of a power-law energy distribution of the resonant particles.

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1. E. Fermi, Astrophys. J. **1**, 119 (1954).
2. G. F. Krymskiĭ, Dokl. Akad. Nauk SSSR **234**, 1306 (1977) [Sov. Phys. Dokl. **22**, 327 (1977)].
3. A. R. Bell, Mon. Not. R. Astron. Soc. **182**, 147 (1978).
4. S. Chapman and T. G. Cowling, Mathematical Theory of Non-Uniform Gases, Cambridge Univ. Press, 1953 (Russ. transl. IIL, Moscow, 1960).
5. D. D. Ryutov and R. Z. Sagdeev, Zh. Eksp. Teor. Fiz. **58**, 739 (1970) [Sov. Phys. JETP **31**, 396 (1970)].
6. A. Achterberg, Astron. Astrophys. **98**, 161 (1981).

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